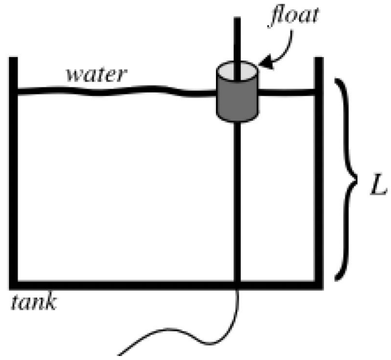
Kalman filtering example

Estimate the level of water in a tank




```
clearvars
close all
```

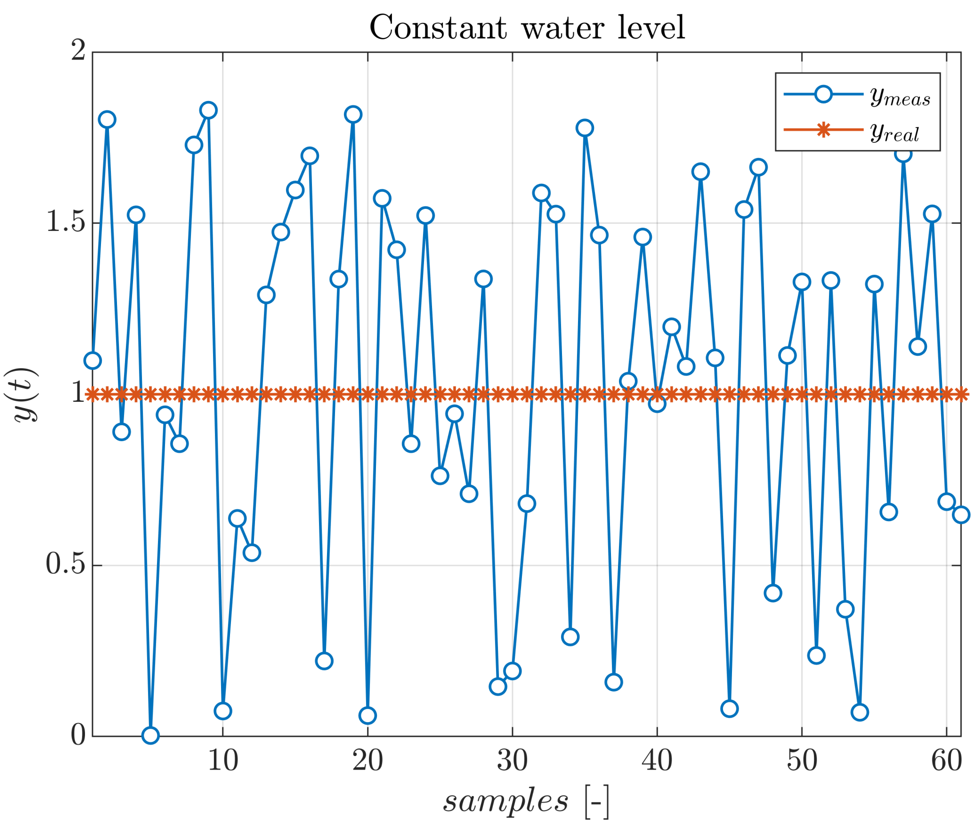
First case: constant level

Import the data

```
data_tab = importdata('Constant_level.xlsx'); % Load data from table
y_true = data_tab.data(:,1); % Get the true value of the output
x_true = y_true; % The state coincides with the output (see later)
y_meas = data_tab.data(:,2); % Get the measurements
u      = zeros(length(y_meas),1); % No input affects the evolution of the level of water
```

Compare the actual state **(generally not available in practice)** and the measurements

```
% Superimpose the plots of the true state and the measured output
figure(1); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(y_meas, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$y_{meas}$');
plot(y_true, '-*', 'Linewidth', 1, 'DisplayName', '$y_{real}$');
legend('Interpreter', 'latex');
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$y(t)$', 'Interpreter', 'latex');
title('Constant water level', 'Interpreter', 'latex');
xlim([1, length(y_true)]); clf;
```



Static model

Define the model $L(t+1) = L(t)$ (constant Level). We can define $L(t) = x(t)$:

$$x(t+1) = x(t) + v_1(t)$$

$$y(t) = x(t) + v_2(t)$$

```
F = 1; % The value of the state at t+1 is equal to that of the state at t
G = 0; % No exogenous input affects the level of water
H = 1; % The state is fully measured
V1 = 1e-4; % We trust the model
V2 = 0.1; % We have less trust on the measurements
```

Note that, V2 **does not correspond** to the **true variance** of the measurement noise (equal to 2). Indeed, we assume we do not know it exactly.

Initialize the predicted state and covariance matrix. We thus firstly perform the correction **step** and, then, the prediction **step** (by initializing the corrected state and covariance matrix the order of the steps can be straightfowardly reversed).

```
x_predInit = 0; % Initial predicted state
P_predInit = 1e3; % Initial predicted state covariance
```

The chosen initialization of the corrected covariance entails that we do not trust the initial predicted state

