As regards the direct kinematics equation in (2.50), the end-effector position and rotation matrix are computed in a unique manner, once the joint variables are known[15]. On the other hand, the inverse kinematics problem is much more complex for the following reasons:

- The equations to solve are in general nonlinear, and thus it is not always possible to find a *closed-form solution*.
- *Multiple solutions* may exist.
- *Infinite solutions* may exist, e.g., in the case of a kinematically redundant manipulator.
- There might be no *admissible* solutions, in view of the manipulator kinematic structure.

The existence of solutions is guaranteed only if the given end-effector position and orientation belong to the manipulator dexterous workspace.

On the other hand, the problem of multiple solutions depends not only on the number of DOFs but also on the number of non-null DH parameters; in general, the greater the number of non-null parameters, the greater the number of admissible solutions. For a six-DOF manipulator without mechanical joint limits, there are in general up to 16 admissible solutions. Such occurrence demands some criterion to choose among admissible solutions (e.g., the elbow-up/elbow-down case of Example 2.6). The existence of mechanical joint limits may eventually reduce the number of admissible multiple solutions for the real structure.

Computation of closed-form solutions requires either *algebraic intuition* to find those significant equations containing the unknowns or *geometric intuition* to find those significant points on the structure with respect to which it is convenient to express position and/or orientation as a function of a reduced number of unknowns. The following examples will point out the ability required to an inverse kinematics problem solver. On the other hand, in all those cases when there are no — or it is difficult to find — closed-form solutions, it might be appropriate to resort to *numerical solution techniques*; these clearly have the advantage of being applicable to any kinematic structure, but in general they do not allow computation of all admissible solutions. In the following chapter, it will be shown how suitable algorithms utilizing the manipulator Jacobian can be employed to solve the inverse kinematics problem.

2.12.1 Solution of Three-link Planar Arm

Consider the arm shown in Fig. 2.20 whose direct kinematics was given in (2.63). It is desired to find the joint variables ϑ_1, ϑ_2, ϑ_3 corresponding to a given end-effector position and orientation.

[15] In general, this cannot be said for (2.82) too, since the Euler angles are not uniquely defined.

As already pointed out, it is convenient to specify position and orientation in terms of a minimal number of parameters: the two coordinates p_x, p_y and the angle ϕ with axis x_0, in this case. Hence, it is possible to refer to the direct kinematics equation in the form (2.83).

A first *algebraic solution* technique is illustrated below. Having specified the orientation, the relation

$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3 \tag{2.90}$$

is one of the equations of the system to solve[16]. From (2.63) the following equations can be obtained:

$$p_{Wx} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12} \tag{2.91}$$

$$p_{Wy} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12} \tag{2.92}$$

which describe the position of point W, i.e., the origin of Frame 2; this depends only on the first two angles ϑ_1 and ϑ_2. Squaring and summing (2.91), (2.92) yields

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 + 2a_1 a_2 c_2$$

from which

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}.$$

The existence of a solution obviously imposes that $-1 \le c_2 \le 1$, otherwise the given point would be outside the arm reachable workspace. Then, set

$$s_2 = \pm\sqrt{1 - c_2^2},$$

where the positive sign is relative to the elbow-down posture and the negative sign to the elbow-up posture. Hence, the angle ϑ_2 can be computed as

$$\vartheta_2 = \mathrm{Atan2}(s_2, c_2).$$

Having determined ϑ_2, the angle ϑ_1 can be found as follows. Substituting ϑ_2 into (2.91), (2.92) yields an algebraic system of two equations in the two unknowns s_1 and c_1, whose solution is

$$s_1 = \frac{(a_1 + a_2 c_2) p_{Wy} - a_2 s_2 p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$

$$c_1 = \frac{(a_1 + a_2 c_2) p_{Wx} + a_2 s_2 p_{Wy}}{p_{Wx}^2 + p_{Wy}^2}.$$

In analogy to the above, it is

$$\vartheta_1 = \mathrm{Atan2}(s_1, c_1).$$

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[16] If ϕ is not specified, then the arm is redundant and there exist infinite solutions to the inverse kinematics problem.

