Photogrammetry & Robotics Lab

Projective 3-Point (P3P) Algorithm / Spatial Resection

Cyrill Stachniss

Camera Localization

Given known 3D control points (X, Y, Z)

Task: estimate the pose of the camera

Camera Localization

Given:

- 3D coordinates of object points $\mathbf{X}_i$

Observed:

- 2D image coordinates $\mathbf{x}_i$ of the object points

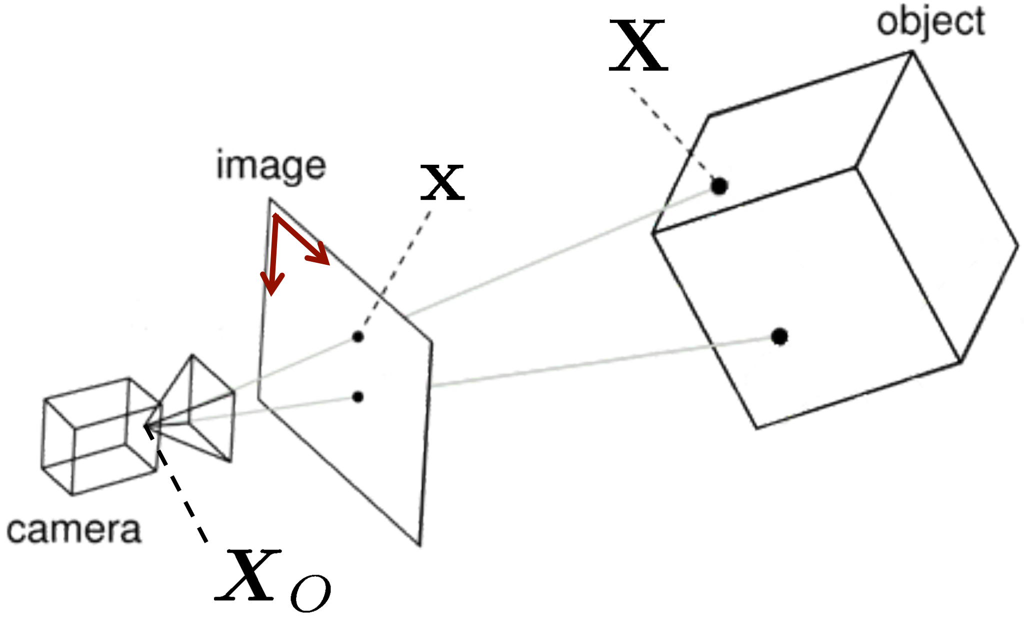
Wanted:

- Extrinsic parameters $R, \boldsymbol{X}_O$ of the calibrated camera

Reminder: Mapping Model

Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$

$$
\begin{aligned}
\mathbf{x} &= \mathsf{K}R[I_3| - \boldsymbol{X}_O]\mathbf{X} \\
&= \mathsf{P}\ \mathbf{X}
\end{aligned}
$$



Reminder: Camera Orientation

$$\mathbf{x} = \mathsf{K}R[I_3| - \boldsymbol{X}_O]\mathbf{X} = \mathsf{P}\ \mathbf{X}$$

- **Intrinsics (interior orientation)**
  - Intrinsic parameters of the camera
  - Given through matrix K
- **Extrinsics (exterior orientation)**
  - Extrinsic parameters of the camera
  - Given through $\boldsymbol{X}_O$ and R

Direct Linear Transform (DLT)

Relation to DLT : Compute the **11 intrinsic and extrinsic parameters**

$$\mathbf{x} = \mathsf{K}R[I_3| - \boldsymbol{X}_O]\mathbf{X}$$

observed image point

c, s, m, x_H, y_H

3 rotations

3 translations

control point coordinates (given)

Projective 3-Point Algorithm (or Spatial Resection)

Given the intrinsic parameters, compute the **6 extrinisic parameters**

$$\mathbf{x} = \mathsf{K} R [I_3 | - \boldsymbol{X}_O] \mathbf{X}$$

observed image points

c, s, m, x_H, y_H (given)

3 rotations

3 translations

control point coordinates (given)

P3P/SR vs. DLT

- **P3P/SR: Calibrated camera**
  - 6 unknowns
  - We need at least **3 points**
- **DLT: Uncalibrated camera**
  - 11 unknowns
  - We need at least **6 points**
  - Assuming an **affine camera** (straight-line preserving projection)

Orienting a calibrated camera by using ≥ 3 points

P3P/Spatial Resection (direct solution)

Problem Formulation

Given:

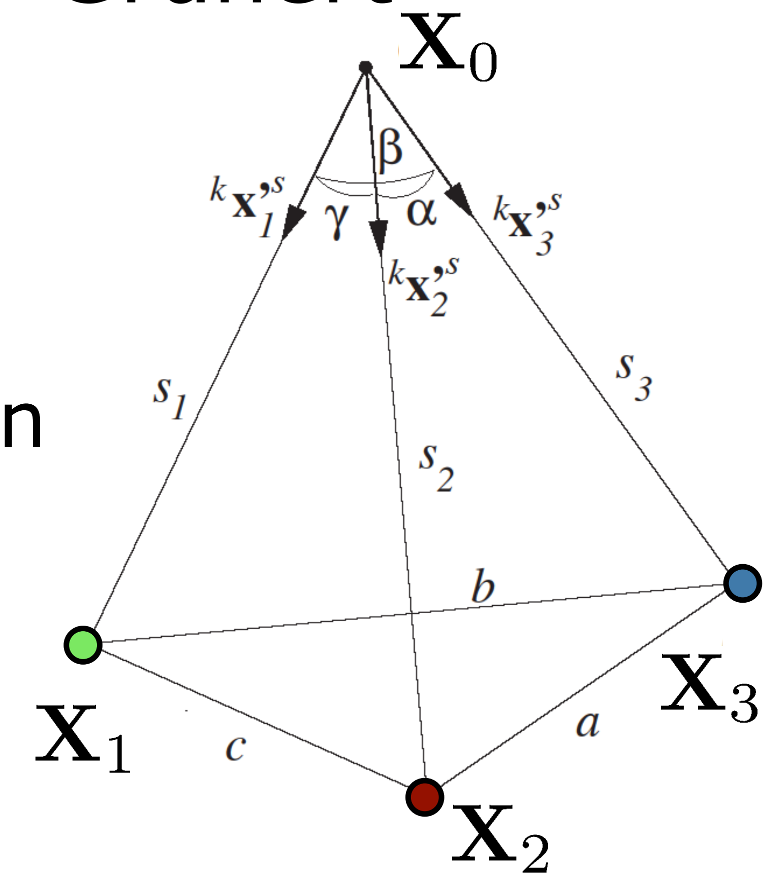
- 3D coordinates $\mathbf{X}_i$ of $I \geq 3$ object points
- Corresponding image coordinates $\mathbf{x}_i$ recorded using a calibrated camera

Task:

- Estimate the 6 parameters $\boldsymbol{X}_O, R$
- Direct solution (no initial guess)

Different Approaches

- Different approaches: Grunert 1841, Killian 1955, Rohrberg 2009, ...
- Here: direct solution by Grunert
- **2-step process**
  1. Estimate length of projection rays
  2. Estimate the orientation

