Introduction to Gaussian Processes

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Linear Regression

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So we could define some strategy to find our model.

Strategy

1 Purpose a **model**, e.g. functions like exponential, polynomial and others.
2 Train our model with the training data set, finding the **unknown parameters**.

Let's fit the points below by polynomial curve fitting
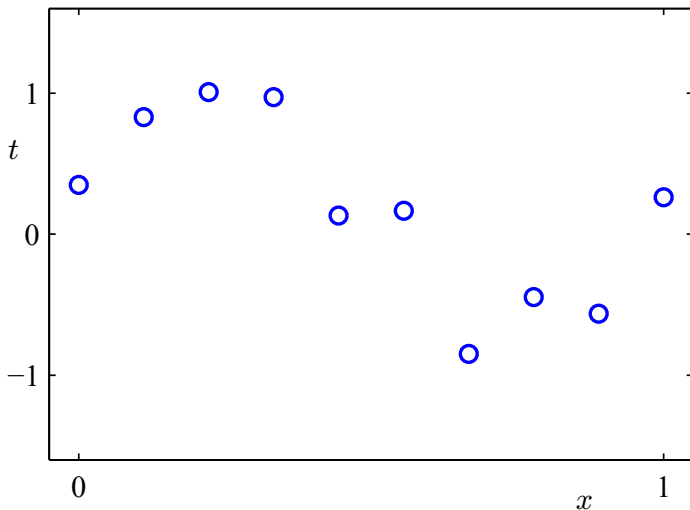
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$$y(x, \mathbf{w}) = w_0 x^0 + w_1 x^1 + w_2 x^2 + ... + w_{M-1} x^{M-1} = \sum_{j=1}^{M-1} w_j x^j$$

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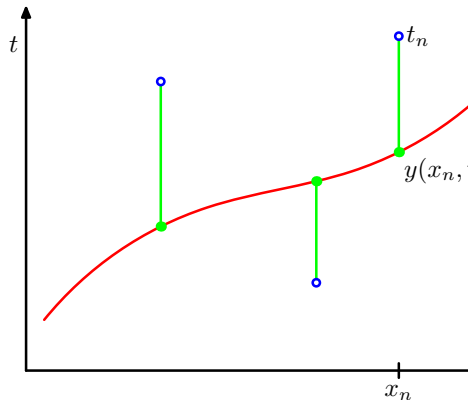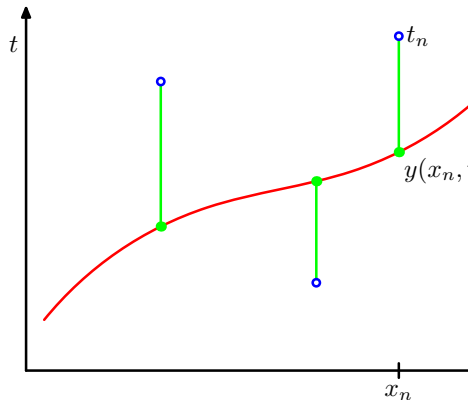
$$y(x, \mathbf{w}) = w_0 \phi_0(x) + w_1 \phi_1(x) + w_2 \phi_2(x) + ... + w_{M-1} \phi_{M-1}(x) = \sum_{j=1}^{M-1} w_j \phi_j(x)$$

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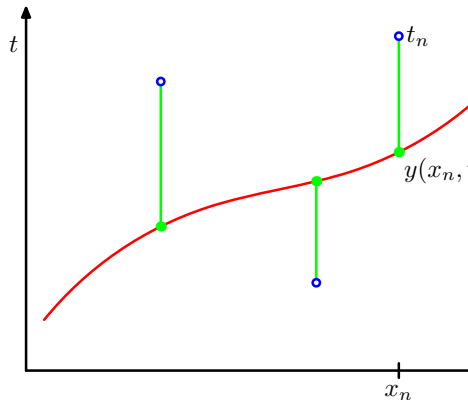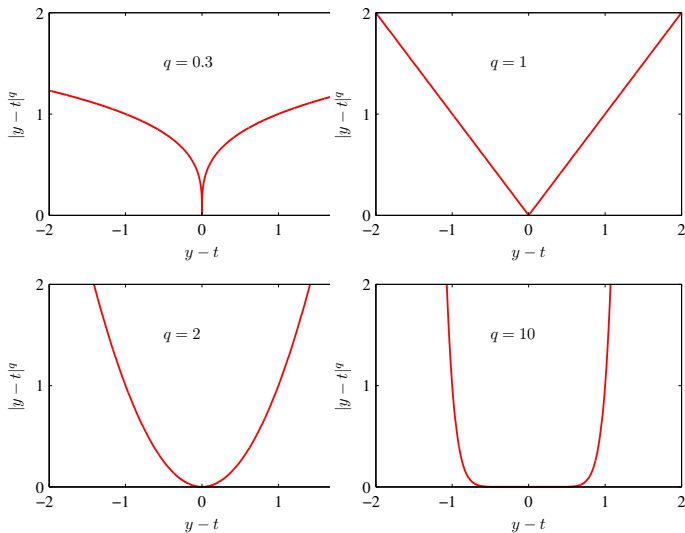
Here, let's define the sum of these distances as *cost function*, or loss function, and write as