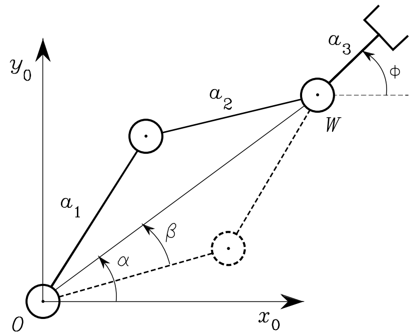

Fig. 2.31. Admissible postures for a two-link planar arm

In the case when $s_2 = 0$, it is obviously $\vartheta_2 = 0, \pi$; as will be shown in the following, in such a posture the manipulator is at a kinematic *singularity*. Yet, the angle ϑ_1 can be determined uniquely, unless $a_1 = a_2$ and it is required $p_{Wx} = p_{Wy} = 0$.

Finally, the angle ϑ_3 is found from (2.90) as

$$\vartheta_3 = \phi - \vartheta_1 - \vartheta_2.$$

An alternative *geometric solution* technique is presented below. As above, the orientation angle is given as in (2.90) and the coordinates of the origin of Frame 2 are computed as in (2.91), (2.92). The application of the cosine theorem to the triangle formed by links a_1, a_2 and the segment connecting points W and O gives

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \vartheta_2);$$

the two admissible configurations of the triangle are shown in Fig. 2.31. Observing that $\cos(\pi - \vartheta_2) = -\cos\vartheta_2$ leads to

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2}.$$

For the existence of the triangle, it must be $\sqrt{p_{Wx}^2 + p_{Wy}^2} \le a_1 + a_2$. This condition is not satisfied when the given point is outside the arm reachable workspace. Then, under the assumption of admissible solutions, it is

$$\vartheta_2 = \pm\cos^{-1}(c_2);$$

the elbow-up posture is obtained for $\vartheta_2 \in (-\pi, 0)$ while the elbow-down posture is obtained for $\vartheta_2 \in (0, \pi)$.

To find ϑ_1 consider the angles α and β in Fig. 2.31. Notice that the determination of α depends on the sign of p_{Wx} and p_{Wy}; then, it is necessary to compute α as

$$\alpha = \mathrm{Atan2}(p_{Wy}, p_{Wx}).$$

To compute β, applying again the cosine theorem yields

$$c_\beta \sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2 c_2$$

and resorting to the expression of c_2 given above leads to

$$\beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

with $\beta \in (0, \pi)$ so as to preserve the existence of triangles. Then, it is

$$\vartheta_1 = \alpha \pm \beta,$$

where the positive sign holds for $\vartheta_2 < 0$ and the negative sign for $\vartheta_2 > 0$. Finally, ϑ_3 is computed from (2.90).

It is worth noticing that, in view of the substantial equivalence between the two-link planar arm and the parallelogram arm, the above techniques can be formally applied to solve the inverse kinematics of the arm in Sect. 2.9.2.

2.12.2 Solution of Manipulators with Spherical Wrist

Most of the existing manipulators are kinematically simple, since they are typically formed by an arm, of the kind presented above, and a spherical wrist; see the manipulators in Sects. 2.9.6–2.9.8. This choice is partly motivated by the difficulty to find solutions to the inverse kinematics problem in the general case. In particular, a *six*-DOF kinematic structure has closed-form inverse kinematics solutions if:

- three consecutive revolute joint axes intersect at a common point, like for the spherical wrist;
- three consecutive revolute joint axes are parallel.

In any case, algebraic or geometric intuition is required to obtain closed-form solutions.

