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# Introduction to Gaussian Processes

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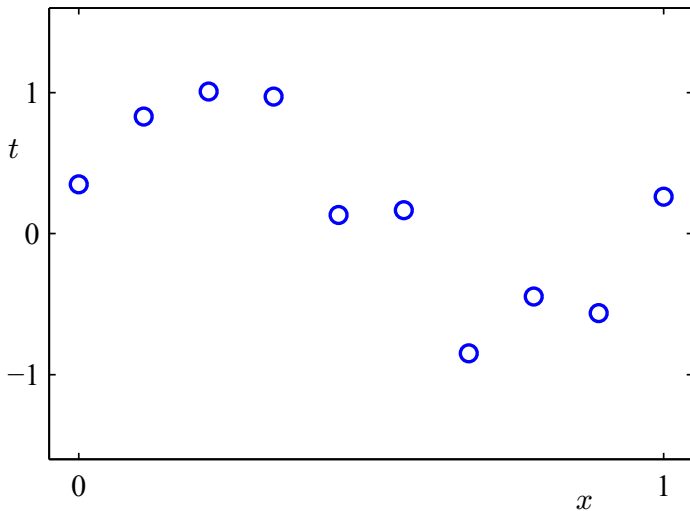
# Linear Regression

- If we have a set of points in a space that comes from observations of an experiment and we want to predict other points, this could be done with **curve fitting**.
- So we could define some strategy to find our model.

## Strategy

- 1 Purpose a **model**, e.g. functions like exponential, polynomial and others.
- 2 Train our model with the training data set, finding the **unknown parameters** or **weights**.

- Let's fit the points below by **polynomial curve fitting**.



- Be the model chosen a **polynomial**, we'll have

$$y(x, \mathbf{w}) = w_0x^0 + w_1x^1 + w_2x^2 + \dots + w_{M-1}x^{M-1} = \sum_{j=1}^{M-1} w_jx^j$$

- In general, we could write this **weighted sum** with any other function. In other words, we can put this in terms of  $\phi_n(x) = x^n$ , where  $\phi$  could be other **basis function**.
- e.g. we could have different  $y$ s for different basis functions.

$$\begin{aligned} y(x, \mathbf{w}) &= w_0\phi_0(x) + w_1\phi_1(x) + w_2\phi_2(x) + \dots + w_{M-1}\phi_{M-1}(x) \\ &= w_0 \exp \left\{ -\frac{(x - \mu_0)^2}{2\sigma^2} \right\} + w_1 \exp \left\{ -\frac{(x - \mu_1)^2}{2\sigma^2} \right\} + \\ &\dots + w_{M-1} \exp \left\{ -\frac{(x - \mu_{M-1})^2}{2\sigma^2} \right\} \\ &= w_0 \sin(0 \cdot x) + w_1 \cos(1 \cdot x) + \\ &\dots + w_{M_2} \sin((M - 2) \cdot x) + w_{M-1} \cos((M - 1) \cdot x) \end{aligned}$$

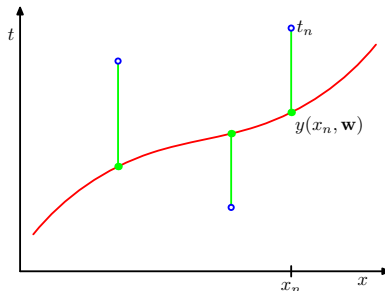
- For simplicity, we'll carry this notation along.

$$\begin{aligned} y(x, \mathbf{w}) &= w_0\phi_0(x) + w_1\phi_1(x) + w_2\phi_2(x) + \dots + w_{M-1}\phi_{M-1}(x) \\ &= \sum_{j=1}^{M-1} w_j\phi_j(x) \end{aligned}$$

- We'll evaluate  $\phi$  for all  $x$ , and then project it in the  $w$  vector space, then our model could be formed by **non-linear** functions. But, remaining **linear on parameters**.

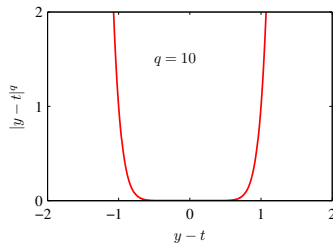
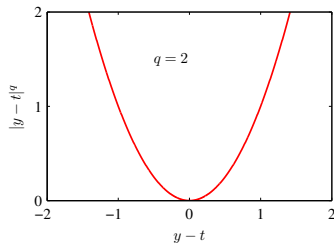
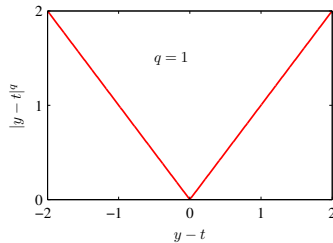
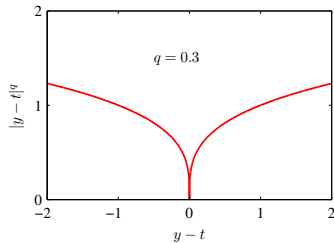
- The chosen model will give us some curve that is needed to adjust such that we'll **minimize its distance** to the **targets** ( $t$ ).
- Here, let's define the sum of these distances as **cost function**, or error function, and write it as