$$E(\mathbf{w}) \triangleq \frac{1}{2} \sum_{n=1}^{N} \{y_n - t_n\}^2$$

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Remembering that

$$y_n(x_n, \mathbf{w}) = w_0\phi_0(x_n) + w_1\phi_1(x_n) + w_2\phi_2(x_n) + ... + w_{M-1}\phi_{M-1}(x_n)$$

We could put $y_n(x_i, \mathbf{w})$ in the matricial form and get

$$y_n = \begin{bmatrix} \phi_0(x_n) & \phi_1(x_n) & ... & \phi_{M-1}(x_n) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix}$$

and then

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \ldots & \phi_{M-1}(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \ldots & \phi_{M-1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_{N-1}) & \phi_1(x_{N-1}) & \ldots & \phi_{M-1}(x_{N-1}) \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}}_{\mathbf{w}}$$

This represents the system $\mathbf{y} = \Phi\mathbf{w}$. If

$$E(\mathbf{w}) = \frac{1}{2}(\mathbf{y} - \mathbf{t})^T(\mathbf{y} - \mathbf{t})$$

where $\mathbf{t} = \begin{bmatrix} t_1 & t_2 & \ldots & t_n \end{bmatrix}^T$

Then we'll have

$$E(\mathbf{w}) = \frac{1}{2}\left(\mathbf{y}^T\mathbf{y} - \mathbf{t}^T\mathbf{y} - \mathbf{y}^T\mathbf{t} + \mathbf{t}^T\mathbf{t}\right)$$
$$= \frac{1}{2}\left((\Phi\mathbf{w})^T(\Phi\mathbf{w}) - \mathbf{t}^T(\Phi\mathbf{w}) - (\Phi\mathbf{w})^T\mathbf{t} + \mathbf{t}^T\mathbf{t}\right)$$
$$= \frac{1}{2}\left(\mathbf{w}^T\Phi^T\Phi\mathbf{w} - 2\mathbf{t}^T\Phi\mathbf{w} + \mathbf{t}^T\mathbf{t}\right)$$

this by the fact that $\alpha = \mathbf{t}^T(\Phi\mathbf{w}) = (\Phi\mathbf{w})^T\mathbf{t}$, being α a scalar.

In sequence, we'll try to minimize it in terms of the weights ($\mathbf{w}$) by

$$0 = \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}}$$

$$0 = \frac{1}{2} \left(2\mathbf{w}^T \Phi^T \Phi - 2\mathbf{t}^T \Phi + 0 \right)$$

$$\mathbf{w}^T = \mathbf{t}^T \Phi \left(\Phi^T \Phi \right)^{-1}$$

$$\mathbf{w} = \left(\Phi^T \Phi \right)^{-1} \Phi^T \mathbf{t}$$

Here, we've obtained $\mathbf{w}$ for the curve fitting.

```
n = 20;
x = linspace(0,1,n)';
y = @(x) sin(2*pi*x);
e = .2*randn(size(x));
t = y(x) + e;

for M = 1:20
    phi = @(a)(bsxfun(@power,a,0:M-1));
    phix = phi(x);
    W = ((phix'*phix)\phix')*t;
end
```

