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Introduction to Gaussian Processes

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October 24, 2018

Preamble

The **Gaussian Processes** are the widely used stochastic processes for modeling dependent data observed over time, space or even time and space. Here, we'll iniciate our study with a **Probability and Random Process Theory Review** taking some point to base our journey, going through **Linear Regression** and finally the **Gaussian Processes**.
The material here presented isn't sufficient to guide you over basic probability, so it's recommended to have some knowledge, once we'll just take a simple review.

Outline

Probability and Random Process Theory Review

Basic Concepts of Probability Theory

A key concept in the field of pattern recognition is that of **uncertainty**, that arises from both through noise on measurements, as well as through the finite size of data sets. Let's begin from a simple example.

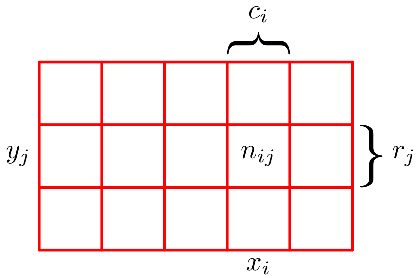
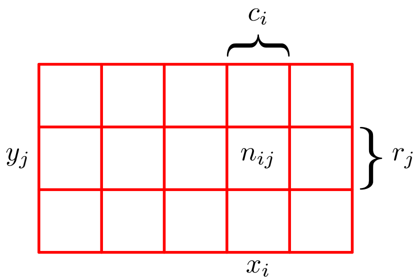


Figure: Considering in the table $X = x_i$ and $Y = y_j$

Basic Concepts of Probability Theory

Let's choose a cell in the 6. We define the probability of choose a cell in a given column is $p(X = x_i) = c_i/N$, being N the total number of cells and $c_i = \sum_j n_{ij}$.



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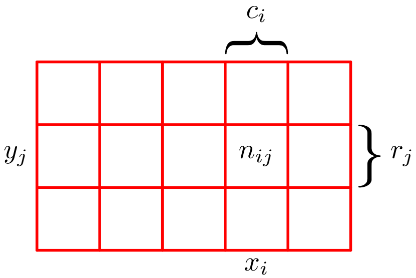
We could say that the probability of choose a cell in a given a row is defined as $p(Y = y_j | X = x_i) = n_{ij}/c_i$.



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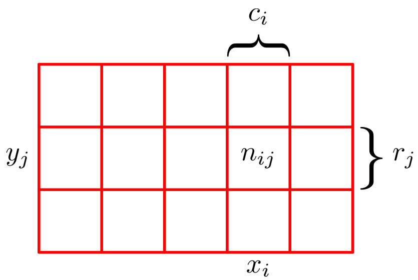
And so, the probability of choose a cell is defined as $p(X = x_i, Y = y_j) = n_{ij}/N$.



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Here, we could see some properties that we call **The Rules of Probability**

The Rules of Probability

- Sum Rule:

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And by the **Product Rule** we prove that

Bayes' Rule

$$p(X,Y) = p(Y|X)p(X) = p(X|Y)p(Y) \Rightarrow p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)} \tag{3}$$

The **Gaussian distribution** is defined as

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} \tag{4}$$

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that means in other words that *if X and Y are independent discrete random variables, then the **joint probability mass function (pmf)** is equal to the product of the marginal pmf's.*

Linear Regression

Bayesian Curve Fitting

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If we could update the ***regression weights*** *as we acquire some new values of the experiment?*

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$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})\,\overbrace{p(\mathbf{w})}^{\text{the weights probability}}}{\underbrace{p(\mathcal{D})}_{\text{the data probability}}} \tag{6}$$

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But, how?

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A good ideia is to express our target values $\mathbf{t}$ in terms of **gaussians distributions** with the mean equals to $y(x, \mathbf{w})$.

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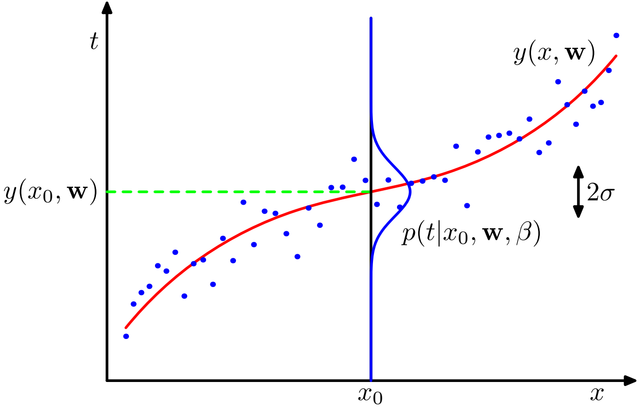

Figure: Schematic of the polynomial function $y(x, \mathbf{w})$ and the gaussian distribution p.

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and then, assume that the training data $\{\mathbf{x}, \mathbf{t}\}$ is indenpendent and identically distributed (i.i.d.) and put on **product form**, i.e. the joint probability is

$$p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t_0|y(x_1, \mathbf{w}), \beta^{-1}) \cap \mathcal{N}(t_n|y(x_0, \mathbf{w}), \beta^{-1}) ... \cap \mathcal{N}(t_n|y(x_0, \mathbf{w}), \beta^{-1}) \quad (8)$$

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$$= \prod_{n=1}^{N} \mathcal{N}(t_n|y(x_n, \mathbf{w}), \beta^{-1}) \tag{9}$$

regarding that $\beta^{-1} = \sigma^2$.

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Applying the **Gaussian distribution** (see 4) will result

$$\ln\left(p(\mathbf{t}|\mathbf{x},\mathbf{w},\beta)\right) = -\frac{\beta}{2}\sum_{n=1}^{N}\{y(x_n,\mathbf{w}-t_n)\}^2 + \frac{N}{2}\ln(\beta) - \frac{N}{2}\ln(2\pi) \tag{11}$$

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(12)

(14)

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$$\frac{1}{N} \sum_{n=1}^{N} \{y(x_n, \mathbf{w} - t_n)\}^2 = \frac{1}{\beta_{ML}} \tag{14}$$

Where β_{ML} is the maximum likelihood.