$$E(\mathbf{w}) \triangleq \frac{1}{2} \sum_{n=1}^N \{y_n - t_n\}^2$$





### Why choose a quadratic norm distance?



### Why choose a quadratic norm distance?

- The first row figures could be used for the derivations, taking care with some **non-continuous derivatives**.
- We'll use the **quadratic norm** because it's the minor integer  $q$  differentiable, and then the error measures  $E$  between the model  $y(x, \mathbf{w})$  and the targets  $t$  will be euclidean.
- More, increasing the value of  $q$ , the smallest than 1 and bigger than 0 errors between the model and the targets that become irrelevant for  $E$ .

- Remembering that

$$y(x, \mathbf{w}) = w_0\phi_0(x) + w_1\phi_1(x) + w_2\phi_2(x) + \dots + w_{M-1}\phi_{M-1}(x)$$

- We'll evaluate for all  $x_i$  values, and then put  $y_n(x_i, \mathbf{w})$  in the matrix form and get

$$y_n = [\phi_0(x_n) \quad \phi_1(x_n) \quad \dots \quad \phi_{M-1}(x_n)] [w_0 \quad w_1 \quad \dots \quad w_{M-1}]^\top$$

- And then

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_{M-1}(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_{M-1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_{N-1}) & \phi_1(x_{N-1}) & \dots & \phi_{M-1}(x_{N-1}) \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}}_{\mathbf{w}}$$

- This represents the system  $\mathbf{y} = \Phi\mathbf{w}$ .

- If  $E(\mathbf{w}) = \frac{1}{2} (\mathbf{y} - \mathbf{t})^\top (\mathbf{y} - \mathbf{t})$  where  $\mathbf{t} = [t_1 \quad t_2 \quad \dots \quad t_n]^\top$
- Then we'll have

$$\begin{aligned} E(\mathbf{w}) &= \frac{1}{2} \left( \mathbf{y}^\top \mathbf{y} - \mathbf{t}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{t} + \mathbf{t}^\top \mathbf{t} \right) \\ &= \frac{1}{2} \left( (\Phi \mathbf{w})^\top (\Phi \mathbf{w}) - \mathbf{t}^\top (\Phi \mathbf{w}) - (\Phi \mathbf{w})^\top \mathbf{t} + \mathbf{t}^\top \mathbf{t} \right) \\ &= \frac{1}{2} \left( \mathbf{w}^\top \Phi^\top \Phi \mathbf{w} - 2\mathbf{t}^\top \Phi \mathbf{w} + \mathbf{t}^\top \mathbf{t} \right) \end{aligned}$$

this by the fact that  $\alpha = \mathbf{t}^\top (\Phi \mathbf{w}) = (\Phi \mathbf{w})^\top \mathbf{t}$ , being  $\alpha$  a scalar.

- Then too,  $E(\mathbf{w})$  remains scalar.

- In sequence, we'll try to minimize it in terms of the weights ( $\mathbf{w}$ ) by

$$0 = \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}}$$

$$0 = \frac{1}{2} \left( 2\mathbf{w}^\top \Phi^\top \Phi - 2\mathbf{t}^\top \Phi + 0 \right)$$

$$\mathbf{w}^\top = \mathbf{t}^\top \Phi \left( \Phi^\top \Phi \right)^{-1}$$

$$\mathbf{w}_{ML} = \left( \Phi^\top \Phi \right)^{-1} \Phi^\top \mathbf{t}$$

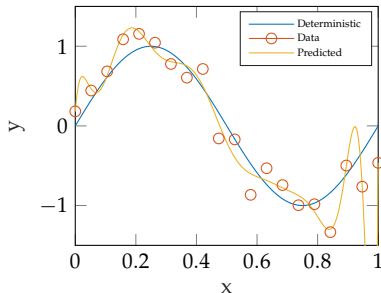
- Here, we've obtained the weights  $\mathbf{w}_{ML}$  with the **maximum likelihood** for fitting the curve.
- We could say that the model **learned** the parameters.

```
n = 20;
x = linspace(0,1,n)';
y = @(x) sin(2*pi*x);
e = .2*randn(size(x));
t = y(x) + e;

for M = 1:20
    phi = @(a)(bsxfun(@power,a,0:M-1));
    phix = phi(x);
    W = ((phix'*phix)\phix')*t;
end
```

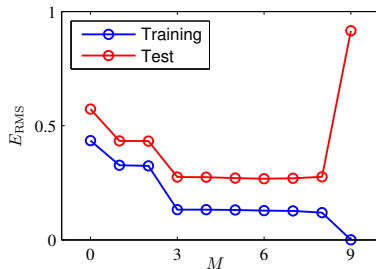
Why the prediction is so distant from the deterministic curve?