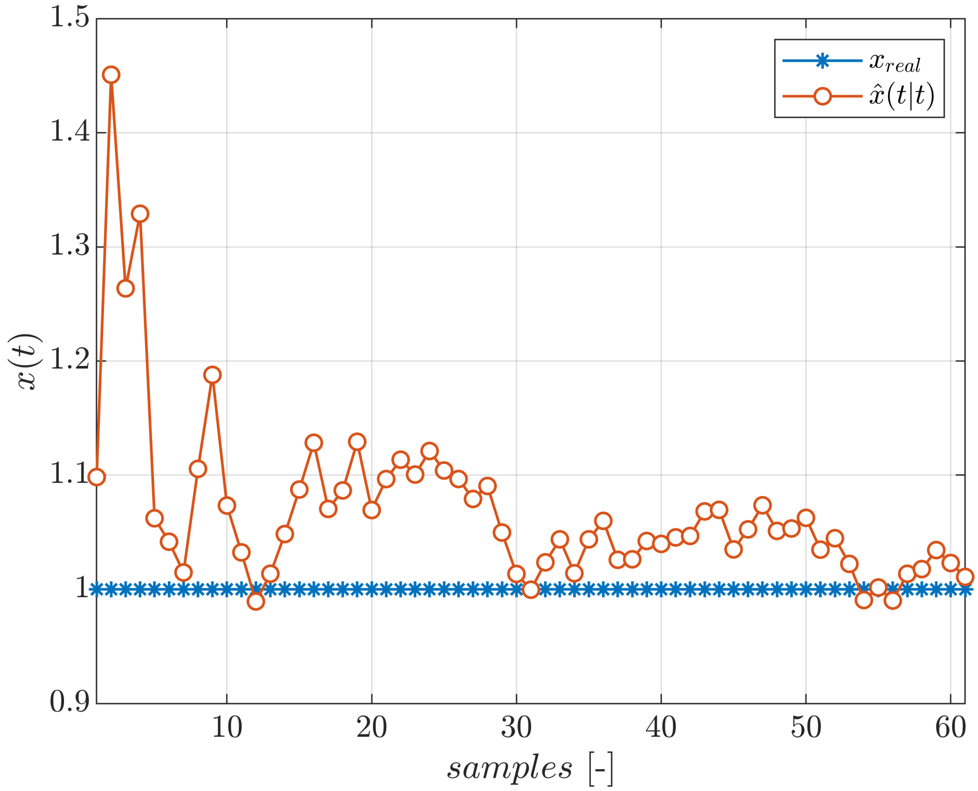
Initialization of all corrected and predicted states, along with the Kalman Gain

```
x_corr = zeros(length(y_meas), 1);
x_pred = zeros(length(y_meas)+1, 1);
P_corr = zeros(length(y_meas), 1);
P_pred = zeros(length(y_meas)+1, 1);
x_pred(1) = x_predInit;
P_pred(1) = P_predInit;
K = zeros(length(y_meas), 1);
```

Kalman filtering steps

```
for t = 1:length(y_meas)
   % Correction step
   [K(t), x_corr(t), P_corr(t)] = kf_correction(y_meas(t), H, V2, x_pred(t), P_pred(t));
   % Prediction step
   [x_pred(t+1), P_pred(t+1)] = kf_prediction(u(t), F, G, V1, x_corr(t), P_corr(t));
end
```

```
% Superimpose the plots of the true state and the corrected state
figure(2); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(x_true, '-*', 'Linewidth', 1, 'DisplayName', '$x_{real}$');
plot(x_corr, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$\hat{x}(t|t)$')
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
xlim([1, length(x_true)]); clf;
```



As it can be seen, the estimate converges to the true value of the state.

Obtain the asymptotic Kalman filter and check the convergence to it. We will use MATLAB's built-in function `kalman`. It has the following inputs and outputs:

`[KF_model,` $\bar{K}$, $\bar{P}$`] = kalman(system_model, V1, V2, V12)`

Being `system_model` thhe system model expressed as follows:

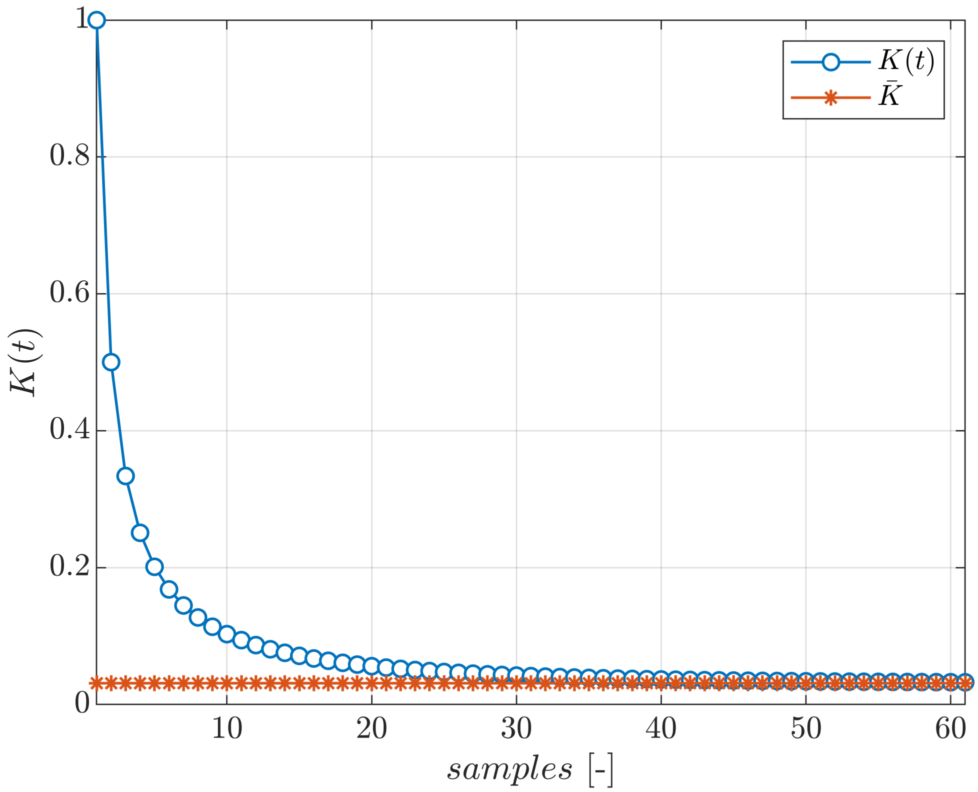
$$x(t+1) = Fx(t) + G_u u(t) + G_{v_1} v_1(t)$$

$$y(t) = Hx(t) + Du(t) + H_{v_1} v_1(t) + v_2(t)$$

```
tank_model = ss(F, [G 1], H, [0 0], -1); % Define the tank model
[KF_model, K_bar, P_bar] = kalman(tank_model, V1, V2, 0); % Find the Kalman filter and compute
```

```
% Check the convergence of the time-varying Kalman gain to the asymptotic value
K_pred = F*K;

figure(3); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(K_pred, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$K(t)$');
plot(K_bar*ones(length(y_meas), 1), '-*', 'Linewidth', 1, 'DisplayName', '$\bar{K}$');
ylabel('$K(t)$', 'Interpreter', 'latex');
xlabel('$samples$ [-]', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
```

```
xlim([1 length(y_meas)]); clf;
```