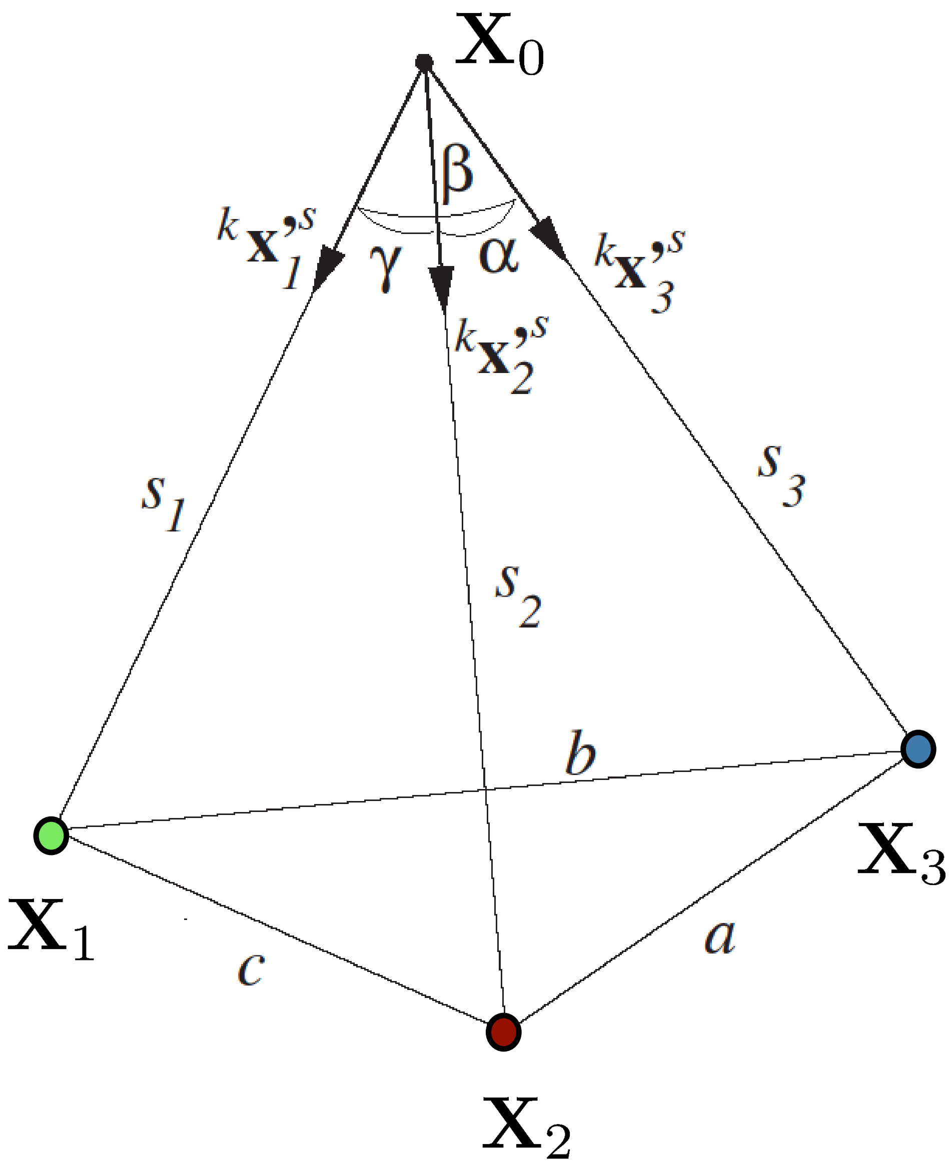


Direct Solution by Grunert

2-Step process

Estimate

1. length of projection rays
2. orientation

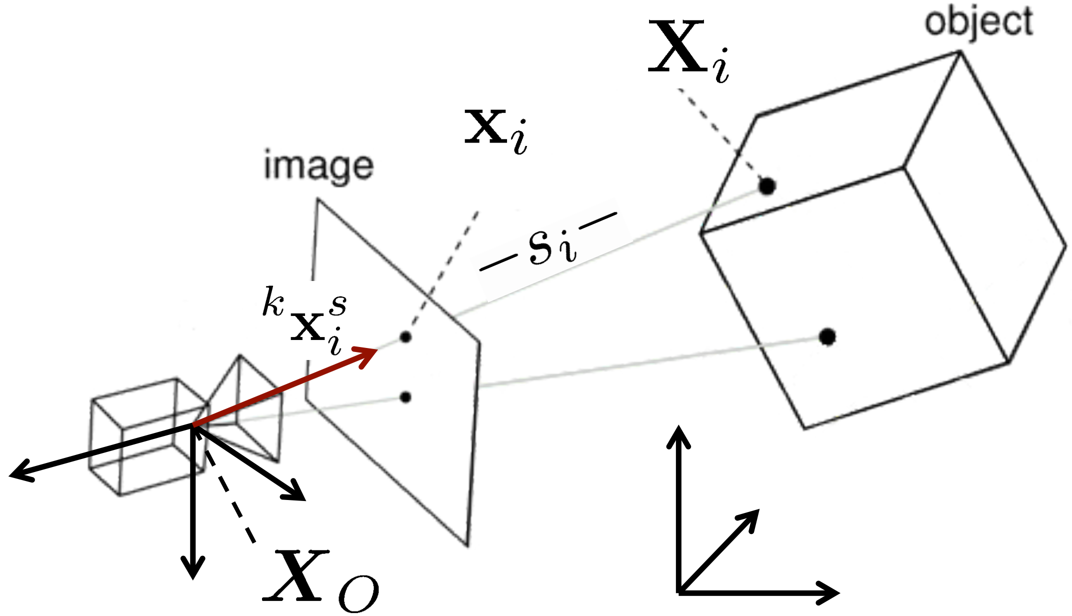


P3P/SR Model

- Coordinates of object points **within the camera system** are given by

$$s_i \; {}^k\mathbf{x}^s_i = \mathsf{R}(\boldsymbol{X}_i - \boldsymbol{X}_O) \qquad\qquad i = 1, 2, 3$$

ray directions pointing to the object points



P3P/SR Model

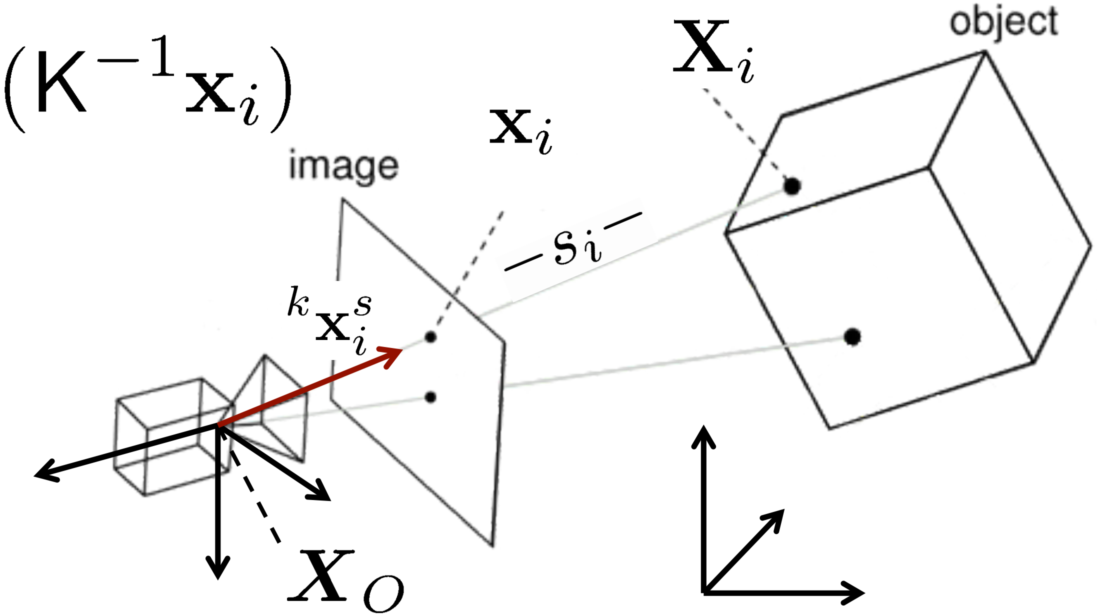
- Coordinates of object points within the camera system are given by

$$s_i \, {}^k\mathbf{x}_i^s = \mathsf{R}(\boldsymbol{X}_i - \boldsymbol{X}_O) \qquad\qquad i = 1, 2, 3$$