- A visible effect of the **increase of the complexity** of the model, represented here by the  $M$ , is the **increase of the weights**.
- This means that the  $M$  number of basis functions is increasing too.
- It's easy to see that our model starts to differ from the  $y$  and starts to interpolate the noise. We call this of **over-fitting**.
- This phenomenon illustrates a method of always search for the **best estimation of the parameters**.



### Could be over-fitting a problem?

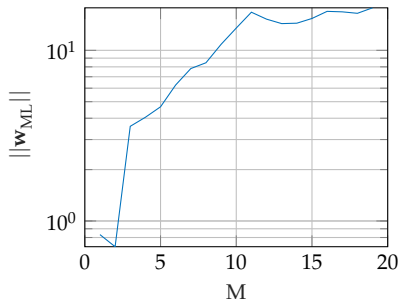
- We could **train** our model, it means evaluate  $\mathbf{w}_{ML}$ , for only a part of our dataset.
- If the model be a good one, the error must be small when its **testing**.
- It means the error must be small when we evaluate all dataset with the  $\mathbf{w}_{ML}$  of the trained part.
- But this does not occur. So the **error increases**.





### How to control the over-fitting?

- With the increase of the model complexity, the value of  $\mathbf{w}_{ML}$  increases too.
- A solution could be add a **penalty term** as the norm of the weights increases.



- To control the over-fitting, we try to **regularize** the weights by adding a penalty term ( $\lambda$ ) to error function, by this we force the coefficients to not reach high values.

$$\begin{aligned}
 \tilde{E}(\mathbf{w}) &= \frac{1}{2}(\mathbf{y} - \mathbf{t})^\top (\mathbf{y} - \mathbf{t}) + \frac{\lambda}{2} \mathbf{w}^\top \mathbf{w} \\
 &= \frac{1}{2} \left( \mathbf{w}^\top \Phi^\top \Phi \mathbf{w} - 2\mathbf{t}^\top \Phi \mathbf{w} + \mathbf{t}^\top \mathbf{t} + \lambda \mathbf{w}^\top \mathbf{I} \mathbf{w} \right) \\
 \Rightarrow \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} &= \frac{1}{2} \left( 2\mathbf{w}^\top \Phi^\top \Phi - 2\mathbf{t}^\top \Phi + 0 + 2\lambda \mathbf{w}^\top \mathbf{I} \right) \\
 0 &= \mathbf{w}^\top \Phi^\top \Phi - \mathbf{t}^\top \Phi + \lambda \mathbf{w}^\top \mathbf{I} \\
 \mathbf{w}_{ML} &= \left( \Phi^\top \Phi + \lambda \mathbf{I} \right)^{-1} \Phi^\top \mathbf{t}
 \end{aligned}$$

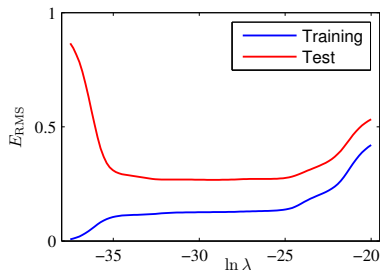
```

n = 10;
x = linspace(0,1,n)';
y = @(x) sin(2*pi*x);
e = .2*randn(size(x));
t = y(x) + e;
for lambda = 75:-1:1
    M = n;
    plot(Xp,y(Xp),'-');hold on; plot(x,t,'o');
    phi = @(a)(bsxfun(@power,a,0:M-1));
    phix = phi(x);
    W = ((phix'*phix+exp(-lambda)*eye(n))\phix')*t;
end

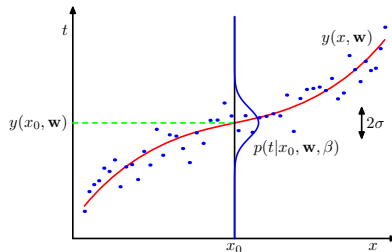
```

### A more sophisticated approach?

- We could **train and test** and evaluate the error for several values of  $\lambda$ .
- Even yet, it's too wasteful partitionate and optimize data to find good model parameters or even a **flexible** one.
- We need a more sophisticated approach.



- Having an **uncertainty** in the measured value, we could represent it with a **probability distribution**.
- Now, each **target** could be expressed as a **random variable**.
- Its **mean** is given by  $y(x, \mathbf{w})$ , and the **variance** by  $\beta$ .