Assess the effect of different parameters

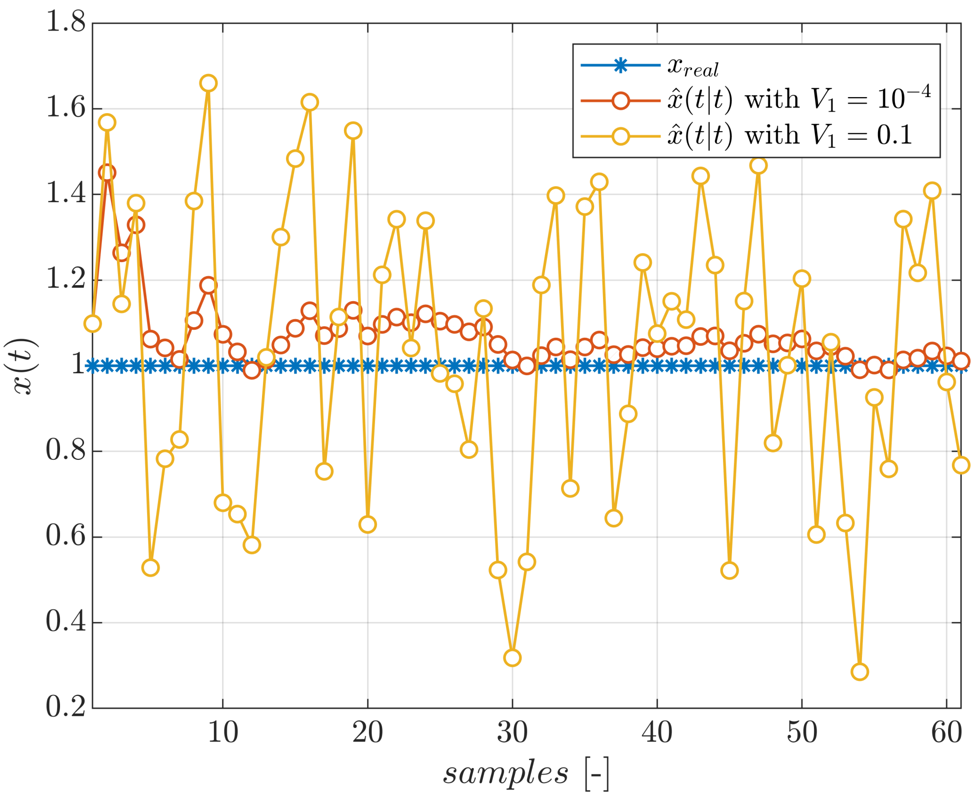
Change the covariance matrix of the process noise

```
V1_2 = 1e-1; % Increase covariance of the process noise
% Initializations
x_corr2 = zeros(length(y_meas), 1);
x_pred2 = zeros(length(y_meas)+1, 1);
P_corr2 = zeros(length(y_meas), 1);
P_pred2 = zeros(length(y_meas)+1, 1);
x_pred2(1) = x_predInit;
P_pred2(1) = P_predInit;
K2 = zeros(length(y_meas), 1);

% Kalman filtering steps
for t=1:length(y_meas)
   % Correction step
   [K2(t), x_corr2(t), P_corr2(t)] = kf_correction(y_meas(t), H, V2, x_pred2(t), P_pred2(t));
   % Prediction step
   [x_pred2(t+1), P_pred2(t+1)] = kf_prediction(u(t), F, G, V1_2, x_corr2(t), P_corr2(t));
end
```

Compare the estimate obtained with the previous model with this one.

```
% Superimpose the plots of the true state and the corrected state
figure(4); hold on; grid on; box on;
```

```
set(gca, 'FontSize', 12);
plot(x_true, '-*', 'Linewidth', 1, 'DisplayName', '$x_{real}$');
plot(x_corr, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$\hat{x}(t|t)$ wi
plot(x_corr2, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$\hat{x}(t|t)$ w
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
xlim([1,length(x_true)]);  clf;
```



Lowering the **trust** on the model (that is, in this case, *correct*) leads to a **poorer estimate** of the state.