A visible effect of the *increase of the complexity* of the model, represented here by M, is the *increase of the weights*. We call it **over-fitting**.
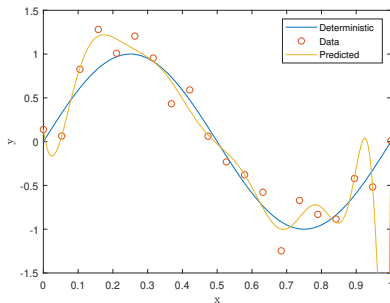
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This phenomenon illustrate a method of ever search for the *best estimation for the parameters*.
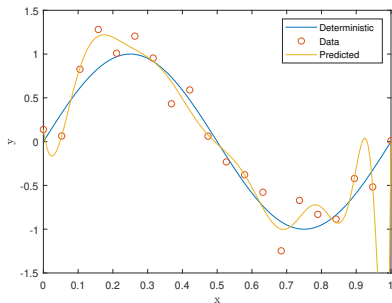
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It's reasonable to see that our model start's to differ from the *y* and starts to interpolate the noise.
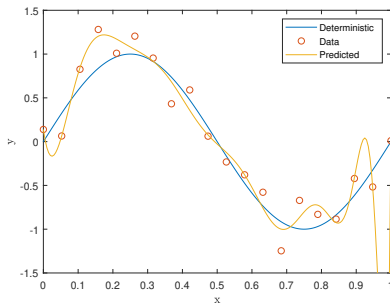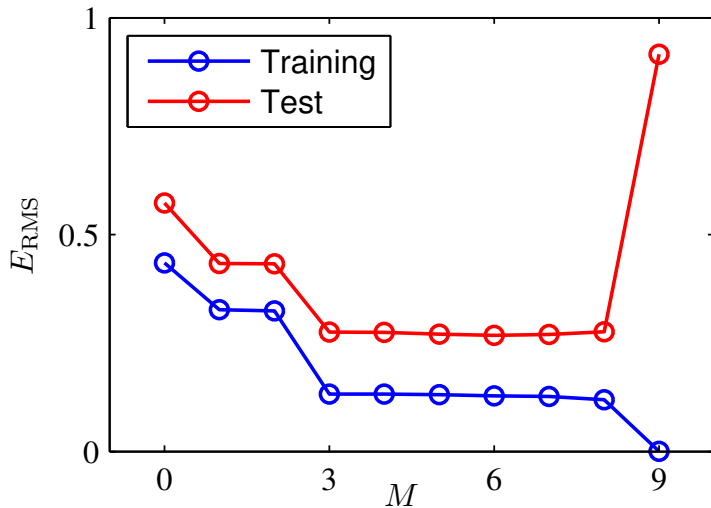
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$$\tilde{E}(\mathbf{w}) = \frac{1}{2}(\mathbf{y} - \mathbf{t})^T(\mathbf{y} - \mathbf{t}) + \frac{\lambda}{2}\mathbf{w}^T\mathbf{w}$$

$$= \frac{1}{2}\left(\mathbf{w}^T\Phi^T\Phi\mathbf{w} - 2\mathbf{t}^T\Phi\mathbf{w} + \mathbf{t}^T\mathbf{t} + \lambda\mathbf{w}^T\mathbf{I}\mathbf{w}\right)$$

$$\Rightarrow \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} = \frac{1}{2}\left(2\mathbf{w}^T\Phi^T\Phi - 2\mathbf{t}^T\Phi + 0 + 2\lambda\mathbf{w}^T\mathbf{I}\right)$$

$$0 = \mathbf{w}^T\Phi^T\Phi - \mathbf{t}^T\Phi + \lambda\mathbf{w}^T\mathbf{I}$$