Inspired by the previous solution to a three-link planar arm, a suitable point along the structure can be found whose position can be expressed both as a function of the given end-effector position and orientation and as a function of a reduced number of joint variables. This is equivalent to articulating the inverse kinematics problem into two subproblems, since the solution for the *position* is *decoupled* from that for the *orientation*.

For a manipulator with spherical wrist, the natural choice is to locate such point W at the intersection of the three terminal revolute axes (Fig. 2.32). In fact, once the end-effector position and orientation are specified in terms of $\boldsymbol{p}_e$ and $\boldsymbol{R}_e = [\,\boldsymbol{n}_e \quad \boldsymbol{s}_e \quad \boldsymbol{a}_e\,]$, the wrist position can be found as

$$\boldsymbol{p}_W = \boldsymbol{p}_e - d_6 \boldsymbol{a}_e \tag{2.93}$$

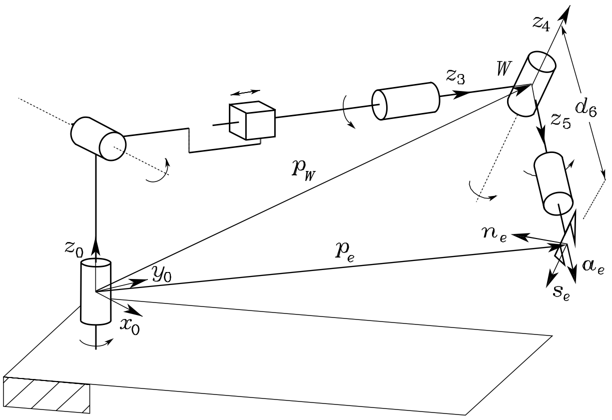

Fig. 2.32. Manipulator with spherical wrist

which is a function of the sole joint variables that determine the arm position[17]. Hence, in the case of a (nonredundant) three-DOF arm, the inverse kinematics can be solved according to the following steps:

- Compute the wrist position $\boldsymbol{p}_W(q_1, q_2, q_3)$ as in (2.93).
- Solve inverse kinematics for (q_1, q_2, q_3).
- Compute $\boldsymbol{R}_3^0(q_1, q_2, q_3)$.
- Compute $\boldsymbol{R}_6^3(\vartheta_4, \vartheta_5, \vartheta_6) = \boldsymbol{R}_3^{0T}\boldsymbol{R}$.
- Solve inverse kinematics for orientation $(\vartheta_4, \vartheta_5, \vartheta_6)$.

Therefore, on the basis of this kinematic decoupling, it is possible to solve the inverse kinematics for the arm separately from the inverse kinematics for the spherical wrist. Below are presented the solutions for two typical arms (spherical and anthropomorphic) as well as the solution for the spherical wrist.

2.12.3 Solution of Spherical Arm

Consider the spherical arm shown in Fig. 2.22, whose direct kinematics was given in (2.65). It is desired to find the joint variables ϑ_1, ϑ_2, d_3 corresponding to a given end-effector position $\boldsymbol{p}_W$.

In order to separate the variables on which $\boldsymbol{p}_W$ depends, it is convenient to express the position of $\boldsymbol{p}_W$ with respect to Frame 1; then, consider the matrix equation

$$(\boldsymbol{A}_1^0)^{-1}\boldsymbol{T}_3^0 = \boldsymbol{A}_2^1\boldsymbol{A}_3^2.$$

[17] Note that the same reasoning was implicitly adopted in Sect. 2.12.1 for the three-link planar arm; $\boldsymbol{p}_W$ described the one-DOF wrist position for the two-DOF arm obtained by considering only the first two links.