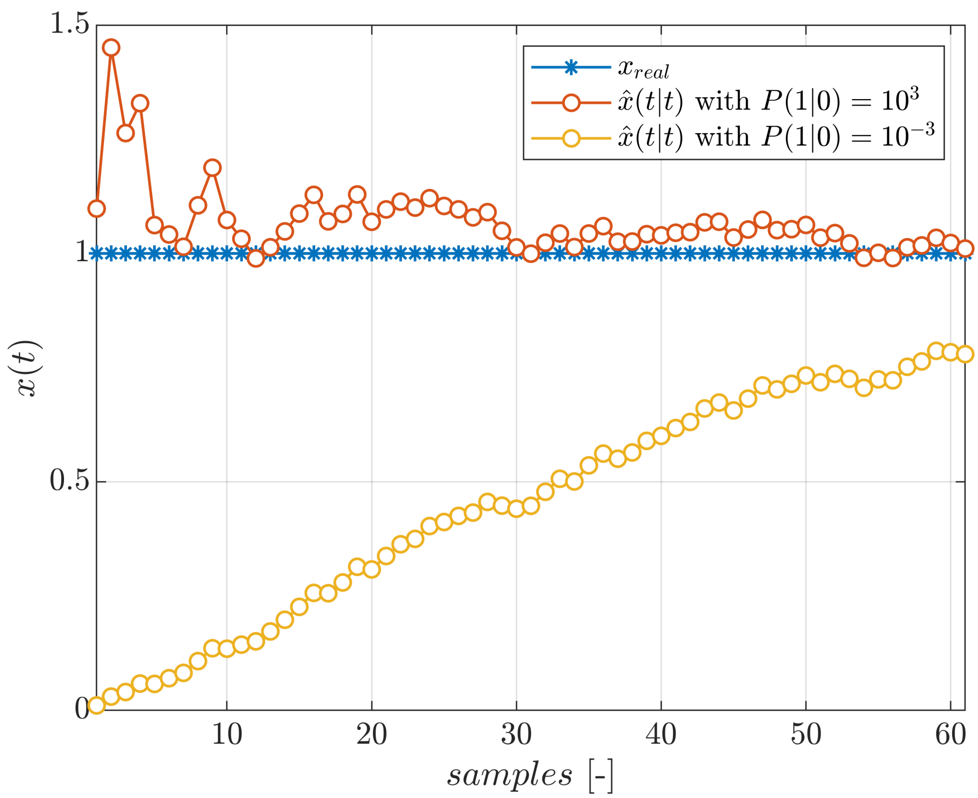
Change the initial trust on the initial estimate of the state

```
P_predInit2 = 1e-3; % Lower predicted covariance of the state
% Initializations
x_corr3 = zeros(length(y_meas),1);
x_pred3 = zeros(length(y_meas)+1,1);
P_corr3 = zeros(length(y_meas),1);
P_pred3 = zeros(length(y_meas)+1,1);
x_pred3(1) = x_predInit;
P_pred3(1) = P_predInit2;
K3 = zeros(length(y_meas), 1);

% Kalman filtering steps
for t = 1:length(y_meas)
   % Correction step
   [K3(t), x_corr3(t), P_corr3(t)] = kf_correction(y_meas(t), H, V2, x_pred3(t), P_pred3(t));
   % Prediction step
```

```
    [x_pred3(t+1), P_pred3(t+1)] = kf_prediction(u(t), F, G, V1, x_corr3(t), P_corr3(t));
end
```

Compare the estimate obtained with the previous model with this one.

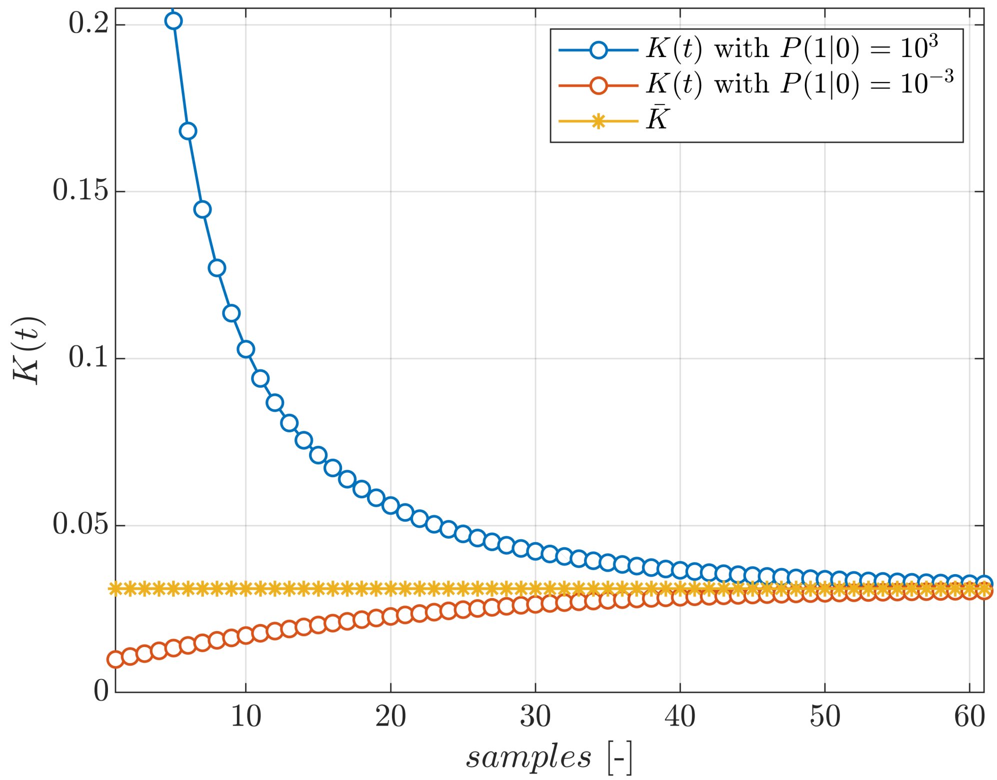
```
% Superimpose the plots of the true state and the corrected state
figure(5); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(x_true, '-*', 'Linewidth', 1, 'DisplayName', '$x_{real}$');
plot(x_corr, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$\hat{x}(t|t)$ wi
plot(x_corr3, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$\hat{x}(t|t)$ w
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
xlim([1, length(x_true)]);  clf;
```



Increasing the level of **trust** on the initial condition **slows down** the **convergence** of the state estimate.

```
% Check the convergence of the time-varying Kalman gain to the asymptotic value
K3_pred = F*K3;

figure; hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(K_pred, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$K(t)$ with $P(1
plot(K3_pred, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$K(t)$ with $P(1
plot(K_bar*ones(length(y_meas), 1), '-*', 'Linewidth', 1, 'DisplayName', '$\bar{K}$');
ylabel('$K(t)$', 'Interpreter', 'latex');
xlabel('$samples$ [-]', 'Interpreter', 'latex');
```

```
legend('Interpreter', 'latex');
ylim([0.000 0.205]);
xlim([1 length(y_meas)]); clf;
```



