Equating the first three elements of the fourth columns of the matrices on both sides yields

$$\boldsymbol{p}_W^1 = \begin{bmatrix} p_{Wx}c_1 + p_{Wy}s_1 \\ -p_{Wz} \\ -p_{Wx}s_1 + p_{Wy}c_1 \end{bmatrix} = \begin{bmatrix} d_3 s_2 \\ -d_3 c_2 \\ d_2 \end{bmatrix} \tag{2.94}$$

which depends only on ϑ_2 and d_3. To solve this equation, set

$$t = \tan\frac{\vartheta_1}{2}$$

so that

$$c_1 = \frac{1-t^2}{1+t^2} \qquad s_1 = \frac{2t}{1+t^2}.$$

Substituting this equation in the third component on the left-hand side of (2.94) gives

$$(d_2 + p_{Wy})t^2 + 2p_{Wx}t + d_2 - p_{Wy} = 0,$$

whose solution is

$$t = \frac{-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2}}{d_2 + p_{Wy}}.$$

The two solutions correspond to two different postures. Hence, it is

$$\vartheta_1 = 2\text{Atan2}\Big(-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2},\ d_2 + p_{Wy}\Big).$$

Once ϑ_1 is known, squaring and summing the first two components of (2.94) yields

$$d_3 = \sqrt{(p_{Wx}c_1 + p_{Wy}s_1)^2 + p_{Wz}^2},$$

where only the solution with $d_3 \geq 0$ has been considered. Note that the same value of d_3 corresponds to both solutions for ϑ_1. Finally, if $d_3 \neq 0$, from the first two components of (2.94) it is

$$\frac{p_{Wx}c_1 + p_{Wy}s_1}{-p_{Wz}} = \frac{d_3 s_2}{-d_3 c_2},$$

from which

$$\vartheta_2 = \text{Atan2}(p_{Wx}c_1 + p_{Wy}s_1, p_{Wz}).$$

Notice that, if $d_3 = 0$, then ϑ_2 cannot be uniquely determined.

2.12.4 Solution of Anthropomorphic Arm

Consider the anthropomorphic arm shown in Fig. 2.23. It is desired to find the joint variables ϑ_1, ϑ_2, ϑ_3 corresponding to a given end-effector position $\boldsymbol{p}_W$. Notice that the direct kinematics for $\boldsymbol{p}_W$ is expressed by (2.66) which can be obtained from (2.70) by setting $d_6 = 0$, $d_4 = a_3$ and replacing ϑ_3 with the angle $\vartheta_3 + \pi/2$ because of the misalignment of the Frames 3 for the structures in Fig. 2.23 and in Fig. 2.26, respectively. Hence, it follows

$$p_{Wx} = c_1(a_2c_2 + a_3c_{23}) \tag{2.95}$$
$$p_{Wy} = s_1(a_2c_2 + a_3c_{23}) \tag{2.96}$$
$$p_{Wz} = a_2s_2 + a_3s_{23}. \tag{2.97}$$

Proceeding as in the case of the two-link planar arm, it is worth squaring and summing (2.95)–(2.97) yielding

$$p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2a_3c_3$$

from which

$$c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3} \tag{2.98}$$

where the admissibility of the solution obviously requires that $-1 \le c_3 \le 1$, or equivalently $|a_2 - a_3| \le \sqrt{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2} \le a_2 + a_3$, otherwise the wrist point is outside the reachable workspace of the manipulator. Hence it is

$$s_3 = \pm\sqrt{1 - c_3^2} \tag{2.99}$$

and thus

$$\vartheta_3 = \mathrm{Atan2}(s_3, c_3)$$

giving the two solutions, according to the sign of s_3,

$$\vartheta_{3,\mathrm{I}} \in [-\pi, \pi] \tag{2.100}$$
$$\vartheta_{3,\mathrm{II}} = -\vartheta_{3,I}. \tag{2.101}$$

Having determined ϑ_3, it is possible to compute ϑ_2 as follows. Squaring and summing (2.95), (2.96) gives

$$p_{Wx}^2 + p_{Wy}^2 = (a_2c_2 + a_3c_{23})^2$$

from which

$$a_2c_2 + a_3c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2}. \tag{2.102}$$

The system of the two Eqs. (2.102), (2.97), for each of the solutions (2.100), (2.101), admits the solutions:

$$c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3c_3) + p_{Wz}a_3s_3}{a_2^2 + a_3^2 + 2a_2a_3c_3} \tag{2.103}$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2}a_3s_3}{a_2^2 + a_3^2 + 2a_2a_3c_3}. \tag{2.104}$$