$$\mathbf{w} = \left(\Phi^T\Phi + \lambda\mathbf{I}\right)^{-1}\Phi^T\mathbf{t}$$

```
n = 10;
x = linspace(0,1,n)';
y = @(x) sin(2*pi*x);
e = .2*randn(size(x));
t = y(x) + e;
for lambda = 75:-1:1
    M = n;
    plot(Xp,y(Xp),'-');hold on; plot(x,t,'o');
    phi = @(a)(bsxfun(@power,a,0:M-1));
    phix = phi(x);
    W = ((phix'*phix+exp(-lambda)*eye(n))\phix')*t;
end
```
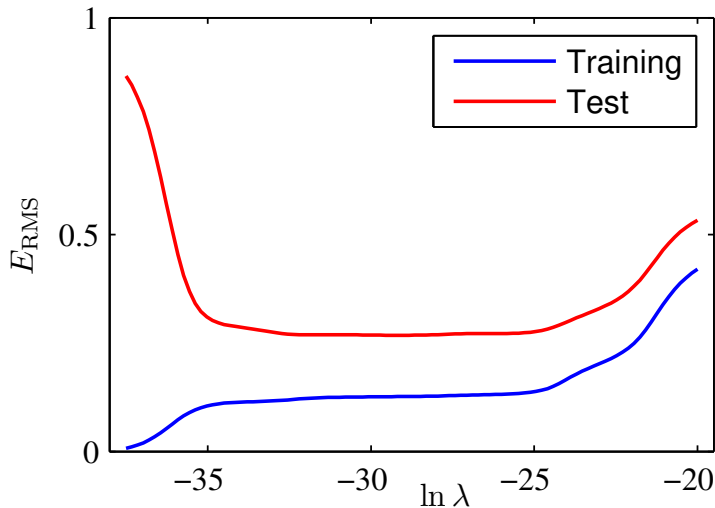
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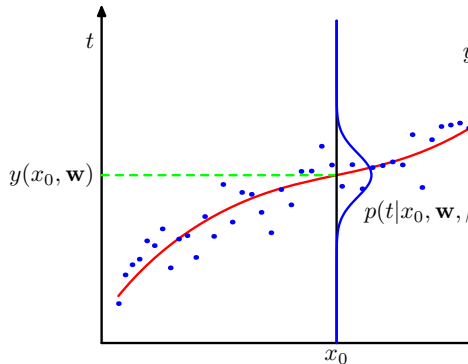
Let's go back to the initial problem of curve fitting. Each observation of the phenomenon is described with a random variable whose *mean* is given by $y(x, \mathbf{w})$, and the *variance* by β.
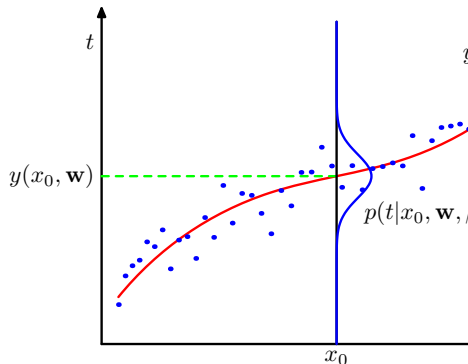
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Then, we want to obtain the probability of the *targets*, given some parameters, in this case $\mathbf{x}$, $\mathbf{w}$ and β.
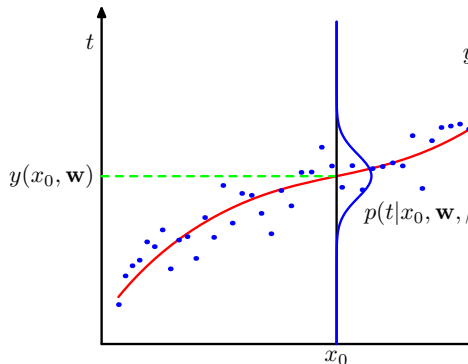
So, if we consider that our conditions are such that being the random variables independent and identically distributed, we can say that our *joint probability* is given by

$$p(t|x, \mathbf{w}, \beta) \Rightarrow p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = \prod_{n=1}^{N} p(t_n|x_n, \mathbf{w}, \beta)$$

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Let's assume we have a distribution such that $p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)$. Our goal is, given the *parameters*, maximize the *probability* of the *targets* given the *parameters*. An approach to do this use the fact that

$$\int_{\infty}^{-\infty} p(x)dx = 1 \text{ and } p(x) \geq 0$$

Seen this, we're supposing that p could assume values much smaller than one. To avoid computational singularity and for future purposes, we'll take the logarithmic probability. And then

$$\ln\left(p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)\right)$$

Reminding that

$$p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = \prod_{n=1}^{N} p\left(t_n|x_n, \mathbf{w}, \beta\right)$$

Implies that

$$\ln\left(p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)\right) = \sum_{n=1}^{N} \ln\left(p\left(t_n|x_n, \mathbf{w}, \beta\right)\right)$$