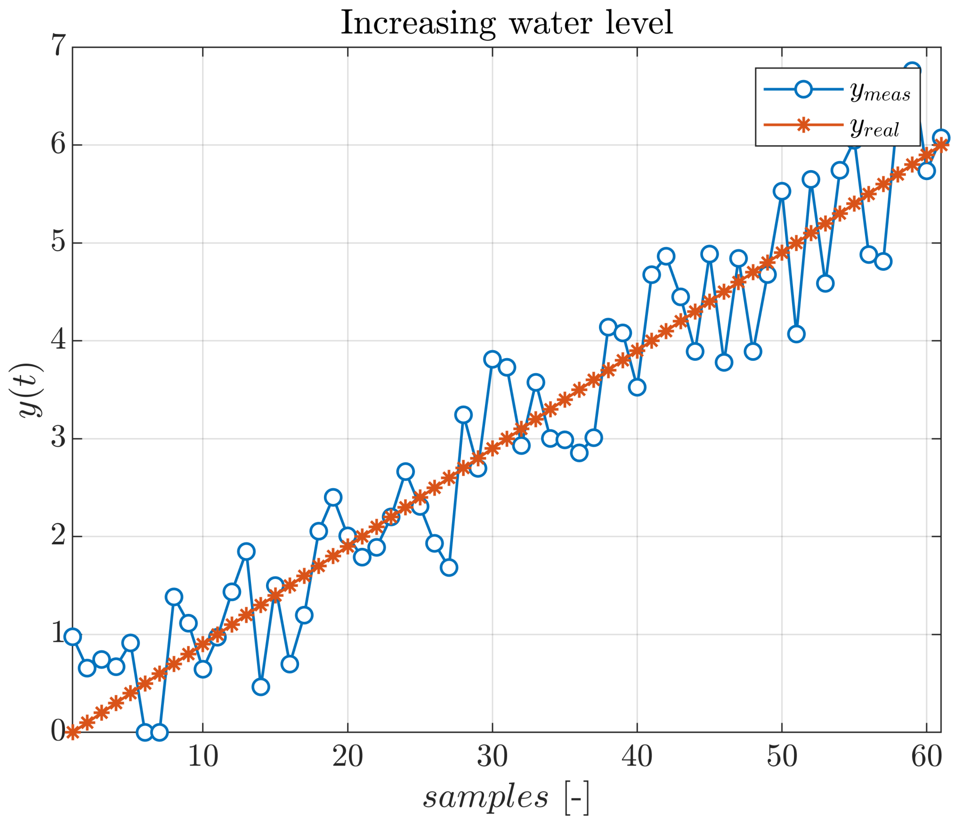
Second case: constant filling

Import the data

```
data_tab = importdata('Filling_tank.xlsx'); % Load data from table
x_true   = data_tab.data(:,1); % Get the true value of the state
y_true   = data_tab.data(:,1); % Get the true value of the state
y_meas   = data_tab.data(:,2); % Get the measurements
u        = zeros(length(y_meas),1); % No input affects the evolution of the level of water
```

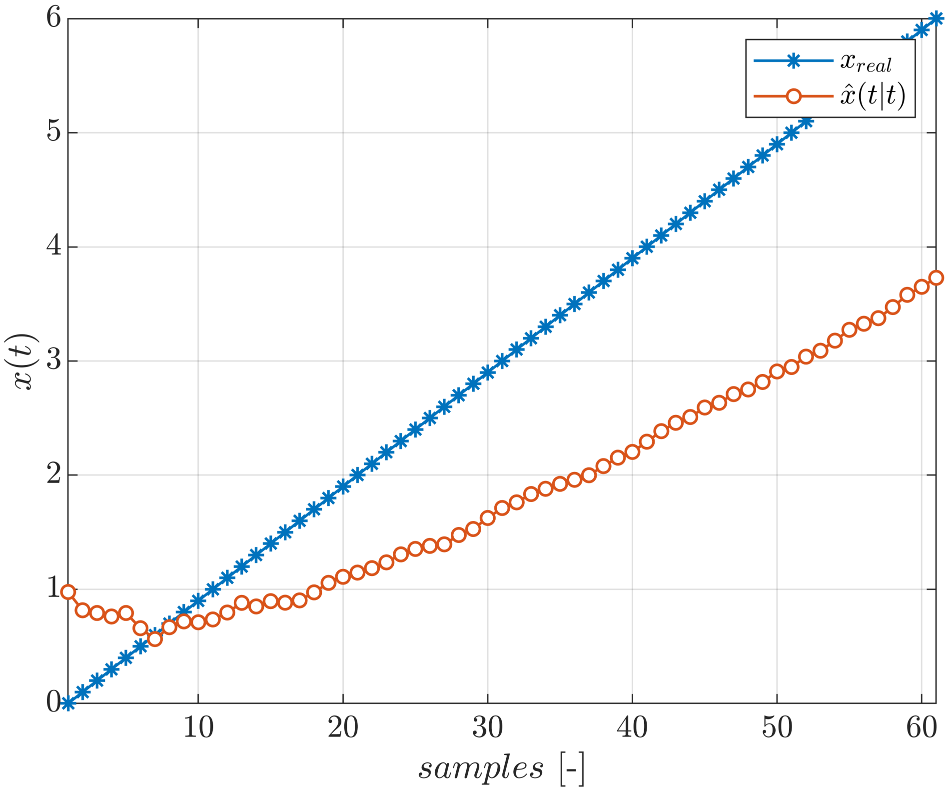
Compare the actual state **(generally not available in practice)** and the measurements

```
% Superimpose the plots of the true state and the measured output
figure(6); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(y_meas, '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$y_{meas}$');
plot(y_true, '-*', 'Linewidth', 1, 'DisplayName', '$y_{real}$');
legend('Interpreter', 'latex');
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$y(t)$', 'Interpreter', 'latex');
title('Increasing water level', 'Interpreter', 'latex');
xlim([1, length(y_true)]); clf;
```