So, if we consider that our conditions are such that being the random variables independent and identically distributed, we can say that our *joint probability* is given by

$$p(t|x, \mathbf{w}, \beta) \Rightarrow p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = \prod_{n=1}^N p(t_n|x_n, \mathbf{w}, \beta)$$

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Let's assume we have a distribution such that  $p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)$ . Our goal is, given the *parameters*, maximize the *probability* of the *targets* given the *parameters*. An approach to do this use the fact that

$$\int_{-\infty}^{\infty} p(x)dx = 1 \text{ and } p(x) \geq 0$$

Seen this, we're supposing that  $p$  could assume values much smaller than one. To avoid computational singularity and for future purposes, we'll take the logarithmic probability. And then

$$\ln (p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta))$$

Reminding that

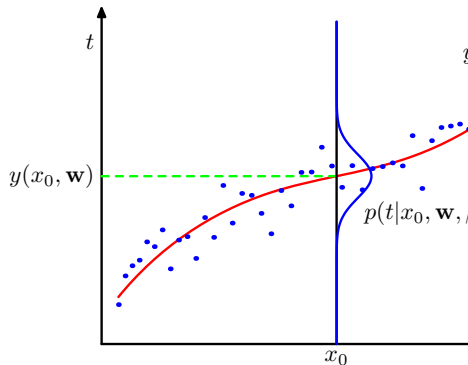
$$p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = \prod_{n=1}^N p(t_n|x_n, \mathbf{w}, \beta)$$

Implies that

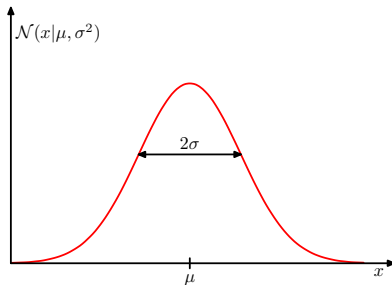
$$\ln (p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)) = \sum_{n=1}^N \ln (p(t_n|x_n, \mathbf{w}, \beta))$$



To proceed, we need to know what distribution  $p$  is. Let's choose the **Gaussian distribution**.



The **Gaussian distribution** comes from many different contexts, as the one that maximize the entropy among of all ones with fixed variance and from the sum of multiple random variables with finite variance.



## One-dimensional Gaussian distribution

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left\{ -\frac{1}{2\sigma^2} (x - \mu)^2 \right\} > 0$$

where  $\mu$  is the mean and  $\sigma^2$  the variance.

Now, back to the discussion of the maximization of

$$\ln (p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)) = \sum_{n=1}^N \ln (p(t_n|x_n, \mathbf{w}, \beta))$$

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we can state a Gaussian distribution for each target and then

$$p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N} \left( t|y(\mathbf{x}, \mathbf{w}), \beta^{-1} \right)$$

And then, from the *joint probability* of the Gaussians distributions

$$\ln(p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)) = \sum_{n=1}^N -\frac{1}{2} \ln(2\pi) + \sum_{n=1}^N \frac{1}{2} \ln \beta - \sum_{n=1}^N \frac{\beta}{2} (x_n - y(x_n, \mathbf{w}))^2$$

From this, we could obtain the **maximum likelihood**, or the *best estimation for the parameters*, taking the derivatives of the log probability to zero, according to the terms  $\beta$  and  $\mathbf{w}$ , our model parameters.

We could observe that taking the derivative with respect to  $\mathbf{w}$ , our expression becomes closer to the *error function* presented previously, added the dependency of  $\beta$

$$E(\mathbf{w}) \triangleq \frac{1}{2} \sum_{n=1}^N \{y_n - t_n\}^2$$

Then some behaviors could be expected, as the **over-fitting**.



We'll obtain the best  $\beta$  by

$$\frac{1}{\beta_{ML}} = \frac{1}{N} \sum_{n=1}^N \{y(x_n, \mathbf{w}_{ML}) - t_n\}^2$$

remembering that  $\mathbf{w}_{ML}$  is already known from the regular linear regression.

At this point, we have a probabilistic model and we may want to predict values for  $x$ . Then, we need a *predictive distribution*.

Let's say we have the probabilities of some idea we desire to update it in the light of some new evidence. This could be done with **Bayes' Rule**, to convert a *prior* probability in a *posterior* probability and put some uncertainty in the parameters too.