To proceed, we need to know what distribution p is. Let's choose the **Gaussian distribution**.
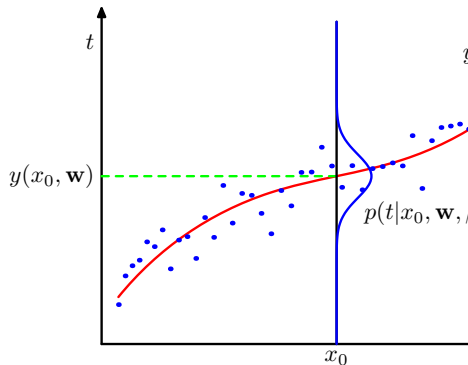
The **Gaussian distribution** comes from many different contexts, as the one that maximize the entropy among of all ones with fixed variance and from the sum of multiple random variables with finite variance.
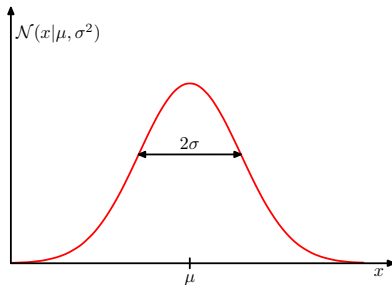
One-dimensional Gaussian distribution

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\} > 0$$

where μ is the mean and σ^2 the variance.

Now, back to the discussion of the maximization of

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$$p(t|\mathbf{x},\mathbf{w},\beta) = \mathcal{N}\left(t|y(\mathbf{x},\mathbf{w}),\beta^{-1}\right)$$

And then, from the *joint probability* of the Gaussians distributions

$$\ln\left(p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)\right) = \sum_{n=1}^{N} -\frac{1}{2}\ln(2\pi) + \sum_{n=1}^{N} \frac{1}{2}\ln\beta - \sum_{n=1}^{N} \frac{\beta}{2}(x_n - y(x_n, \mathbf{w}))^2$$

From this, we could obtain the **maximum likelihood**, or the *best estimation for the parameters*, taking the derivatives of the log probability to zero, according to the terms β and $\mathbf{w}$, our model parameters.

We could observe that taking the derivative with respect to **w**, our expression becomes closer to the *error function* presented previously, added the dependency of β

$$E(\mathbf{w}) \triangleq \frac{1}{2} \sum_{n=1}^{N} \{y_n - t_n\}^2$$

Then some behaviors could be expected, as the **over-fitting**.

We'll obtain the best β by

$$\frac{1}{\beta_{ML}} = \frac{1}{N} \sum_{n=1}^{N} \{y(x_n, \mathbf{w}_{ML}) - t_n\}^2$$

remembering that $\mathbf{w}_{ML}$ is already known from the regular linear regression.

At this point, we have a probabilistic model and we may want to predict values for *x*. Then, we need a *predictive distribution*.

Let's say we have the probabilities of some idea we desire to update it in the light of some new evidence. This could be done with **Bayes' Rule**, to convert a *prior* probability in a *posterior* probability and put some uncertainty in the parameters too.

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$$\underbrace{p\left(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta\right)}_{\text{posterior}} \propto \underbrace{p\left(\mathbf{t}|\mathbf{w}, \mathbf{x}, \beta\right)}_{\text{likelihood}} \underbrace{p\left(\mathbf{w}|\alpha\right)}_{\text{prior}}$$

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and for simplicity, consider the follow prior for $\mathbf{w}$

$$p\left(\mathbf{w}|\alpha\right) = \mathcal{N}\left(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I}\right) = \left(\frac{\alpha}{2\pi}\right)^{(M+1)/2} \exp\left\{-\frac{\alpha}{2}\mathbf{w}^T\mathbf{w}\right\}$$

where α the precision of the distribution and $M + 1$ is the dimension of $\mathbf{w}$, for a polynomial of Mth order. Variables such α are called *hyperparameters* and control the distribuition of model parameters.

By this, we can find a distribution and its maximum, or most probable value of $\mathbf{w}$ given the data taking the minimum of the negative logarithm of the infered expression, that will lead us to a term

$$\sum_{n=1}^{N} \{y(x_n, \mathbf{w}) - t_n\}^2 + \frac{\alpha}{2} \mathbf{w}^T \mathbf{w} + \text{const.}$$

Note that if we consider $\lambda = \alpha/\beta$, this will back to the regularized form of *least squares*. This technique is called *maximum posterior* (MAP).

So, observe that even making some probabilistic assumptions, we don't have yet a fully bayesian model, given that finding the *maximum likelihood*, we're finding only the parameters given one model such that maximize our targets probabilities. Furthermore, even with some probabilistic assumptions, our model still have a **over-fitting** problem, given that we obtained the same expressions for the simple regression, adding some constants.

The next step is put some **uncertainty in predictive model**, and makes adjustments in the light of our new evidences. By that we could obtain a "more Bayesian" model, in other words, a Bayesian Linear Regression.