Use again the **static model**

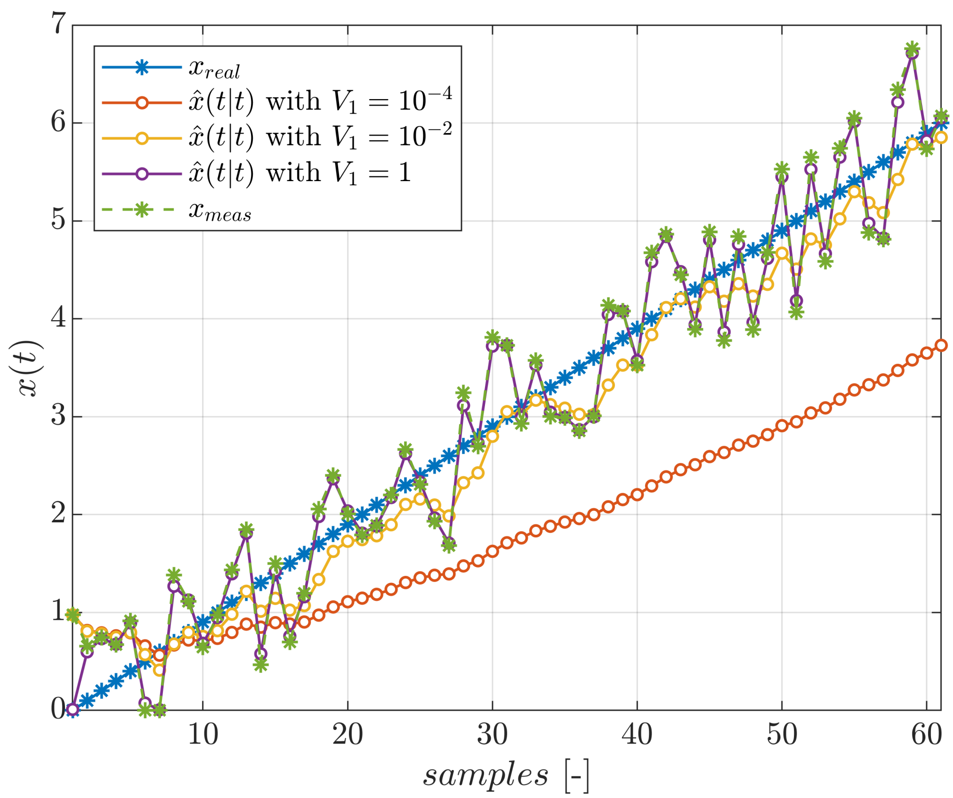
```
x_corr = zeros(length(y_meas), 1);
x_pred = zeros(length(y_meas)+1, 1);
P_corr = zeros(length(y_meas), 1);
P_pred = zeros(length(y_meas)+1, 1);
x_pred(1) = x_predInit;
P_pred(1) = P_predInit;
K = zeros(length(y_meas), 1);

% Kalman filtering steps
for t = 1:length(y_meas)
   % Correction step
   [K(t), x_corr(t), P_corr(t)] = kf_correction(y_meas(t), H, V2, x_pred(t), P_pred(t));
   % Prediction step
   [x_pred(t+1), P_pred(t+1)] = kf_prediction(u(t), F, G, V1, x_corr(t), P_corr(t));
end

figure(7); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(x_true, '-*', 'Linewidth', 1, 'DisplayName', '$x_{real}$');
plot(x_corr, '-o', 'MarkerFaceColor', 'auto', 'MarkerSize', 5,'Linewidth', 1, 'DisplayName', '$
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
xlim([1, length(x_true)]); clf;
```



Check the effect of a progressive **increase** of the **process noise covariance**

```
V1_2 = 1e-2;
V1_3 = 1;

x_corr2 = zeros(length(y_meas), 1);
x_pred2 = zeros(length(y_meas)+1, 1);
P_corr2 = zeros(length(y_meas), 1);
P_pred2 = zeros(length(y_meas)+1, 1);
x_pred2(1) = x_predInit;
P_pred2(1) = P_predInit;
K2 = zeros(length(y_meas), 1);

%Kalman filtering steps
for t = 1:length(y_meas)
   % Correction step
   [K2(t), x_corr2(t), P_corr2(t)] = kf_correction(y_meas(t), H, V2, x_pred2(t), P_pred2(t));
   % Prediction step
   [x_pred2(t+1), P_pred2(t+1)]=kf_prediction(u(t), F, G, V1_2, x_corr2(t), P_corr2(t));
end

%Kalman filtering steps
for t = 1:length(y_meas)
   % Correction step
   [K3(t), x_corr3(t), P_corr3(t)] = kf_correction(y_meas(t), H, V2, x_pred3(t), P_pred3(t));
   % Prediction step
   [x_pred3(t+1), P_pred3(t+1)] = kf_prediction(u(t), F, G, V1_3, x_corr3(t), P_corr3(t));
```

```
end

% Superimpose the plots of the true state and the corrected state
figure(8); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(x_true, '-*', 'Linewidth', 1, 'DisplayName', '$x_{real}$');
plot(x_corr, '-o', 'MarkerFaceColor', 'auto', 'MarkerSize', 4, 'Linewidth', 1, 'DisplayName',
plot(x_corr2, '-o', 'MarkerFaceColor', 'auto', 'MarkerSize', 4, 'Linewidth', 1, 'DisplayName',
plot(x_corr3, '-o', 'MarkerFaceColor', 'auto', 'MarkerSize', 4, 'Linewidth', 1, 'DisplayName',
plot(y_meas,'--*', 'Linewidth', 1, 'DisplayName', '$x_{meas}$');
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
legend('Interpreter', 'latex', 'Location', 'northwest');
xlim([1,length(x_true)]);  clf;
```



The more the process noise covariance is increased, the more the estimate resambles the measurement (I am progressively attributing more trust to the measurements).