- From image coordinates, we obtain the directional vector of projection ray

$$ {}^k\mathbf{x}_i^s = -\mathrm{sign}(c)\mathsf{N}(\mathsf{K}^{-1}\mathbf{x}_i)$$

ensure ray directions are pointing to the object points



P3P/SR Model

- Coordinates of object points within the camera system are given by

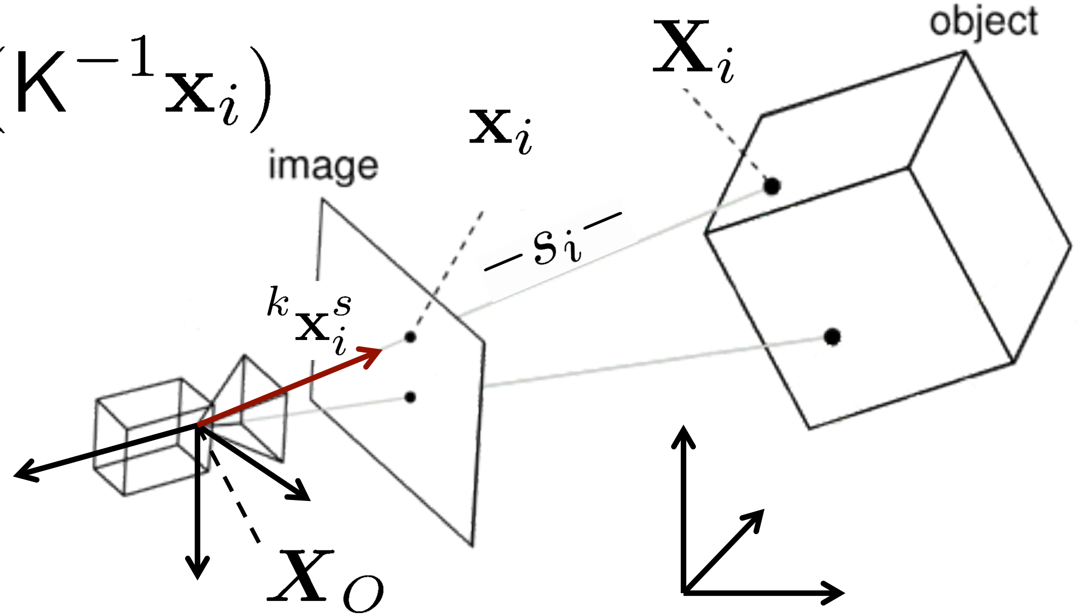
$$s_i \, {}^k\mathbf{x}_i^s = \mathsf{R}(\boldsymbol{X}_i - \boldsymbol{X}_O) \qquad\qquad i = 1, 2, 3$$

- From image coordinates, we obtain the directional vector of projection ray

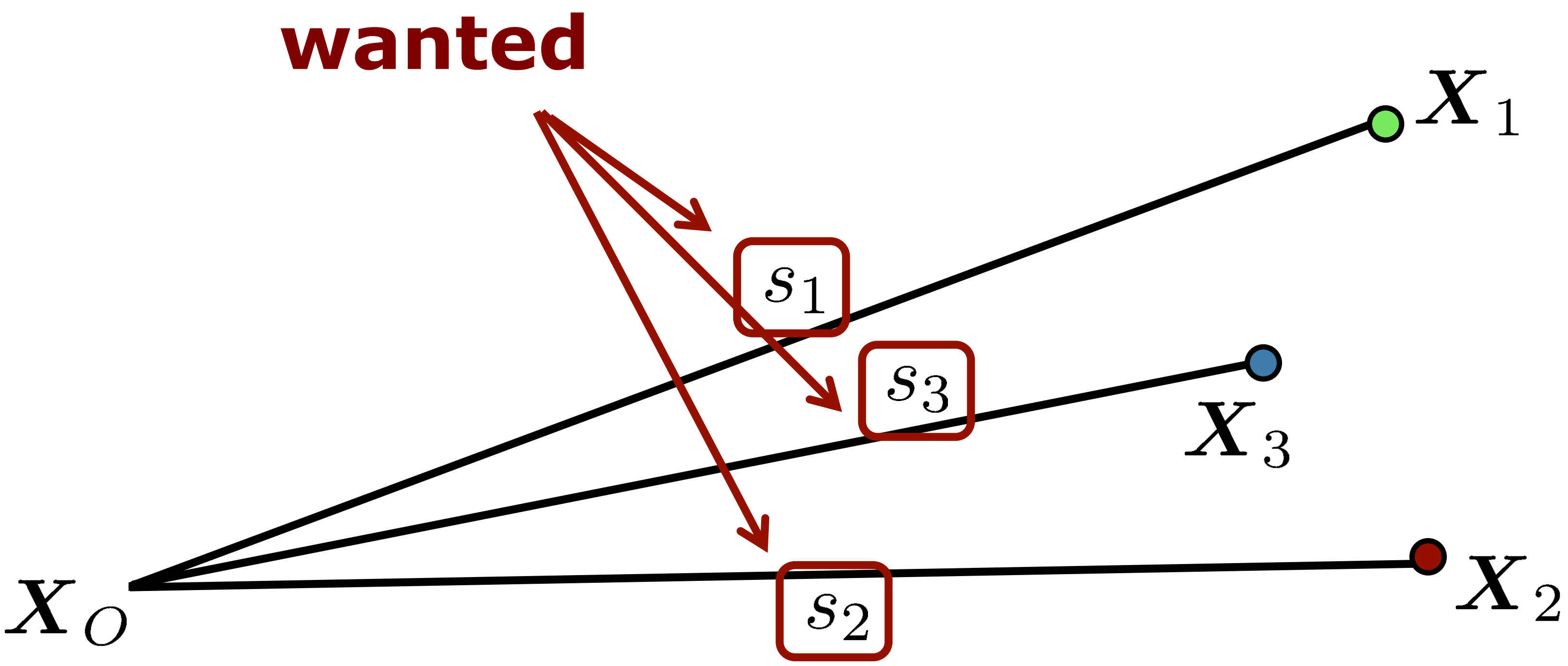
$${}^k\mathbf{x}_i^s = -\mathrm{sign}(c)\mathsf{N}(\mathsf{K}^{-1}\mathbf{x}_i)$$

spherical normalization

$$\mathsf{N}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$$



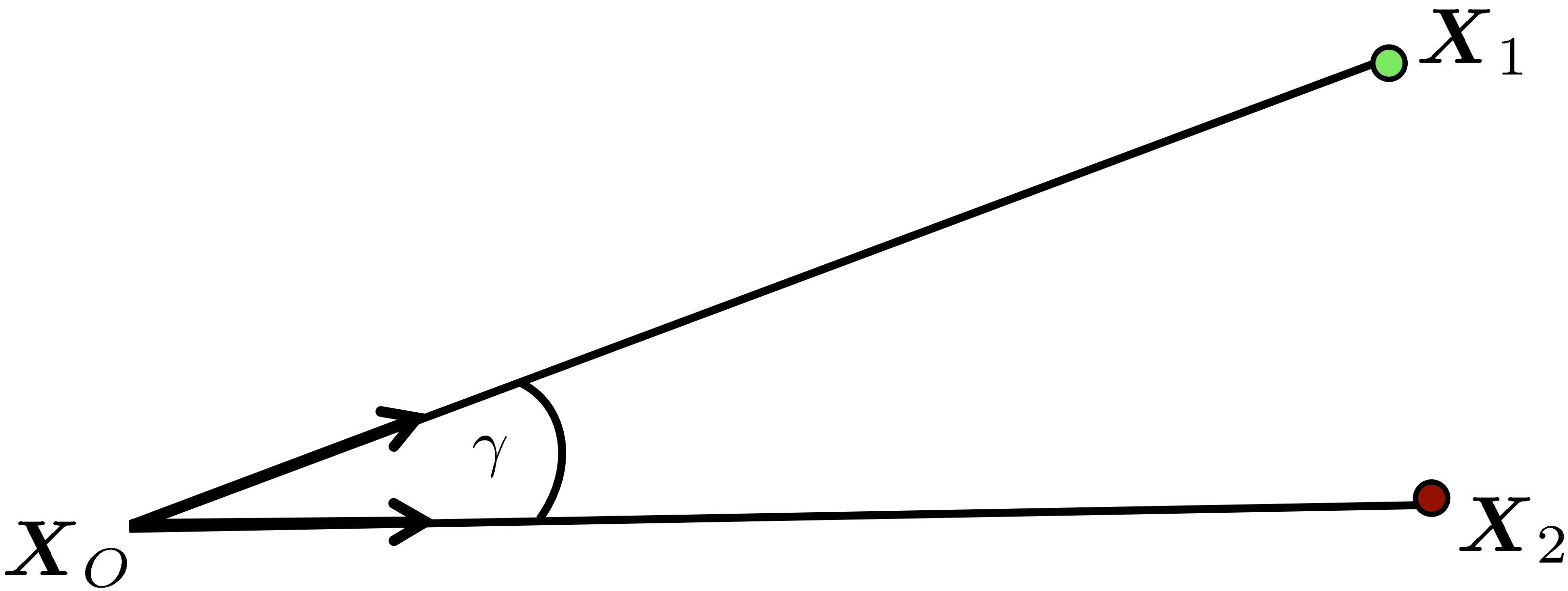
1. Get Length of Projection Rays



1. Get Length of Projection Rays

- Start with computing the angle between rays:

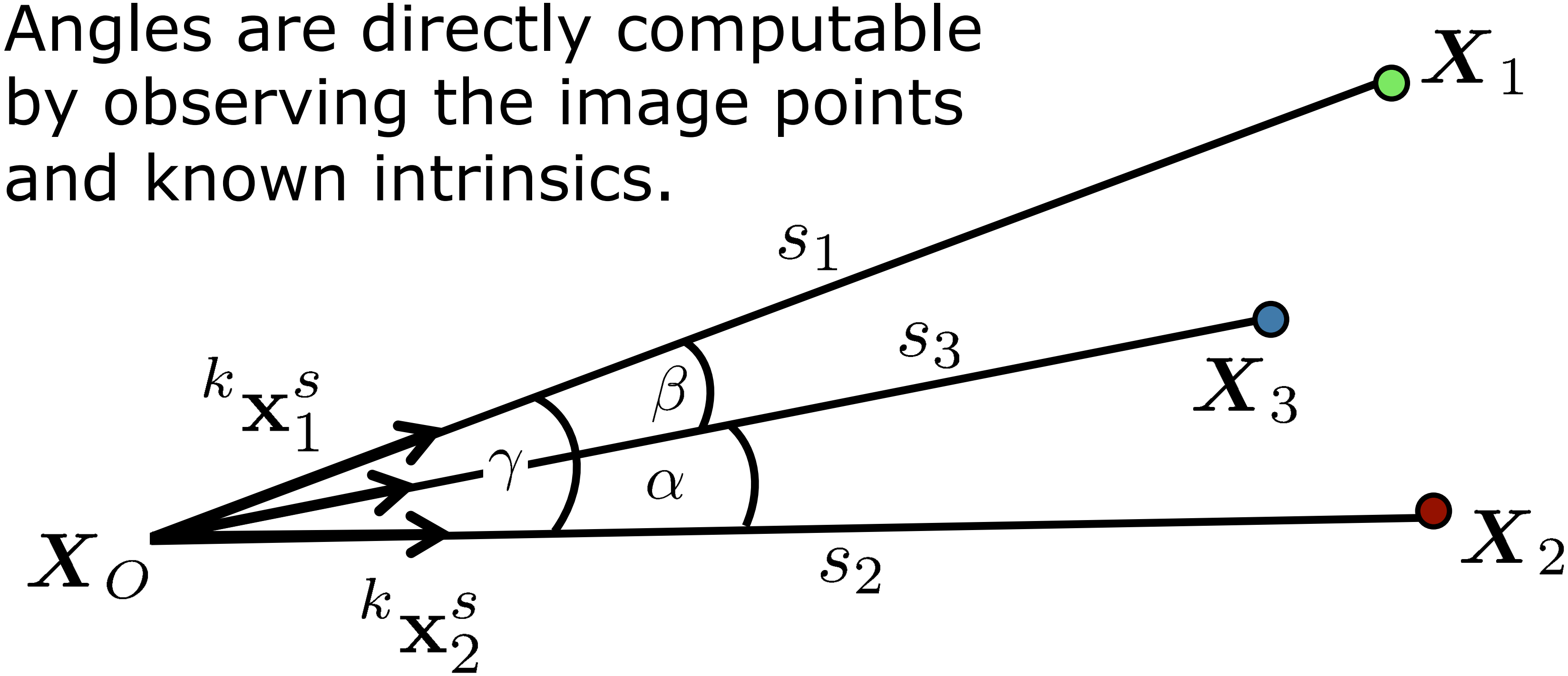
$$\cos\gamma = \frac{(\boldsymbol{X}_1 - \boldsymbol{X}_0) \cdot (\boldsymbol{X}_2 - \boldsymbol{X}_0)}{||\boldsymbol{X}_1 - \boldsymbol{X}_0||\,||\boldsymbol{X}_2 - \boldsymbol{X}_0||}$$



1. Get Length of Projection Rays

$$\begin{aligned}\alpha &= \arccos\left({}^k\mathbf{x}_2^s,\ {}^k\mathbf{x}_3^s\right)\\ \beta &= \arccos\left({}^k\mathbf{x}_3^s,\ {}^k\mathbf{x}_1^s\right)\\ \gamma &= \arccos\left({}^k\mathbf{x}_1^s,\ {}^k\mathbf{x}_2^s\right)\end{aligned}$$

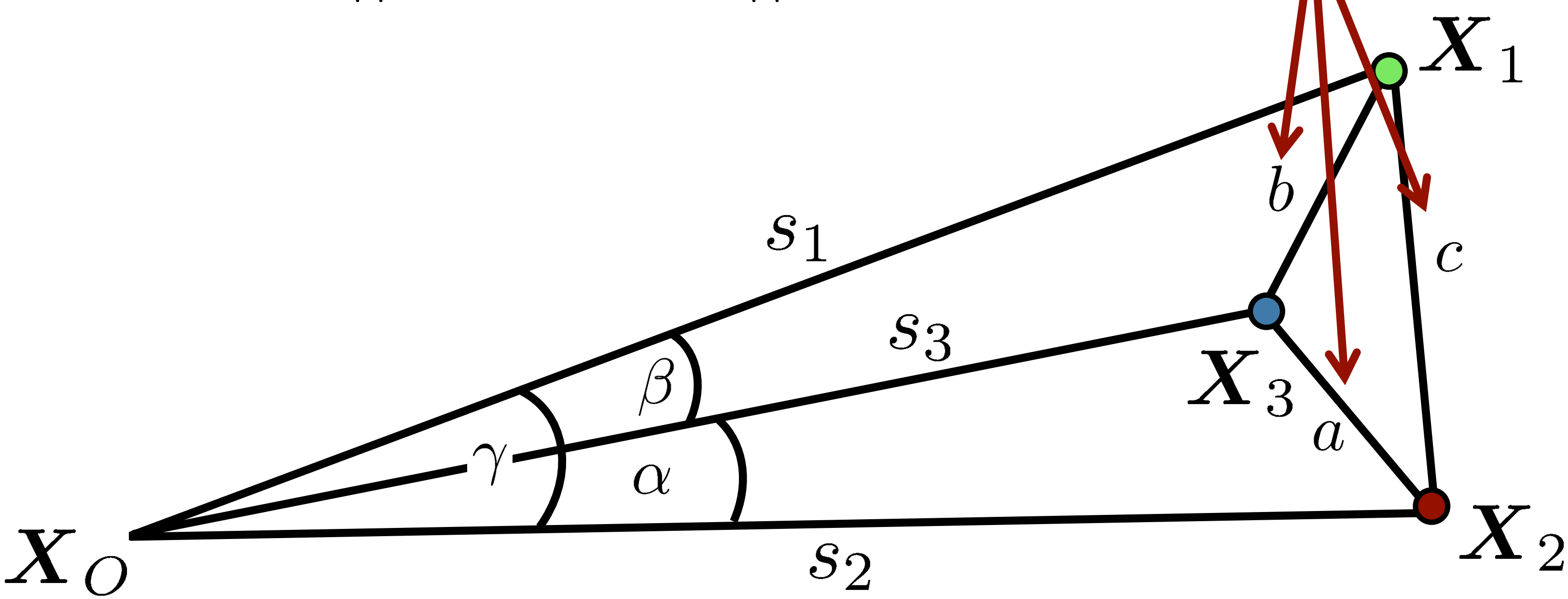
Angles are directly computable by observing the image points and known intrinsics.



1. Get Length of Projection Rays

$$\begin{aligned} a &= \|\boldsymbol{X}_3 - \boldsymbol{X}_2\| \\ b &= \|\boldsymbol{X}_1 - \boldsymbol{X}_3\| \\ c &= \|\boldsymbol{X}_2 - \boldsymbol{X}_1\| \end{aligned}$$