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$$\underbrace{p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta)}_{\text{posterior}} \propto \underbrace{p(\mathbf{t}|\mathbf{w}, \mathbf{x}, \beta)}_{\text{likelihood}} \underbrace{p(\mathbf{w}|\alpha)}_{\text{prior}}$$

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and for simplicity, consider the follow prior for  $\mathbf{w}$

$$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I}) = \left(\frac{\alpha}{2\pi}\right)^{(M+1)/2} \exp\left\{-\frac{\alpha}{2}\mathbf{w}^\top \mathbf{w}\right\}$$

where  $\alpha$  the precision of the distribution and  $M + 1$  is the dimension of  $\mathbf{w}$ , for a polynomial of  $M^{th}$  order. Variables such  $\alpha$  are called *hyperparameters* and control the distribution of model parameters.

By this, we can find a distribution and its maximum, or most probable value of  $\mathbf{w}$  given the data taking the minimum of the negative logarithm of the inferred expression, that will lead us to a term

$$\sum_{n=1}^N \{y(x_n, \mathbf{w}) - t_n\}^2 + \frac{\alpha}{2} \mathbf{w}^\top \mathbf{w} + \text{const.}$$

Note that if we consider  $\lambda = \alpha/\beta$ , this will back to the regularized form of *least squares*. This technique is called *maximum posterior* (MAP).

So, observe that even making some probabilistic assumptions, we don't have yet a fully bayesian model, given that finding the *maximum likelihood*, we're finding only the parameters given one model such that maximize our targets probabilities. Furthermore, even with some probabilistic assumptions, our model still have a **over-fitting** problem, given that we obtained the same expressions for the simple regression, adding some constants.

The next step is put some **uncertainty in predictive model**, and makes adjustments in the light of our new evidences. By that we could obtain a "more Bayesian" model, in other words, a **Bayesian Linear Regression**.

# Bayesian Linear Regression



Seeking a Bayesian approach, the next steps consists to apply the **sum** and **product** rules of probability to evaluate the predictive distribution. By now we assume that the hyperparameters are fixed, but they could assume a distribution too.

We saw that the posterior distribution for  $\mathbf{w}$  could be given by

$$\underbrace{p(\mathbf{w}|\mathbf{x}, \mathbf{t})}_{\text{posterior}} \propto \underbrace{p(\mathbf{t}|\mathbf{w}, \mathbf{x})}_{\text{likelihood}} \underbrace{p(\mathbf{w})}_{\text{prior}}$$

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where  $\mu$  is the mean and  $\sigma^2$  the variance.

First we'll consider a geometrical approach by the quadratic distance  $(x - \mu)^2$  normalized by the variance  $\sigma^2$ . This comprehension will help us with the D-dimensional case.

To more than one dimensions, we'll consider the points ( $\mathbf{x}$ ) distance for the mean of the distribution, as we done in the one dimensional case, by adding a term to prioritize some dimension distribution in particular. Then

$$\Delta^2 = (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})$$

called *Mahalanobis distance*. And it's becomes the *Euclidean distance*, when  $\boldsymbol{\Sigma}$  is the identity matrix. This means that the all the distances are equally normalized. The matrix  $\boldsymbol{\Sigma}$  is the covariance matrix of the distributions, by definition.

And then

## D-dimensional Gaussian distribution

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

where  $\boldsymbol{\mu}$  is the D-dimensional mean vector,  $\boldsymbol{\Sigma}$  the  $D \times D$ -dimensional variance matrix and  $|\boldsymbol{\Sigma}|$  its determinant.

## Partitioned Gaussians

Given a joint Gaussian distribution  $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$  with  $\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}^{-1}$  and

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}.$$

Will give us

- Conditional distribution:

$$p(\mathbf{x}_a|\mathbf{x}_b) = \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_{a|b}, \boldsymbol{\Lambda}_{aa}^{-1}), \quad \boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{aa}^{-1}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)$$

- Marginal distribution:

$$p(\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa})$$

To proceed we'd like to prove that the Gaussians are **closed under linear transformations**. This will allow us to transform the Gaussians under the likelihood distribution given a prior. For example, given a distribution

$$p(\mathbf{z}) = p(\mathbf{x}, \mathbf{y})$$

In other words, we're trying to find the marginal distribution  $p(\mathbf{y})$  and the conditional distribution  $p(\mathbf{x}|\mathbf{y})$ , given

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1})$$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{Ax} + \mathbf{b}, \mathbf{L}^{-1})$$



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$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{Ax} + \mathbf{b}, \mathbf{L}^{-1})$$

So, applying the joint distribution and the its ln after

$$p(\mathbf{z}) = p(\mathbf{x}, \mathbf{y}) = p(\mathbf{y}|\mathbf{x}) p(\mathbf{x})$$

$$\ln p(\mathbf{z}) = \ln p(\mathbf{y}|\mathbf{x}) + \ln p(\mathbf{x})$$

$$= -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Lambda} (\mathbf{x} - \boldsymbol{\mu})$$

$$- \frac{1}{2} (\mathbf{y} - \mathbf{Ax} - \mathbf{b})^\top \mathbf{L} (\mathbf{y} - \mathbf{Ax} - \mathbf{b}) + \text{const}$$