Equating the first three elements of the fourth columns of the matrices on both sides yields

$$\boldsymbol{p}_W^1 = \begin{bmatrix} p_{Wx}c_1 + p_{Wy}s_1 \\ -p_{Wz} \\ -p_{Wx}s_1 + p_{Wy}c_1 \end{bmatrix} = \begin{bmatrix} d_3 s_2 \\ -d_3 c_2 \\ d_2 \end{bmatrix} \tag{2.94}$$

which depends only on ϑ_2 and d_3. To solve this equation, set

$$t = \tan\frac{\vartheta_1}{2}$$

so that

$$c_1 = \frac{1-t^2}{1+t^2} \qquad s_1 = \frac{2t}{1+t^2}.$$

Substituting this equation in the third component on the left-hand side of (2.94) gives

$$(d_2 + p_{Wy})t^2 + 2p_{Wx}t + d_2 - p_{Wy} = 0,$$

whose solution is

$$t = \frac{-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2}}{d_2 + p_{Wy}}.$$

The two solutions correspond to two different postures. Hence, it is

$$\vartheta_1 = 2\mathrm{Atan2}\Big(-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2},\ d_2 + p_{Wy}\Big).$$

Once ϑ_1 is known, squaring and summing the first two components of (2.94) yields

$$d_3 = \sqrt{(p_{Wx}c_1 + p_{Wy}s_1)^2 + p_{Wz}^2},$$

where only the solution with $d_3 \geq 0$ has been considered. Note that the same value of d_3 corresponds to both solutions for ϑ_1. Finally, if $d_3 \neq 0$, from the first two components of (2.94) it is

$$\frac{p_{Wx}c_1 + p_{Wy}s_1}{-p_{Wz}} = \frac{d_3 s_2}{-d_3 c_2},$$

from which

$$\vartheta_2 = \mathrm{Atan2}(p_{Wx}c_1 + p_{Wy}s_1, p_{Wz}).$$

Notice that, if $d_3 = 0$, then ϑ_2 cannot be uniquely determined.

2.12.4 Solution of Anthropomorphic Arm

Consider the anthropomorphic arm shown in Fig. 2.23. It is desired to find the joint variables ϑ_1, ϑ_2, ϑ_3 corresponding to a given end-effector position $\boldsymbol{p}_W$. Notice that the direct kinematics for $\boldsymbol{p}_W$ is expressed by (2.66) which can be obtained from (2.70) by setting $d_6 = 0$, $d_4 = a_3$ and replacing ϑ_3 with the angle $\vartheta_3 + \pi/2$ because of the misalignment of the Frames 3 for the structures in Fig. 2.23 and in Fig. 2.26, respectively. Hence, it follows

$$p_{Wx} = c_1(a_2 c_2 + a_3 c_{23}) \tag{2.95}$$
$$p_{Wy} = s_1(a_2 c_2 + a_3 c_{23}) \tag{2.96}$$
$$p_{Wz} = a_2 s_2 + a_3 s_{23}. \tag{2.97}$$

Proceeding as in the case of the two-link planar arm, it is worth squaring and summing (2.95)–(2.97) yielding

$$p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2 a_3 c_3$$

from which

$$c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2 a_3} \tag{2.98}$$

where the admissibility of the solution obviously requires that $-1 \leq c_3 \leq 1$, or equivalently $|a_2 - a_3| \leq \sqrt{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2} \leq a_2 + a_3$, otherwise the wrist point is outside the reachable workspace of the manipulator. Hence it is

$$s_3 = \pm\sqrt{1 - c_3^2} \tag{2.99}$$

and thus

$$\vartheta_3 = \text{Atan2}(s_3, c_3)$$

giving the two solutions, according to the sign of s_3,

$$\vartheta_{3,\mathrm{I}} \in [-\pi, \pi] \tag{2.100}$$
$$\vartheta_{3,\mathrm{II}} = -\vartheta_{3,I}. \tag{2.101}$$

Having determined ϑ_3, it is possible to compute ϑ_2 as follows. Squaring and summing (2.95), (2.96) gives

$$p_{Wx}^2 + p_{Wy}^2 = (a_2 c_2 + a_3 c_{23})^2$$

from which

$$a_2 c_2 + a_3 c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2}. \tag{2.102}$$