From (2.103), (2.104) it follows

$$\vartheta_2 = \text{Atan2}(s_2, c_2)$$

which gives the four solutions for ϑ_2, according to the sign of s_3 in (2.99):

$$\begin{aligned}\vartheta_{2,\text{I}} = \text{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} - a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big) \qquad (2.105)\\ \vartheta_{2,\text{II}} = \text{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} + a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &-(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big) \qquad (2.106)\end{aligned}$$

corresponding to $s_3^+ = \sqrt{1 - c_3^2}$, and

$$\begin{aligned}\vartheta_{2,\text{III}} = \text{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} - a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big) \qquad (2.107)\\ \vartheta_{2,\text{IV}} = \text{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} + a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &-(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big) \qquad (2.108)\end{aligned}$$

corresponding to $s_3^- = -\sqrt{1 - c_3^2}$.

Finally, to compute ϑ_1, it is sufficient to rewrite (2.95), (2.96), using (2.102), as

$$\begin{aligned}p_{Wx} &= \pm c_1\sqrt{p_{Wx}^2 + p_{Wy}^2}\\ p_{Wy} &= \pm s_1\sqrt{p_{Wx}^2 + p_{Wy}^2}\end{aligned}$$

which, once solved, gives the two solutions:

$$\begin{aligned}\vartheta_{1,\text{I}} &= \text{Atan2}(p_{Wy}, p_{Wx}) \qquad (2.109)\\ \vartheta_{1,\text{II}} &= \text{Atan2}(-p_{Wy}, -p_{Wx}). \qquad (2.110)\end{aligned}$$

Notice that (2.110) gives[18]

$$\vartheta_{1,\text{II}} = \begin{cases}\text{Atan2}(p_{Wy}, p_{Wx}) - \pi & p_{Wy} \geq 0\\ \text{Atan2}(p_{Wy}, p_{Wx}) + \pi & p_{Wy} < 0.\end{cases}$$

[18] It is easy to show that $\text{Atan2}(-y, -x) = -\text{Atan2}(y, -x)$ and

$$\text{Atan2}(y, -x) = \begin{cases}\pi - \text{Atan2}(y, x) & y \geq 0\\ -\pi - \text{Atan2}(y, x) & y < 0.\end{cases}$$

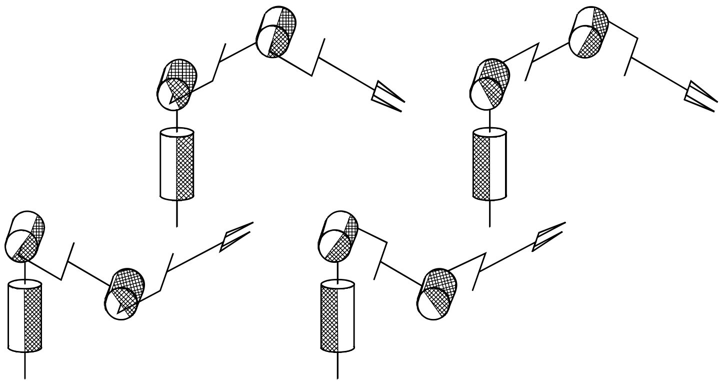

Fig. 2.33. The four configurations of an anthropomorphic arm compatible with a given wrist position

As can be recognized, there exist four solutions according to the values of ϑ_3 in (2.100), (2.101), ϑ_2 in (2.105)–(2.108) and ϑ_1 in (2.109), (2.110):

$$(\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{I}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{III}}, \vartheta_{3,\mathrm{II}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{II}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{IV}}, \vartheta_{3,\mathrm{II}}),$$

which are illustrated in Fig. 2.33: shoulder–right/elbow–up, shoulder–left/elbow–up, shoulder–right/elbow–down, shoulder–left/elbow–down; obviously, the forearm orientation is different for the two pairs of solutions.

Notice finally how it is possible to find the solutions only if at least

$$p_{Wx} \neq 0 \qquad \text{or} \qquad p_{Wy} \neq 0.$$

In the case $p_{Wx} = p_{Wy} = 0$, an infinity of solutions is obtained, since it is possible to determine the joint variables ϑ_2 and ϑ_3 independently of the value of ϑ_1; in the following, it will be seen that the arm in such configuration is kinematically *singular* (see Problem 2.18).

2.12.5 Solution of Spherical Wrist

Consider the spherical wrist shown in Fig. 2.24, whose direct kinematics was given in (2.67). It is desired to find the joint variables ϑ_4, ϑ_5, ϑ_6 corresponding to a given end-effector orientation $\boldsymbol{R}_6^3$. As previously pointed out, these angles constitute a set of Euler angles ZYZ with respect to Frame 3. Hence, having computed the rotation matrix

$$\boldsymbol{R}_6^3 = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix},$$