The "const" is the term independent of  $\mathbf{x}$  and  $\mathbf{y}$ . Then, expanding the quadratic form

$$\begin{aligned}\ln p(\mathbf{z}) &= -\frac{1}{2}\mathbf{x}^\top \left( \mathbf{\Lambda} + \mathbf{A}^\top \mathbf{L} \mathbf{A} \right) \mathbf{x} - \frac{1}{2}\mathbf{y}^\top \mathbf{L} \mathbf{y} + \frac{1}{2}\mathbf{y}^\top \mathbf{L} \mathbf{A} \mathbf{x} + \frac{1}{2}\mathbf{x}^\top \mathbf{A}^\top \mathbf{L} \mathbf{y} \\ &= -\frac{1}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^\top \begin{pmatrix} \mathbf{\Lambda} + \mathbf{A}^\top \mathbf{L} \mathbf{A} & -\mathbf{A}^\top \mathbf{L} \\ -\mathbf{L} \mathbf{A} & \mathbf{L} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = -\frac{1}{2} \mathbf{z}^\top \mathbf{R} \mathbf{z}\end{aligned}$$

The "const" is the term independent of  $\mathbf{x}$  and  $\mathbf{y}$ . Then, expanding the quadratic form

$$\begin{aligned}\ln p(\mathbf{z}) &= -\frac{1}{2}\mathbf{x}^\top \left( \mathbf{\Lambda} + \mathbf{A}^\top \mathbf{L} \mathbf{A} \right) \mathbf{x} - \frac{1}{2}\mathbf{y}^\top \mathbf{L} \mathbf{y} + \frac{1}{2}\mathbf{y}^\top \mathbf{L} \mathbf{A} \mathbf{x} + \frac{1}{2}\mathbf{x}^\top \mathbf{A}^\top \mathbf{L} \mathbf{y} \\ &= -\frac{1}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^\top \begin{pmatrix} \mathbf{\Lambda} + \mathbf{A}^\top \mathbf{L} \mathbf{A} & -\mathbf{A}^\top \mathbf{L} \\ -\mathbf{L} \mathbf{A} & \mathbf{L} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = -\frac{1}{2} \mathbf{z}^\top \mathbf{R} \mathbf{z}\end{aligned}$$

We'll apply the partitioned matrices inversion to obtain  $\mathbf{R}^{-1}$

$$\mathbf{R}^{-1} = \begin{pmatrix} \mathbf{\Lambda}^{-1} & \mathbf{\Lambda}^{-1} \mathbf{A}^\top \\ \mathbf{A} \mathbf{\Lambda}^{-1} & \mathbf{L}^{-1} + \mathbf{A} \mathbf{\Lambda}^{-1} \mathbf{A}^\top \end{pmatrix}$$

The expanded form of  $\ln p(\mathbf{z})$  give us the mean too by the linear terms, then

$$\mathbf{x}^\top \Lambda \boldsymbol{\mu} - \mathbf{x}^\top \mathbf{A}^\top \mathbf{L} \mathbf{b} + \mathbf{y}^\top \mathbf{L} \mathbf{b} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^\top \begin{pmatrix} \Lambda \boldsymbol{\mu} - \mathbf{A}^\top \mathbf{L} \mathbf{b} \\ \mathbf{L} \mathbf{b} \end{pmatrix}$$

The expanded form of  $\ln p(\mathbf{z})$  give us the mean too by the linear terms, then

$$\mathbf{x}^\top \mathbf{\Lambda} \boldsymbol{\mu} - \mathbf{x}^\top \mathbf{A}^\top \mathbf{L} \mathbf{b} + \mathbf{y}^\top \mathbf{L} \mathbf{b} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^\top \begin{pmatrix} \mathbf{\Lambda} \boldsymbol{\mu} - \mathbf{A}^\top \mathbf{L} \mathbf{b} \\ \mathbf{L} \mathbf{b} \end{pmatrix}$$

By inspection of the linear terms

$$\mathbb{E}[\mathbf{z}] = \mathbf{R}^{-1} \begin{pmatrix} \mathbf{\Lambda} \boldsymbol{\mu} - \mathbf{A}^\top \mathbf{L} \mathbf{b} \\ \mathbf{L} \mathbf{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{A} \boldsymbol{\mu} + \mathbf{b} \end{pmatrix}$$

And then we we'll have that

$$\begin{aligned}\mathbb{E}[\mathbf{y}] &= \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \\ \text{cov}[\mathbf{y}] &= \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^\top\end{aligned}$$