The system of the two Eqs. (2.102), (2.97), for each of the solutions (2.100), (2.101), admits the solutions:

$$c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3 c_3) + p_{Wz} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \tag{2.103}$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3 c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2}\, a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}. \tag{2.104}$$

From (2.103), (2.104) it follows

$$\vartheta_2 = \mathrm{Atan2}(s_2, c_2)$$

which gives the four solutions for ϑ_2, according to the sign of s_3 in (2.99):

$$\begin{aligned}\vartheta_{2,\mathrm{I}} = \mathrm{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} - a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big)\end{aligned} \tag{2.105}$$

$$\begin{aligned}\vartheta_{2,\mathrm{II}} = \mathrm{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} + a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &-(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big)\end{aligned} \tag{2.106}$$

corresponding to $s_3^+ = \sqrt{1 - c_3^2}$, and

$$\begin{aligned}\vartheta_{2,\mathrm{III}} = \mathrm{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} - a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big)\end{aligned} \tag{2.107}$$

$$\begin{aligned}\vartheta_{2,\mathrm{IV}} = \mathrm{Atan2}\Big(&(a_2 + a_3c_3)p_{Wz} + a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},\\ &-(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big)\end{aligned} \tag{2.108}$$

corresponding to $s_3^- = -\sqrt{1 - c_3^2}$.

Finally, to compute ϑ_1, it is sufficient to rewrite (2.95), (2.96), using (2.102), as

$$\begin{aligned}p_{Wx} &= \pm c_1\sqrt{p_{Wx}^2 + p_{Wy}^2}\\ p_{Wy} &= \pm s_1\sqrt{p_{Wx}^2 + p_{Wy}^2}\end{aligned}$$

which, once solved, gives the two solutions:

$$\vartheta_{1,\mathrm{I}} = \mathrm{Atan2}(p_{Wy}, p_{Wx}) \tag{2.109}$$

$$\vartheta_{1,\mathrm{II}} = \mathrm{Atan2}(-p_{Wy}, -p_{Wx}). \tag{2.110}$$

Notice that (2.110) gives[18]

$$\vartheta_{1,\mathrm{II}} = \begin{cases}\mathrm{Atan2}(p_{Wy}, p_{Wx}) - \pi & p_{Wy} \geq 0\\ \mathrm{Atan2}(p_{Wy}, p_{Wx}) + \pi & p_{Wy} < 0.\end{cases}$$

[18] It is easy to show that $\mathrm{Atan2}(-y, -x) = -\mathrm{Atan2}(y, -x)$ and

$$\mathrm{Atan2}(y, -x) = \begin{cases}\pi - \mathrm{Atan2}(y, x) & y \geq 0\\ -\pi - \mathrm{Atan2}(y, x) & y < 0.\end{cases}$$

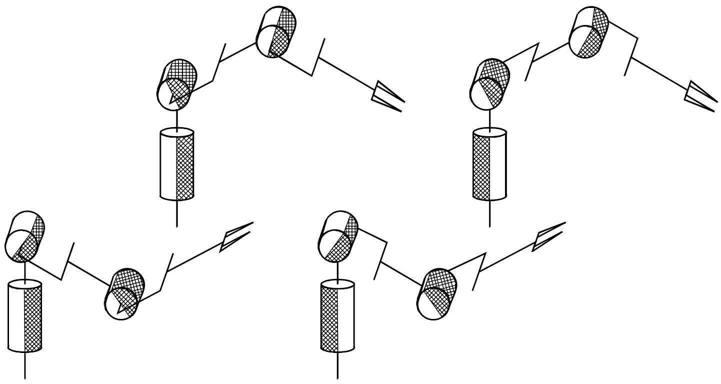

Fig. 2.33. The four configurations of an anthropomorphic arm compatible with a given wrist position