Given through known control point coordinates

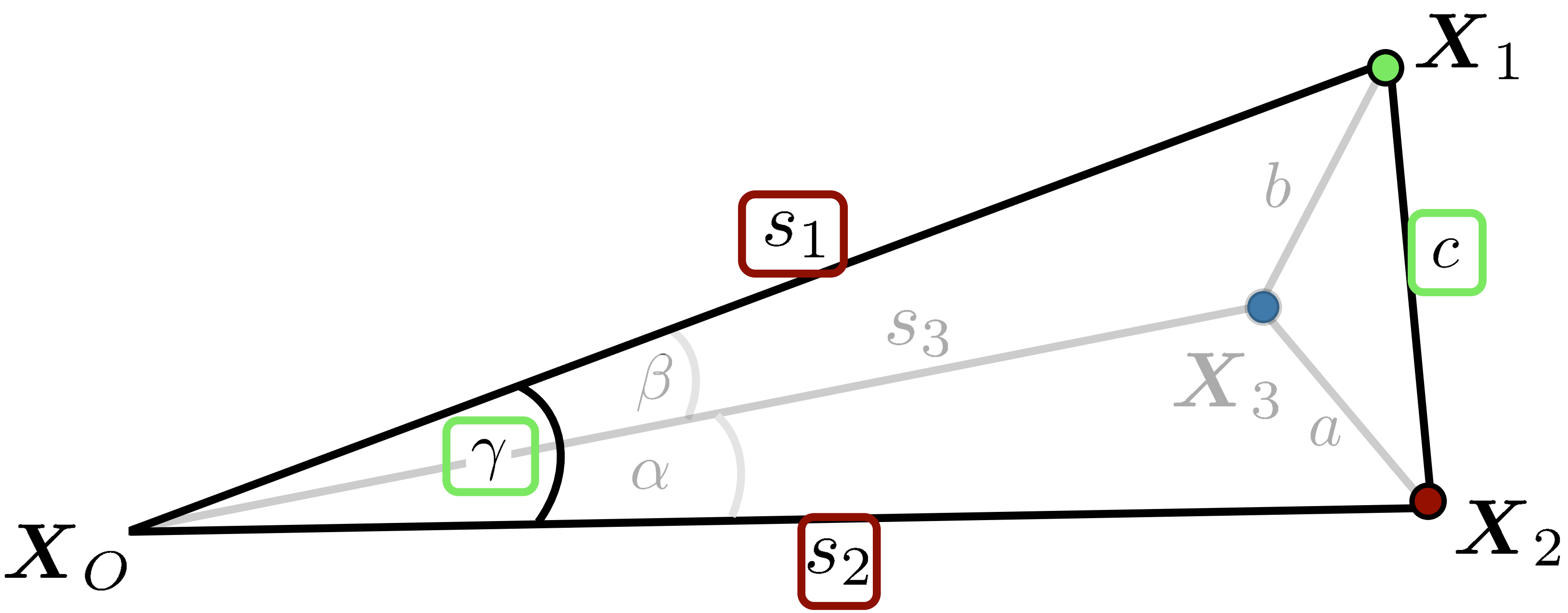


Use the Law of Cosines

In triangle $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2$

$$s_1^2 + s_2^2 - 2\boxed{s_1}\,\boxed{s_2}\cos\boxed{\gamma} = \boxed{c^2}$$

wanted known



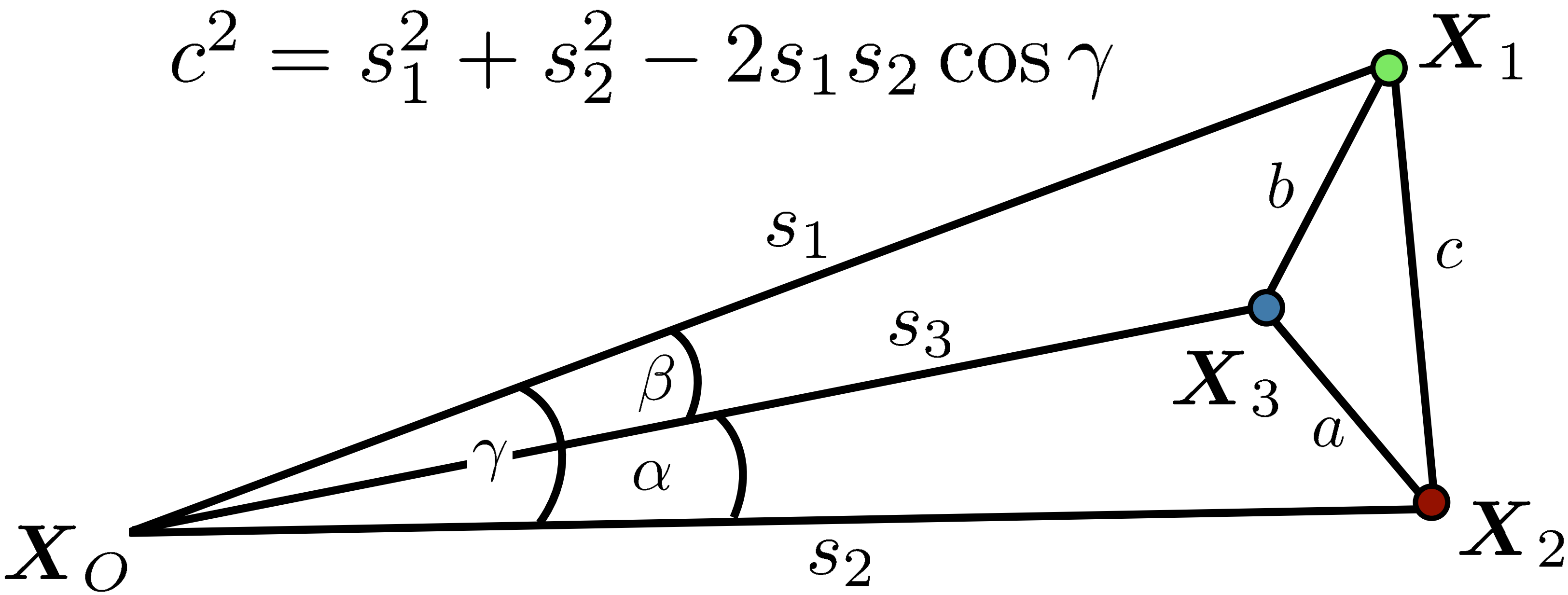
Use the Law of Cosines

Analogously in all three triangles

$$a^2 = s_2^2 + s_3^2 - 2 s_2 s_3 \cos\alpha$$

$$b^2 = s_1^2 + s_3^2 - 2 s_1 s_3 \cos\beta$$

$$c^2 = s_1^2 + s_2^2 - 2 s_1 s_2 \cos\gamma$$



Compute Distances

- We start from:

$$a^2 = s_2^2 + s_3^2 - 2s_2 s_3 \cos\alpha$$

- Define: $u = \frac{s_2}{s_1}$ $\quad v = \frac{s_3}{s_1}$

- Substitution leads to:

$$a^2 = s_1^2(u^2 + v^2 - 2uv\cos\alpha)$$

- Rearrange to: $s_1^2 = \frac{a^2}{u^2+v^2-2uv\cos\alpha}$

Compute Distances

- Use the same definition

$$u = \frac{s_2}{s_1} \qquad v = \frac{s_3}{s_1}$$

- And perform the substitution again for:

$$b^2 = s_1^2 + s_3^2 - 2s_1 s_3 \cos\beta$$

$$c^2 = s_1^2 + s_2^2 - 2s_1 s_2 \cos\gamma$$

Compute Distances

Analogously, we obtain

$$
\begin{aligned}
s_1^2 &= \frac{a^2}{u^2 + v^2 - 2uv\cos\alpha} \\
&= \frac{b^2}{1 + v^2 - 2v\cos\beta} \\
&= \frac{c^2}{1 + u^2 - 2u\cos\gamma}
\end{aligned}
$$

Rearrange Again

Solve one equation for u put into the other

$$s_1^2 = \frac{a^2}{u^2+v^2-2uv\cos\alpha}$$

$$s_1^2 = \frac{b^2}{1+v^2-2v\cos\beta}$$

$$s_1^2 = \frac{c^2}{1+u^2-2u\cos\gamma}$$

Results in a fourth degree polynomial

$$A_4v^4 + A_3v^3 + A_2v^2 + A_1v + A_0 = 0$$

Forth Degree Polynomial

$$\boxed{A_4} v^4 + \boxed{A_3} v^3 + A_2 v^2 + A_1 v + A_0 = 0$$

$$\boxed{A_4} = \left(\frac{a^2 - c^2}{b^2} - 1\right)^2 - \frac{4c^2}{b^2} \cos^2 \alpha$$

$$\begin{aligned} \boxed{A_3} \quad &= \quad 4\left[\frac{a^2 - c^2}{b^2}\left(1 - \frac{a^2 - c^2}{b^2}\right) \cos\beta \right. \\ &\left. - \left(1 - \frac{a^2 + c^2}{b^2}\right) \cos\alpha \cos\gamma + 2\frac{c^2}{b^2} \cos^2\alpha \cos\beta \right] \end{aligned}$$