And then we we'll have that

$$\begin{aligned}\mathbb{E}[\mathbf{y}] &= \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \\ \text{cov}[\mathbf{y}] &= \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^\top\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\mathbf{x}|\mathbf{y}] &= \left(\boldsymbol{\Lambda} + \mathbf{A}^\top\mathbf{L}\mathbf{A}\right)^{-1} \left\{ \mathbf{A}^\top\mathbf{L}(\mathbf{y} - \mathbf{b}) + \boldsymbol{\Lambda}\boldsymbol{\mu} \right\} \\ \text{cov}[\mathbf{x}|\mathbf{y}] &= \left(\boldsymbol{\Lambda} + \mathbf{A}^\top\mathbf{L}\mathbf{A}\right)^{-1}\end{aligned}$$

In the next step, we'll assume a **prior distribution over parameters**,  $p(\mathbf{w})$ , and define it as a Gaussian distribution, then

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} | \mathbf{m}_0, \mathbf{S}_0)$$

with mean  $\mathbf{m}_0$  and variance  $\mathbf{S}_0$ .



## Marginal and Conditioned Gaussians

- For  $\mathbf{y}$  given  $\mathbf{x}$ :

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x} | \boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1})$$

$$p(\mathbf{y} | \mathbf{x}) = \mathcal{N}(\mathbf{y} | \mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1})$$

- For  $\mathbf{x}$  given  $\mathbf{y}$ :

$$p(\mathbf{x} | \mathbf{y}) = \mathcal{N}(\mathbf{y} | \boldsymbol{\Sigma} \{ \mathbf{A}^\top \mathbf{L}(\mathbf{y} - \mathbf{b} + \boldsymbol{\Sigma} \boldsymbol{\mu}) \}, \boldsymbol{\Sigma})$$

$$p(\mathbf{y}) = \mathcal{N}(\mathbf{y} | \mathbf{A} \boldsymbol{\mu} + \mathbf{b}, \mathbf{L}^{-1} + \mathbf{A} \boldsymbol{\Lambda}^{-1} \mathbf{A}^\top), \text{ where } \boldsymbol{\Sigma} = (\boldsymbol{\Lambda} + \mathbf{A}^\top \mathbf{L} \mathbf{A})^{-1}$$

By the derivations, we make the assumptions of given  $p(\mathbf{w})$  and for  $p(\mathbf{t}|\mathbf{w})$  such that

$$\begin{aligned} p(\mathbf{t}|\mathbf{w}) &= \mathcal{N}(\mathbf{t}|y(\mathbf{x}, \mathbf{w}), \beta^{-1}) \\ &= \mathcal{N}(\mathbf{t}|\Phi^\top \mathbf{w}, \beta^{-1}) \end{aligned}$$

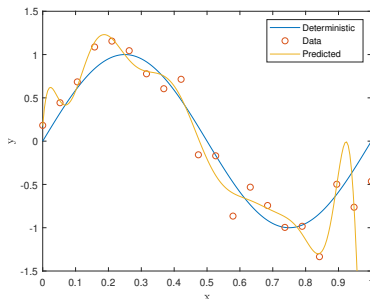
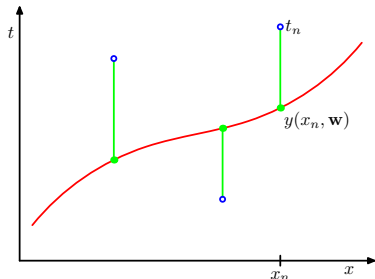
And then  $p(\mathbf{w}|\mathbf{t}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N)$  where

$$\begin{aligned} \mathbf{m}_N &= \mathbf{S}_N (\mathbf{S}_0^{-1} \mathbf{m}_0 + \beta \Phi^\top \mathbf{t}) \\ \mathbf{S}_N^{-1} &= \mathbf{S}_0^{-1} + \beta \Phi^\top \Phi \end{aligned}$$

# Gaussian Processes

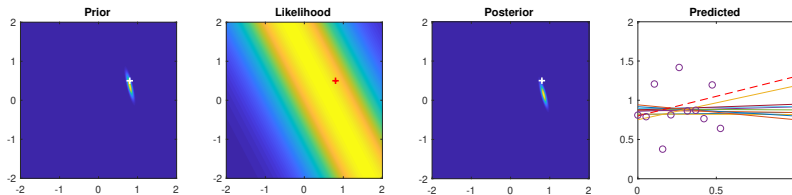
### What was done until here?

- We assumed that our targets  $t$  were **i.i.d.** and given by  $t = y(\mathbf{x}) + \varepsilon$ , where  $\varepsilon \sim \mathcal{N}(0, \beta)$ .
- Our model is given by  $y(\mathbf{x}) = \Phi^\top \mathbf{w}$ , where  $\Phi$  is the **design matrix**, and this characterizes our model as **linear in parameters**.
- The **design matrix** was defined as  $\phi_{i,j} = \phi_i(\mathbf{x}_j)$ .
- The **parameters** were given by  $\mathbf{w} = (\Phi^\top \Phi)^{-1} \Phi^\top \mathbf{t}$ .
- These **parameters** calculated at the minimum of the cost function are called **maximum likelihood**.