Bayesian Linear Regression

Seeking a Bayesian approach, the next steps consists to apply the **sum** and **product** rules of probability to evaluate the predictive distribution. By now we assume that the hyperparameters are fixed, but they could assume a distribution too.

We saw that the posterior distribution for **w** could be given by

$$\underbrace{p\left(\mathbf{w}|\mathbf{x}, \mathbf{t}\right)}_{\text{posterior}} \propto \underbrace{p\left(\mathbf{t}|\mathbf{w}, \mathbf{x}\right)}_{\text{likelihood}} \underbrace{p\left(\mathbf{w}\right)}_{\text{prior}}$$

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First we'll consider a geometrical approach by the quadratic distance $(x-\mu)^2$ normalized by the variance σ^2. This comprehension will help us with the D-dimensional case.

To more than one dimensions, we'll consider the points (x) distance for the mean of the distribution, as we done in the one dimensional case, by adding a term to prioritize some dimension distribution in particular. Then

$$\Delta^2 = (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})$$

called *Mahalanobis distance*. And it's becomes the *Euclidean distance*, when $\boldsymbol{\Sigma}$ is the indentity matrix. This means that the all the distances are equally normalized. The matrix $\boldsymbol{\Sigma}$ is the covariance matrix of the distributions, by definition.

And then

D-dimensional Gaussian distribution

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right\}$$

where $\boldsymbol{\mu}$ is the D-dimensional mean vector, $\boldsymbol{\Sigma}$ the D×D-dimensional variance matrix and $|\boldsymbol{\Sigma}|$ its determinant.

Partitioned Gaussians

Given a joint Gaussian distribution $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}^{-1}$ and

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}.$$

Will give us

- Conditional distribution:

$$p\left(\mathbf{x}_a|\mathbf{x}_b\right) = \mathcal{N}\left(\mathbf{x}_a|\boldsymbol{\mu}_{a|b}, \boldsymbol{\Lambda}_{aa}^{-1}\right), \; \boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{aa}^{-1}\boldsymbol{\Lambda}_{ab}\left(\mathbf{x}_b - \boldsymbol{\mu}_b\right)$$

- Marginal distribution:

$$p\left(\mathbf{x}_a\right) = \mathcal{N}\left(\mathbf{x}_a|\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa}\right)$$

To proceed we'd like to prove that the Gaussians are **closed under linear transformations**. This will allow us to transform the Gaussians under the likelihood distribution given a prior. For example, given a distribution

$$p(\mathbf{z}) = p(\mathbf{x}, \mathbf{y})$$

In other words, we're trying to find the parginal distribution $p(\mathbf{y})$ and the conditional distribution $p(\mathbf{x}|\mathbf{y})$, given

$$p(\mathbf{x}) = \mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}\right)$$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}\left(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1}\right)$$

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So, applying the joint distribution and the its ln after

$$p\left(\mathbf{z}\right) = p\left(\mathbf{x}, \mathbf{y}\right) = p\left(\mathbf{y}|\mathbf{x}\right) p\left(\mathbf{x}\right)$$

$$\ln p\left(\mathbf{z}\right) = \ln p\left(\mathbf{y}|\mathbf{x}\right) + \ln p\left(\mathbf{x}\right)$$

$$= -\frac{1}{2}\left(\mathbf{x} - \boldsymbol{\mu}\right)^T \boldsymbol{\Lambda}\left(\mathbf{x} - \boldsymbol{\mu}\right)$$

$$- \frac{1}{2}\left(\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{b}\right)^T \mathbf{L}\left(\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{b}\right) + \text{const}$$

The "const" is the term independent of $\mathbf{x}$ and $\mathbf{y}$. Then, expanding the quadratic form

$$\ln p\left(\mathbf{z}\right) = -\frac{1}{2}\mathbf{x}^T\left(\mathbf{\Lambda} + \mathbf{A}^T\mathbf{L}\mathbf{A}\right)\mathbf{x} - \frac{1}{2}\mathbf{y}^T\mathbf{L}\mathbf{y} + \frac{1}{2}\mathbf{y}^T\mathbf{L}\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}^T\mathbf{L}\mathbf{y}$$

$$= -\frac{1}{2}\begin{pmatrix}\mathbf{x}\\\mathbf{y}\end{pmatrix}^T\begin{pmatrix}\mathbf{\Lambda} + \mathbf{A}^T\mathbf{L}\mathbf{A} & -\mathbf{A}^T\mathbf{L}\\ -\mathbf{L}\mathbf{A} & \mathbf{L}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\\mathbf{y}\end{pmatrix} = -\frac{1}{2}\mathbf{z}^T\mathbf{R}\mathbf{z}$$