Forth Degree Polynomial

$$A_4 v^4 + A_3 v^3 + \boxed{A_2} v^2 + A_1 v + A_0 = 0$$

$$\begin{aligned}
\boxed{A_2} \quad = \quad & 2\left[\left(\frac{a^2 - c^2}{b^2}\right)^2 - 1 + 2\left(\frac{a^2 - c^2}{b^2}\right)^2 \cos^2\beta\right. \\
& +2\left(\frac{b^2 - c^2}{b^2}\right)\cos^2\alpha \\
& -4\left(\frac{a^2 + c^2}{b^2}\right)\cos\alpha\cos\beta\cos\gamma \\
& \left.+2\left(\frac{b^2 - a^2}{b^2}\right)\cos^2\gamma\right]
\end{aligned}$$

Forth Degree Polynomial

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + \boxed{A_1} v + \boxed{A_0} = 0$$

$$\begin{aligned}\boxed{A_1} \quad = \quad & 4\left[-\left(\frac{a^2 - c^2}{b^2}\right)\left(1 + \frac{a^2 - c^2}{b^2}\right)\cos\beta \right. \\ & + \frac{2a^2}{b^2}\cos^2\gamma\cos\beta \\ & \left. - \left(1 - \left(\frac{a^2 + c^2}{b^2}\right)\right)\cos\alpha\cos\gamma \right]\end{aligned}$$

$$\boxed{A_0} = \left(1 + \frac{a^2 - c^2}{b^2}\right)^2 - \frac{4a^2}{b^2}\cos^2\gamma$$

Forth Degree Polynomial

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$$

Solve for v to get s_1, s_2, s_3 through:

$$s_1^2 = \frac{b^2}{1+v^2-2v\cos\beta}$$

$$s_3 = v\ s_1$$

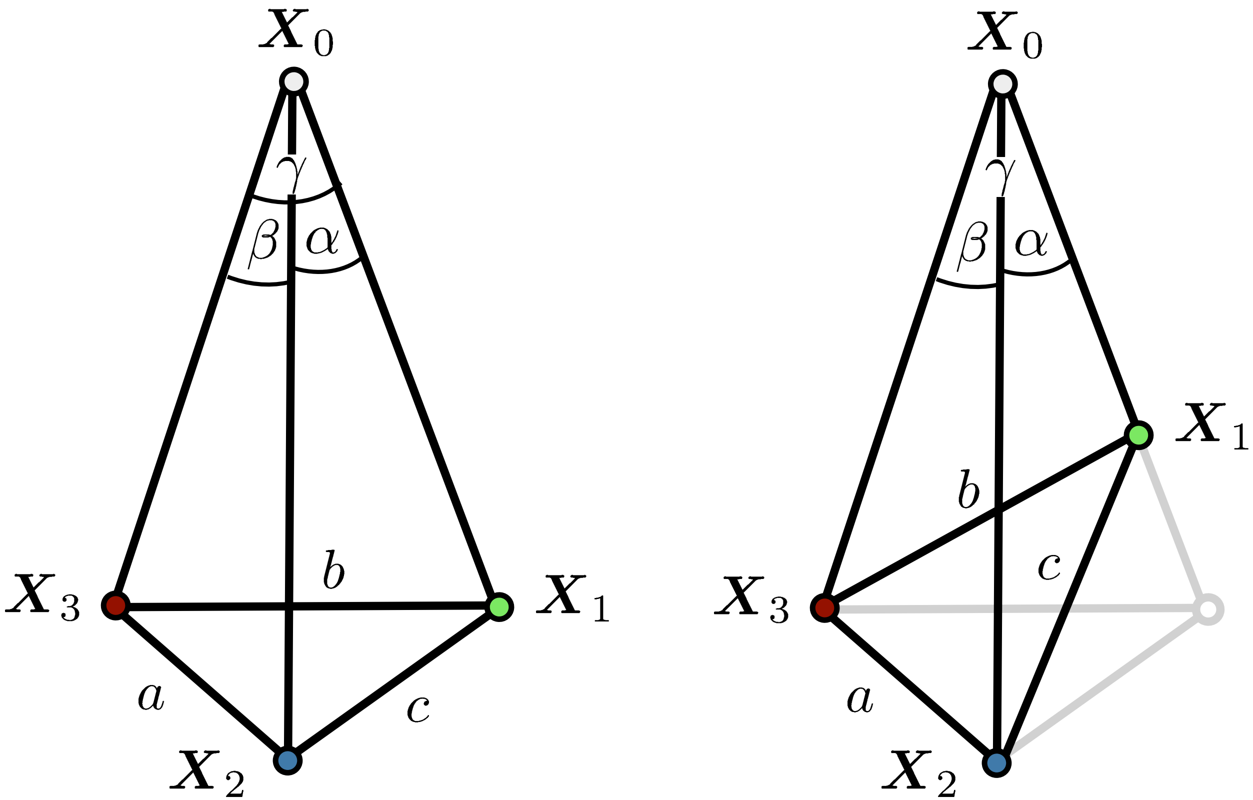
$$a^2 = s_2^2 + s_3^2 - 2 s_2 s_3 \cos\alpha \Rightarrow s_2 = \ \cdots$$

Problem:
up to 4 possible solutions !

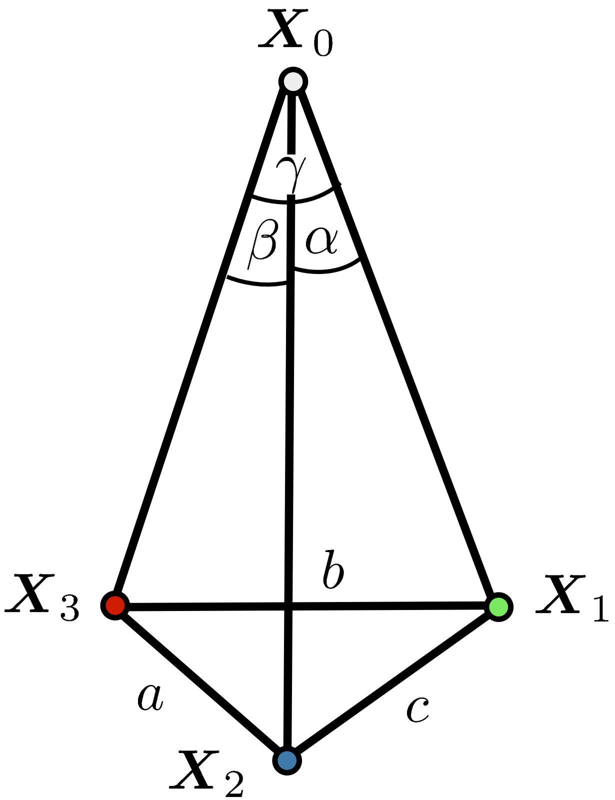
$$\{s_1, s_2, s_3\}_{1\ldots 4}$$

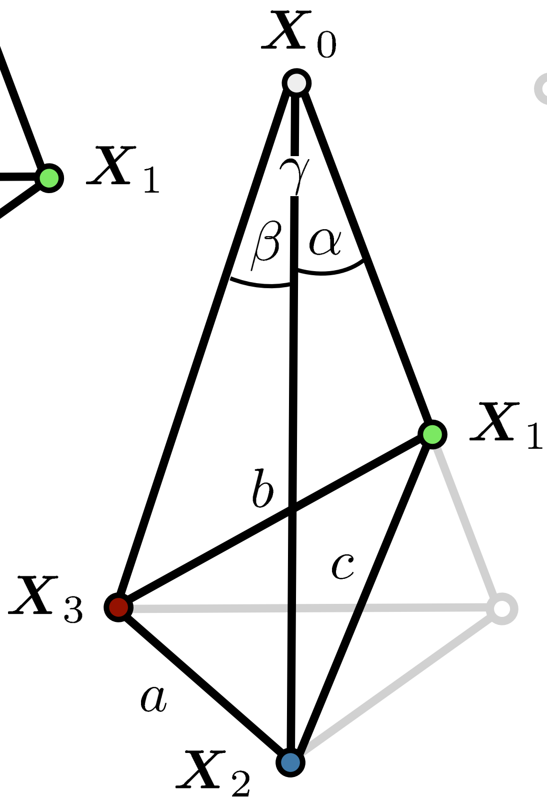
Example for Multiple Solutions

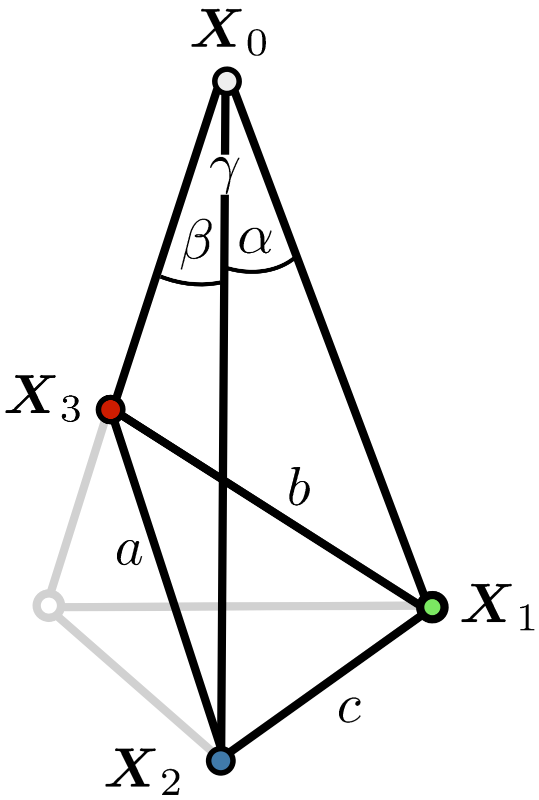
- Assume $a = b = c$ and $\alpha = \beta = \gamma$
- Tilting the triangle $(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)$ has no effect on (a, b, c) and (α, β, γ)