As can be recognized, there exist four solutions according to the values of ϑ_3 in (2.100), (2.101), ϑ_2 in (2.105)–(2.108) and ϑ_1 in (2.109), (2.110):

$$(\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{I}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{III}}, \vartheta_{3,\mathrm{II}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{II}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{IV}}, \vartheta_{3,\mathrm{II}}),$$

which are illustrated in Fig. 2.33: shoulder–right/elbow–up, shoulder–left/elbow–up, shoulder–right/elbow–down, shoulder–left/elbow–down; obviously, the forearm orientation is different for the two pairs of solutions.

Notice finally how it is possible to find the solutions only if at least

$$p_{Wx} \neq 0 \qquad \text{or} \qquad p_{Wy} \neq 0.$$

In the case $p_{Wx} = p_{Wy} = 0$, an infinity of solutions is obtained, since it is possible to determine the joint variables ϑ_2 and ϑ_3 independently of the value of ϑ_1; in the following, it will be seen that the arm in such configuration is kinematically *singular* (see Problem 2.18).

2.12.5 Solution of Spherical Wrist

Consider the spherical wrist shown in Fig. 2.24, whose direct kinematics was given in (2.67). It is desired to find the joint variables ϑ_4, ϑ_5, ϑ_6 corresponding to a given end-effector orientation $\boldsymbol{R}_6^3$. As previously pointed out, these angles constitute a set of Euler angles ZYZ with respect to Frame 3. Hence, having computed the rotation matrix

$$\boldsymbol{R}_6^3 = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix},$$

from its expression in terms of the joint variables in (2.67), it is possible to compute the solutions directly as in (2.19), (2.20), i.e.,

$$\begin{aligned}\vartheta_4 &= \mathrm{Atan2}(a_y^3, a_x^3)\\ \vartheta_5 &= \mathrm{Atan2}\Big(\sqrt{(a_x^3)^2+(a_y^3)^2}, a_z^3\Big)\\ \vartheta_6 &= \mathrm{Atan2}(s_z^3, -n_z^3)\end{aligned} \tag{2.111}$$

for $\vartheta_5 \in (0,\pi)$, and

$$\begin{aligned}\vartheta_4 &= \mathrm{Atan2}(-a_y^3, -a_x^3)\\ \vartheta_5 &= \mathrm{Atan2}\Big(-\sqrt{(a_x^3)^2+(a_y^3)^2}, a_z^3\Big)\\ \vartheta_6 &= \mathrm{Atan2}(-s_z^3, n_z^3)\end{aligned} \tag{2.112}$$

for $\vartheta_5 \in (-\pi, 0)$.

Bibliography

The treatment of kinematics of robot manipulators can be found in several classical robotics texts, such as [180, 10, 200, 217]. Specific texts are [23, 6, 151].

For the descriptions of the orientation of a rigid body, see [187]. Quaternion algebra can be found in [46]; see [204] for the extraction of quaternions from rotation matrices.

The Denavit–Hartenberg convention was first introduced in [60]. A modified version is utilized in [53, 248, 111]. The use of homogeneous transformation matrices for the computation of open-chain manipulator direct kinematics is presented in [181], while in [183] sufficient conditions are given for the closed-form computation of the inverse kinematics problem. For kinematics of closed chains see [144, 111]. The design of the Stanford manipulator is due to [196].

The problem of kinematic calibration is considered in [188, 98]. Methods which do not require the use of external sensors for direct measurement of end-effector position and orientation are proposed in [68].

The kinematic decoupling deriving from the spherical wrist is utilized in [76, 99, 182]. Numerical methods for the solution of the inverse kinematics problem based on iterative algorithms are proposed in [232, 86].

Problems

2.1. Find the rotation matrix corresponding to the set of Euler angles ZXZ.

2.2. Discuss the inverse solution for the Euler angles ZYZ in the case $s_\vartheta = 0$.