The "const" is the term independent of $\mathbf{x}$ and $\mathbf{y}$. Then, expanding the quadratic form

$$\ln p\left(\mathbf{z}\right) = -\frac{1}{2}\mathbf{x}^T\left(\boldsymbol{\Lambda} + \mathbf{A}^T\mathbf{L}\mathbf{A}\right)\mathbf{x} - \frac{1}{2}\mathbf{y}^T\mathbf{L}\mathbf{y} + \frac{1}{2}\mathbf{y}^T\mathbf{L}\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}^T\mathbf{L}\mathbf{y}$$

$$= -\frac{1}{2}\begin{pmatrix}\mathbf{x}\\\mathbf{y}\end{pmatrix}^T\begin{pmatrix}\boldsymbol{\Lambda} + \mathbf{A}^T\mathbf{L}\mathbf{A} & -\mathbf{A}^T\mathbf{L}\\-\mathbf{L}\mathbf{A} & \mathbf{L}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\\mathbf{y}\end{pmatrix} = -\frac{1}{2}\mathbf{z}^T\mathbf{R}\mathbf{z}$$

We'll apply the partitioned matrices inversion to obtain $\mathbf{R}^{-1}$

$$\mathbf{R}^{-1} = \begin{pmatrix}\boldsymbol{\Lambda}^{-1} & \boldsymbol{\Lambda}^{-1}\mathbf{A}^T\\\mathbf{A}\boldsymbol{\Lambda}^{-1} & \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^T\end{pmatrix}$$

The expanded form of $\ln p\left(\mathbf{z}\right)$ give us the mean too by the linear terms, then

$$\mathbf{x}^{T}\mathbf{\Lambda}\boldsymbol{\mu} - \mathbf{x}^{T}\mathbf{A}^{T}\mathbf{Lb} + \mathbf{y}^{T}\mathbf{Lb} = \begin{pmatrix}\mathbf{x}\\\mathbf{y}\end{pmatrix}^{T}\begin{pmatrix}\mathbf{\Lambda}\boldsymbol{\mu} - \mathbf{A}^{T}\mathbf{Lb}\\\mathbf{Lb}\end{pmatrix}$$

The expanded form of $\ln p\left(\mathbf{z}\right)$ give us the mean too by the linear terms, then

$$\mathbf{x}^T \mathbf{\Lambda} \boldsymbol{\mu} - \mathbf{x}^T \mathbf{A}^T \mathbf{L} \mathbf{b} + \mathbf{y}^T \mathbf{L} \mathbf{b} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \begin{pmatrix} \mathbf{\Lambda} \boldsymbol{\mu} - \mathbf{A}^T \mathbf{L} \mathbf{b} \\ \mathbf{L} \mathbf{b} \end{pmatrix}$$

By inspection of the linear terms

$$\mathbb{E}[\mathbf{z}] = \mathbf{R}^{-1} \begin{pmatrix} \mathbf{\Lambda} \boldsymbol{\mu} - \mathbf{A}^T \mathbf{L} \mathbf{b} \\ \mathbf{L} \mathbf{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{A} \boldsymbol{\mu} + \mathbf{b} \end{pmatrix}$$

And then we we'll have that

$$\mathbb{E}[\mathbf{y}] = \mathbf{A}\boldsymbol{\mu} + \mathbf{b}$$
$$\text{cov}[\mathbf{y}] = \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{T}$$

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$$\mathbb{E}[\mathbf{x}|\mathbf{y}] = \left(\boldsymbol{\Lambda} + \mathbf{A}^T\mathbf{L}\mathbf{A}\right)^{-1} \left\{\mathbf{A}^T\mathbf{L}(\mathbf{y} - \mathbf{b}) + \boldsymbol{\Lambda}\boldsymbol{\mu}\right\}$$
$$\text{cov}[\mathbf{x}|\mathbf{y}] = \left(\boldsymbol{\Lambda} + \mathbf{A}^T\mathbf{L}\mathbf{A}\right)^{-1}$$

In the next step, we'll assume a **prior distribution over parameters**, $p(\mathbf{w})$, and define it as a Gaussian distribution, then

$$p(\mathbf{w}) = \mathcal{N}\left(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0\right)$$

with mean $\mathbf{m}_0$ and variance $\mathbf{S}_0$.