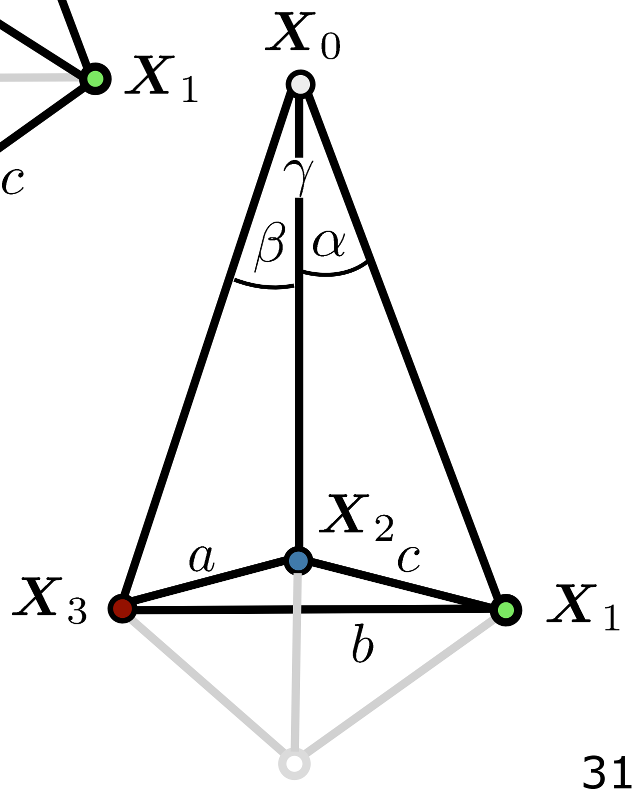


Four Solutions









How to Eliminate This Ambiguity?

- Known approximate solution (e.g., from GPS) or
- Use 4th points to confirm identify the correct solution

Unique solution for

$$s_1, s_2, s_3$$

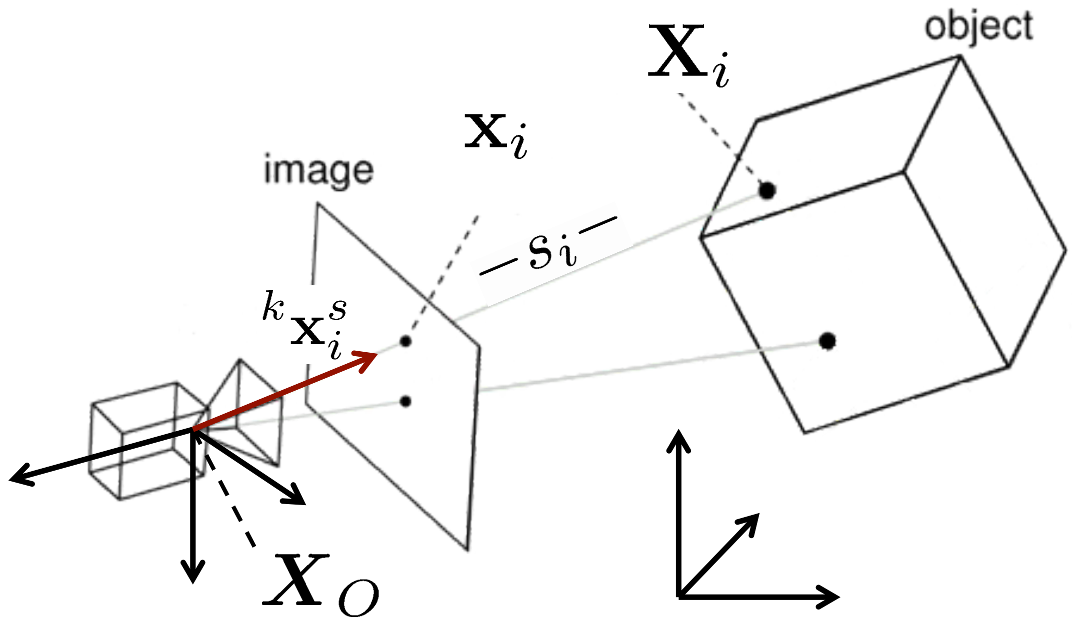
2. Orientation of the Camera

Given:

- Distances and direction vectors to the control points

Task:

- Estimate 6 extrinsic parameters

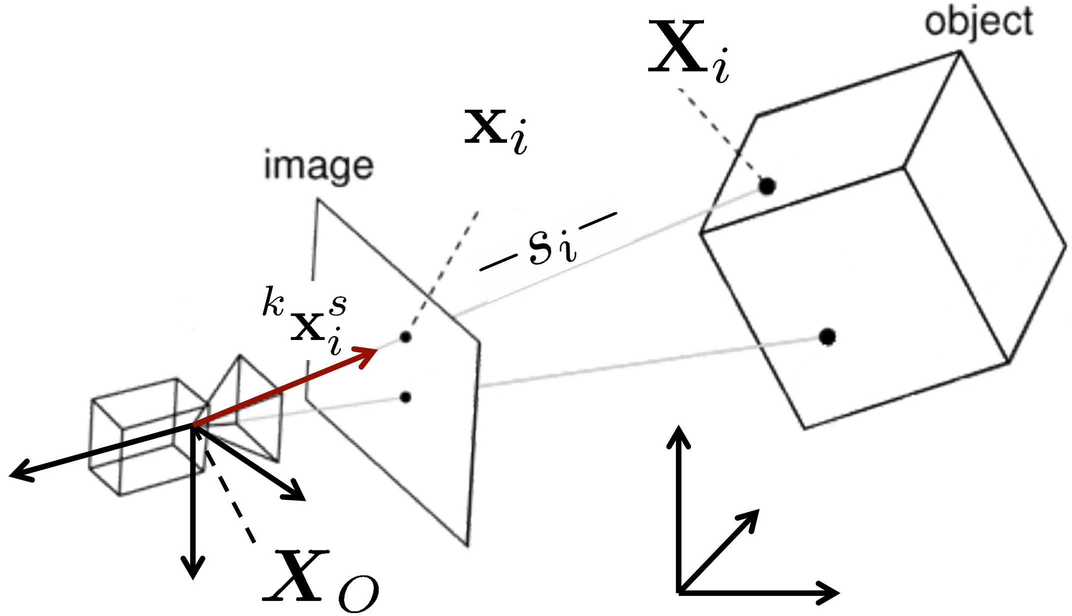


2. Orientation of the Camera

1. Compute 3D coordinates of the control points in the camera system

$$ {}^k\boldsymbol{X}_i = s_i \; {}^k\mathbf{x}_i^s \qquad\qquad i = 1, 2, 3 $$

That's what we just discussed!