In general we can state that:

- if $\frac{V_1}{V_2} \gg 1$ we are trusting the measurement over the state (the correction over the prediction), hence the filter will be faster (but noisier)
- if $\frac{V_1}{V_2} \ll 1$we are trusting the state over the measurement (the prediction over the correction), hence the filter will be slower

Linear model

Define the model $L(t+1) = L(t) + f$

$$x_1(t+1) = x_1(t) + T_s x_2(t) + w_1(t)$$

$$x_2(t+1) = x_2(t) + w_2(t)$$

$$y(t) = x_1(t) + v(t)$$

```
Ts = 1; % Sampling time
F = [1 Ts; 0 1]; % The value of the state at t+1 is equal to that of the state at t
G = zeros(2,1); % No exogenous input affects the level of water
H = [1 0]; % The state is fully measured
V1 = [1e-4/3 1e-4/2; 1e-4/2 1e-4]; % We trust the model (obtained dicretizing [0 0; 0 1e-4])
V2 = 0.1; % We have less trust on the measurements
```

Note that, V2 **does not correspond** to the **true variance** of the measurement noise (equal to 2). Indeed, we assume we do not know it exactly.

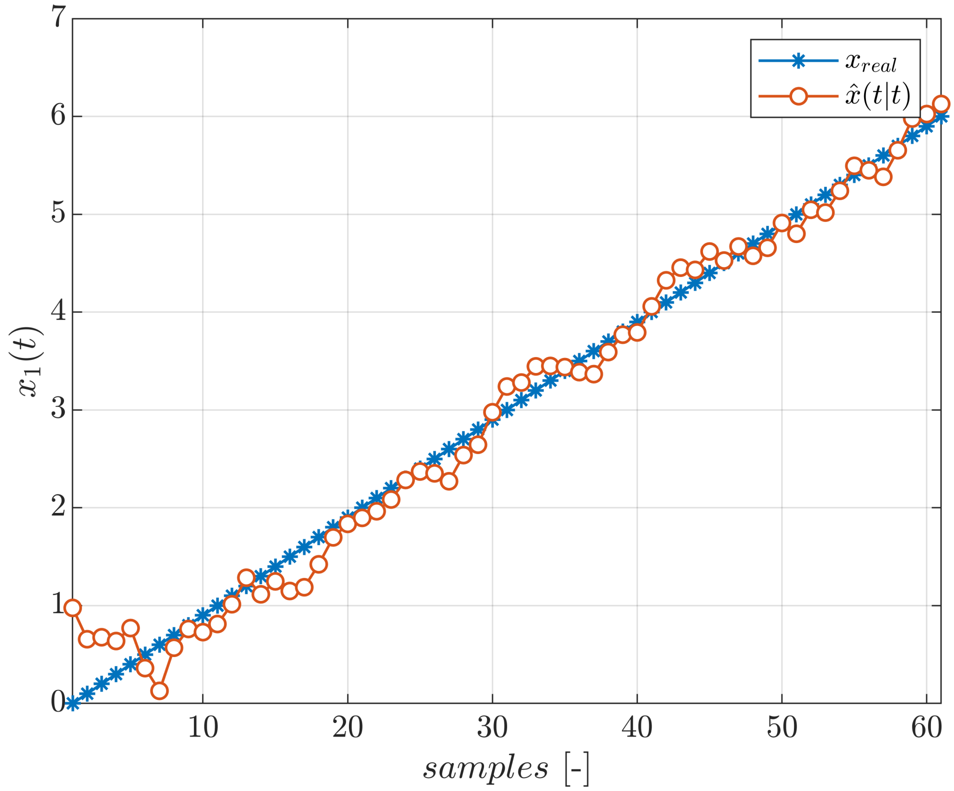
Initialization of all corrected and predicted states, along with the Kalman Gain

```
x_corr_l = zeros(2, length(y_meas));
x_pred_l = zeros(2, length(y_meas)+1);
P_corr_l = zeros(2, 2, length(y_meas));
P_pred_l = zeros(2, 2, length(y_meas)+1);
x_pred_l(:,1) = x_predInit*[1; 1];
P_pred_l(:,:,1) = P_predInit*eye(2);
K_l = zeros(2, length(y_meas));
```

Kalman filtering steps

```
for t = 1:length(y_meas)
   % Correction step
   [K_l(:,t), x_corr_l(:,t), P_corr_l(:,:,t)] = kf_correction(y_meas(t), H, V2, squeeze(x_pred_
   % Prediction step
   [x_pred_l(:,t+1), P_pred_l(:,:,t+1)] = kf_prediction(u(t), F, G, V1, squeeze(x_corr_l(:,t)),
end
```

```
% Superimpose the plots of the true state and the corrected state
figure(9); hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(x_true, '-*', 'Linewidth', 1, 'DisplayName', '$x_{real}$');
plot(x_corr_l(1,:), '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$\hat{x}(t
xlabel('$samples$ [-]', 'Interpreter', 'latex');
ylabel('$x_1(t)$', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
xlim([1, length(x_true)]); clf;
```