Marginal and Conditioned Gaussians

- For $\mathbf{y}$ given $\mathbf{x}$:

$$p\left(\mathbf{x}\right) = \mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}\right)$$

$$p\left(\mathbf{y}|\mathbf{x}\right) = \mathcal{N}\left(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1}\right)$$

- For $\mathbf{x}$ given $\mathbf{y}$:

$$p\left(\mathbf{x}|\mathbf{y}\right) = \mathcal{N}\left(\mathbf{y}|, \boldsymbol{\Sigma}\left\{\mathbf{A}^{T}\mathbf{L}(\mathbf{y} - \mathbf{b} + \boldsymbol{\Sigma}\boldsymbol{\mu})\right\}, \boldsymbol{\Sigma}\right)$$

$$p\left(\mathbf{y}\right) = \mathcal{N}\left(\mathbf{y}|\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{T}\right), \text{ where } \boldsymbol{\Sigma} = \left(\boldsymbol{\Lambda} + \mathbf{A}^{T}\mathbf{L}\mathbf{A}\right)^{-1}$$

By the derivations, we make the assumptions of given $p(\mathbf{w})$ and for $p(\mathbf{t}|\mathbf{w})$ such that

$$p(\mathbf{t}|\mathbf{w}) = \mathcal{N}\left(\mathbf{t}|y(\mathbf{x}, \mathbf{w}), \beta^{-1}\right)$$
$$= \mathcal{N}\left(\mathbf{t}|\boldsymbol{\Phi}^T\mathbf{w}, \beta^{-1}\right)$$

And then $p(\mathbf{w}|\mathbf{t}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N)$ where

$$\mathbf{m}_N = \mathbf{S}_N\left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\boldsymbol{\Phi}^T\mathbf{t}\right)$$
$$\mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta\boldsymbol{\Phi}^T\boldsymbol{\Phi}$$

Gaussian Processes

What was done until here?

- We assumed that our targets t were **i.i.d.** and given by $t = y(\mathbf{x}) + \varepsilon$, where $\varepsilon \sim \mathcal{N}(0, \beta)$.

- Our model is given by $\mathbf{y}(\mathbf{x}) = \Phi^T \mathbf{w}$, where Φ is the **design matrix**, and this caracterize our model as **linear in parameters**.

- The **design matrix** was defined as $\phi_{i,j} = \phi_i(\mathbf{x}_j)$.

- The **parameters** were given by $\mathbf{w} = \left(\Phi^T \Phi\right)^{-1} \Phi^T \mathbf{t}$.

- These **parameters** calculated at the minimum of the cost function are called **maximum likelihood**.
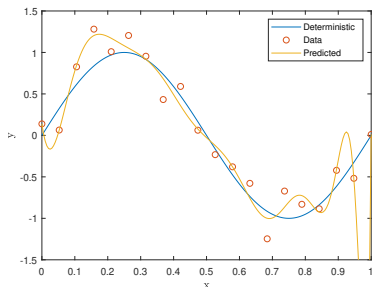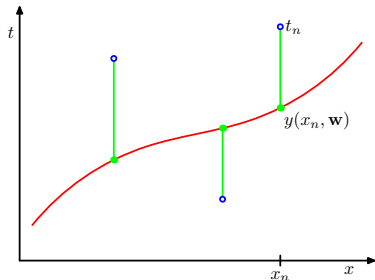
What was done until here?

- We put an **uncertainty** over the targets t and the parameters $\mathbf{w}$.
- We assumed that targets being **distributed** as $p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$.
- By **Bayes' Rule** we obtained that $p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta) \propto p(\mathbf{t}|\mathbf{w}, \mathbf{x}, \beta) p(\mathbf{w}|\alpha)$
- This allowed to make an **inference** to obtain a **prediction** of the parameters.
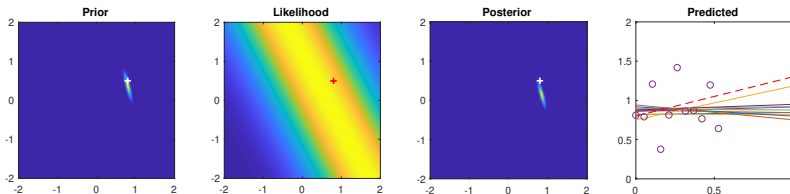
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