### What was done until here?

- We put an **uncertainty** over the targets  $t$  and the parameters  $\mathbf{w}$ .
- We assumed that targets being **distributed** as  $p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$ .
- By **Bayes' Rule** we obtained that  $p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta) \propto p(\mathbf{t}|\mathbf{w}, \mathbf{x}, \beta) p(\mathbf{w}|\alpha)$
- This allowed to make an **inference** to obtain a **prediction** of the parameters in the **weight-space**.



# Recap

A more clear way to see what is happening...

## From Bayesian inference

- We have

$$p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta) \propto p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)p(\mathbf{w}|\alpha)$$

- We'll change from **weight-space**

$$p(\mathbf{t}_*|\mathbf{x}_*, \mathbf{x}, \mathbf{t}) = \int p(\mathbf{t}_*|\mathbf{x}_*, \mathbf{w})p(\mathbf{w}|\mathbf{x}, \mathbf{t})d\mathbf{w}$$

- To **feature-space**

$$\begin{aligned} p(f_*|\mathbf{x}_*, \Phi, \mathbf{t}) &= \int p(f_*|\mathbf{x}_*, \mathbf{w})p(\mathbf{w}|\Phi, \mathbf{t})d\mathbf{w} = \int \mathbf{x}_*^\top \mathbf{w} p(\mathbf{w}|\Phi, \mathbf{t})d\mathbf{w} \\ &= \mathcal{N}\left(\beta \phi(\mathbf{x}_*)^\top \mathbf{S}_N \Phi \mathbf{t}, \phi(\mathbf{x}_*)^\top \mathbf{S}_N^{-1} \phi(\mathbf{x}_*)\right) \end{aligned}$$

where  $f_* \triangleq f(\mathbf{x}_*)$  at  $\mathbf{x}_*$  and  $\Phi = \Phi(\mathbf{x})$

## Alternative formulation

$$f_* | \mathbf{x}_*, \Phi, \mathbf{t} \sim \mathcal{N} \left( \phi_*^\top \mathbf{S}_0 \Phi \left( K + \beta^{-2} I \right)^{-1} \mathbf{t}, \phi_*^\top \mathbf{S}_0 \phi_* - \phi_*^\top \mathbf{S}_0 \Phi \left( K + \beta^{-2} I \right)^{-1} \Phi^\top \mathbf{S}_0 \phi_* \right)$$

where  $K = \Phi^\top \mathbf{S}_0 \Phi$



### What is kernel?

$$f_* | \mathbf{x}_*, \Phi, \mathbf{t} \sim \mathcal{N} \left( \phi_*^\top \mathbf{S}_0 \Phi \left( K + \beta^{-2} I \right)^{-1} \mathbf{t}, \phi_*^\top \mathbf{S}_0 \phi_* - \phi_*^\top \mathbf{S}_0 \Phi \left( K + \beta^{-2} I \right)^{-1} \Phi^\top \mathbf{S}_0 \phi_* \right)$$

- We could observe the appearance of terms like  $\Phi^\top \mathbf{S}_0 \Phi$ ,  $\phi_*^\top \mathbf{S}_0 \Phi$ , or  $\phi_*^\top \mathbf{S}_0 \phi_*$ .
- The common term between these operations is  $k(\mathbf{x}, \mathbf{x}') = \phi(\mathbf{x})^\top \mathbf{S}_0 \phi(\mathbf{x}')$
- Then we define  $k(\cdot, \cdot)$  as **kernel function**
- This technique is particularly valuable in situations where it is more convenient to compute the kernel than the design matrix vectors themselves.

- Previously we make the inference in the **feature-space** and then we find the function distribution.
- Now we'll make the inference directly on **function-space**.
- Let's define

## Definition

*A **Gaussian process** is a collection of random variables which any finite number of them have a joint Gaussian distribution.*

## Mean and covariance function

- As the Gaussian distribution, the  $\mathcal{GP}$  is characterized by its **mean function**  $m(\mathbf{x})$  and its **covariance function**  $k(\mathbf{x}, \mathbf{x}')$  of a real process  $f(\mathbf{x})$ .
- For a Gaussian processes

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}'))$$

- We have

$$m(\mathbf{x}) = \mathbb{E}[f(\mathbf{x})]$$

$$k(\mathbf{x}, \mathbf{x}') = \mathbb{E}[(f(\mathbf{x}) - m(\mathbf{x}))(f(\mathbf{x}') - m(\mathbf{x}'))]$$



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