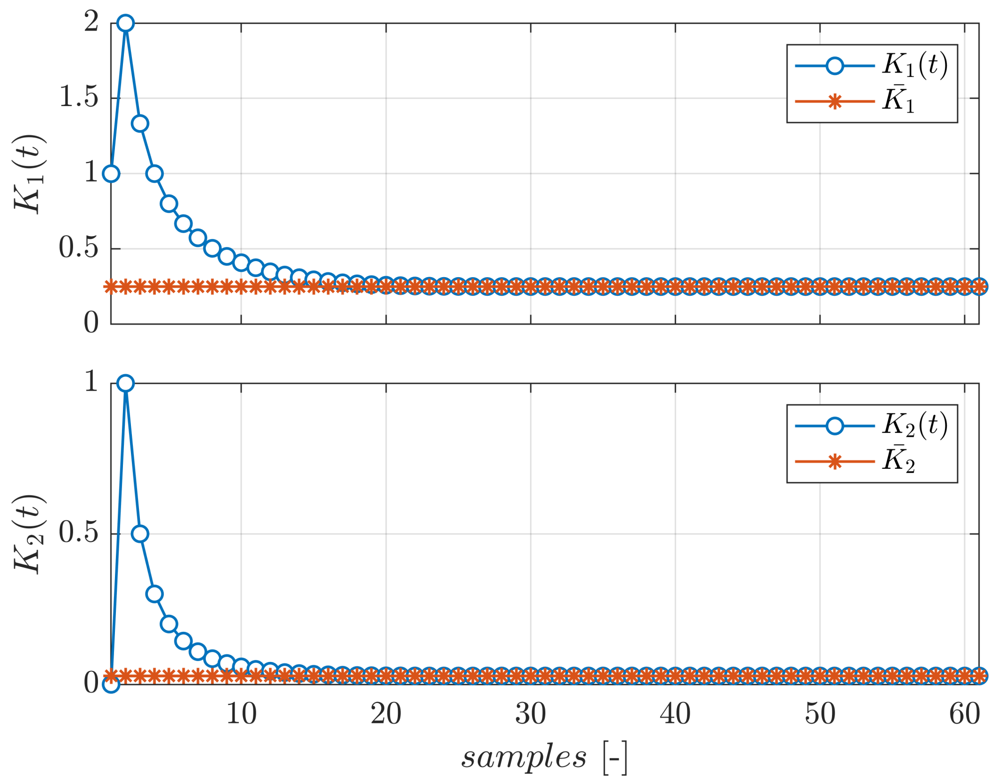


Obtain the asymptotic Kalman filter and check the convergence to it.

```
tank_model = ss(F, [G diag([1 1])], H, [0 0 0], -1); % Define the tank model
[kalmf, K_bar, P_bar] = kalman(tank_model, V1, V2, 0); % Find the Kalman filter and compute its
```

```
% Check the convergence of the time-varying Kalman gain to the asymptotic
% value
K_pred_l = F*K_l;

% First component
figure(10);
tiledlayout(2, 1, 'TileSpacing', 'compact');
nexttile; hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(K_pred_l(1,:), '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$K_1(t)$')
plot(K_bar(1)*ones(length(y_meas),1), '-*', 'Linewidth', 1, 'DisplayName', '$\bar{K_1}$');
ylabel('$K_1(t)$', 'Interpreter', 'latex');
xticklabels('');
legend('Interpreter', 'latex');
xlim([1 length(y_meas)]);

% Second component
nexttile; hold on; grid on; box on;
set(gca, 'FontSize', 12);
plot(K_pred_l(2,:), '-o', 'MarkerFaceColor', 'auto', 'Linewidth', 1, 'DisplayName', '$K_2(t)$')
plot(K_bar(2)*ones(length(y_meas),1), '-*', 'Linewidth', 1, 'DisplayName', '$\bar{K_2}$');
```

```
ylabel('$K_2(t)$', 'Interpreter', 'latex');
xlabel('$samples$ [-]', 'Interpreter', 'latex');
legend('Interpreter', 'latex');
xlim([1 length(y_meas)]); clf;
```



Utils

Define the functions embedding the main steps of the Kalman filtering

It can be proven that if $V_{12} = 0$, Kalman's Filter and Predictor can be splitted as follows:

1. Prediction step:

$$\hat{x}(t+1|\,t) = F\,\hat{x}(t|\,t) + G\,u(t)$$
$$P(t+1) = F\,P(t|\,t)F^T + Q(t)$$

1. Correction step:

$$K(t) = P(t|\,t-1)H^T\left(HP(t|\,t-1)H^T + V_2\right)^{-1}$$
$$P(t|\,t) = (I - K(t)H)\,P(t|\,t-1)$$
$$\hat{x}(t|\,t) = \hat{x}(t|\,t-1) + K(t)\,(y(t) - H\hat{x}(t|\,t-1))$$

```
function [x_pred, P_pred] = kf_prediction(u, F, G, V1, x_corr, P_corr)
```

```
% Prediction step of the Kalman filter. Given the corrected state x_corr and
% its variance, both are predicted based on the system's model.
% Inputs:
%   - u: current input [mx1]
%   - F: state transition matrix [nxn]
%   - G: input-to-state matrix [nxm]
%   - V1: covariance of the process noise [nxn]
%   - x_corr: corrected state at the previous time step [nx1]
%   - P_corr: corrected covaraince of the state at the pevious time step [nxn]
% Outputs:
%   - x_pred: predicted state estimate
%   - P_pred: predicted variance of the state

    x_pred = F*x_corr + G*u;
    P_pred = F*P_corr*F' + V1;
end

function [K, x_corr, P_corr] = kf_correction(y_meas, H, V2, x_pred, P_pred)
% Correction step of the Kalman filter. Given the predicted state x_pred and
% its variance, both are corrected based on the measurements.
% Inputs:
%   - y_meas: current measured output [px1]
%   - H: state-to-output matrix [pxn]
%   - V2: covariance of the measurement noise [pxp]
%   - x_pred: predicted state at the previous time step [nx1]
%   - P_pred: predicted covaraince of the state at the pevious time step [nxn]
% Outputs:
%   - K: updated Kalman filter gain
%   - x_corr: corrected state estimate
%   - P_corr: corrected covariance of the state

    n = length(x_pred); % Obtain the dimension of the state
    K = (P_pred*H')/(H*P_pred*H' + V2); % Update the Kalman filter gain
    P_corr = (eye(n) - K*H)*P_pred; % Correct the state variance based on the new Kalman gain.
    x_corr = x_pred + K*(y_meas - H*x_pred); % Correct the state estimate based on the current
end
```