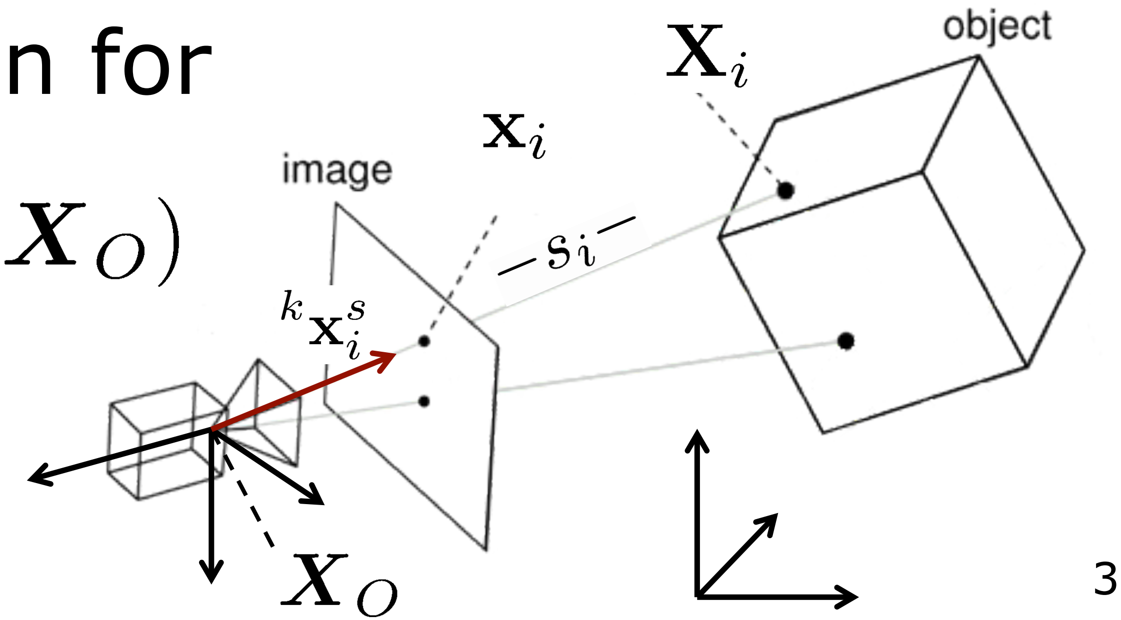
2. Orientation of the Camera

1. Compute 3D coordinates of the control points in the camera system

$$ {}^k\boldsymbol{X}_i = s_i \, {}^k\mathbf{x}_i^s \qquad i = 1, 2, 3 $$

2. Compute coordinate transformation for

$$ {}^k\boldsymbol{X}_i = R(\boldsymbol{X}_i - \boldsymbol{X}_O) $$



P3P/SR in a Nutshell

$\mathbf{X}_i, \mathbf{x}_i, \mathsf{K} \qquad i = 1, 2, 3, 4$

$\downarrow$

Compute angles

$\downarrow \; \alpha, \beta, \gamma$

Computes distances

$\downarrow \; \{s_1, s_2, s_3\}_{1 \ldots 4}$

Identify correct solution through 4th point

$\downarrow \; s_1, s_2, s_3$

Compute coordinate transformation

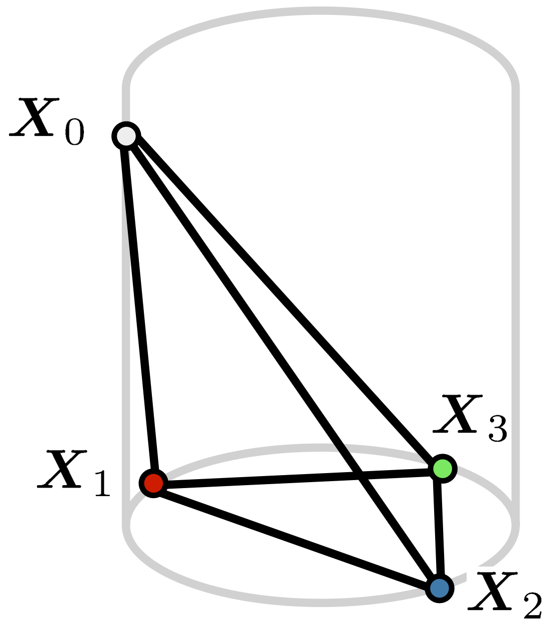
$\downarrow$

$R, \boldsymbol{X}_0$

Critical Surfaces

"Critical cylinder"

- If the projection center lies on a cylinder defined by the control points
- Small changes in angles lead to large changes in coordinates
- Instable solution



Outlier Handling with RANSAC

Use **direct solution** to find correct solution among set of corrupted points

- Assume $I \geq 3$ points

Repeat

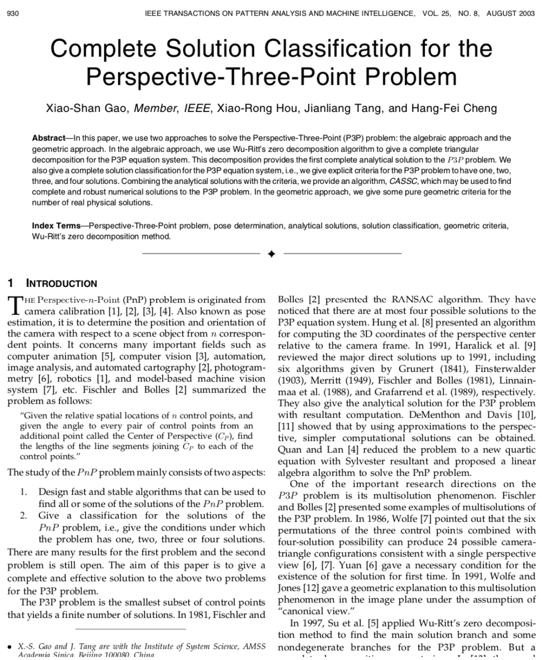
1. Select 3 points randomly
2. Estimate parameters of SR/P3P
3. Count the number of other points that support current hypotheses
4. Select best solution

- Can deal with large numbers of outliers in data

More Recent Solutions

- Further solutions gave been proposed after Grunert's solutions of 1841
- New methods still have ambiguities when using 3 control points only
- 4th point needed for disambiguation
- Faster to compute
- Numerically more stable
- Partially less complex

Recent Approaches



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Complete Solution Classification for the Perspective-Three-Point Problem

Xiao-Shan Gao, *Member, IEEE*, Xiao-Rong Hou, Jianliang Tang, and Hang-Fei Cheng

Abstract—In this paper, we use two approaches to solve the Perspective-Three-Point (P3P) problem: the algebraic approach and the geometric approach. In the algebraic approach, we use Wu-Ritt's zero decomposition algorithm to give a complete triangular decomposition for the P3P equation system. This decomposition provides the first complete analytical solution to the $P3P$ problem. We also give a complete solution classification for the P3P equation system, i.e., we give explicit criteria for the P3P problem to have one, two, three, and four solutions. Combining the analytical solutions with the criteria, we provide an algorithm, *CASSC*, which may be used to find complete and robust numerical solutions to the P3P problem. In the geometric approach, we give some pure geometric criteria for the number of real physical solutions.

Index Terms—Perspective-Three-Point problem, pose determination, analytical solutions, solution classification, geometric criteria, Wu-Ritt's zero decomposition method.

1 Introduction

The Perspective-n-Point (PnP) problem is originated from camera calibration [1], [2], [3], [4]. Also known as pose estimation, it is to determine the position and orientation of the camera with respect to a scene object from n correspondent points. It concerns many important fields such as computer animation [5], computer vision [3], automation, image analysis, and automated cartography [2], photogrammetry [6], robotics [1], and model-based machine vision system [7], etc. Fischler and Bolles [2] summarized the problem as follows:

"Given the relative spatial locations of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective (C_P), find the lengths of the line segments joining C_P to each of the control points."

The study of the PnP problem mainly consists of two aspects:

1. Design fast and stable algorithms that can be used to find all or some of the solutions of the PnP problem.
2. Give a classification for the solutions of the PnP problem, i.e., give the conditions under which the problem has one, two, three or four solutions.

There are many results for the first problem and the second problem is still open. The aim of this paper is to give a complete and effective solution to the above two problems for the P3P problem.

The P3P problem is the smallest subset of control points that yields a finite number of solutions. In 1981, Fischler and Bolles [2] presented the RANSAC algorithm. They have noticed that there are at most four possible solutions to the P3P equation system. Hung et al. [8] presented an algorithm for computing the 3D coordinates of the perspective center relative to the camera frame. In 1991, Haralick et al. [9] reviewed the major direct solutions up to 1991, including six algorithms given by Grunert (1841), Finsterwalder (1903), Merritt (1949), Fischler and Bolles (1981), Linnainmaa et al. (1988), and Grafarrend et al. (1989), respectively. They also give the analytical solution for the P3P problem with resultant computation. DeMenthon and Davis [10], [11] showed that by using approximations to the perspective, simpler computational solutions can be obtained. Quan and Lan [4] reduced the problem to a new quartic equation with Sylvester resultant and proposed a linear algebra algorithm to solve the PnP problem.

One of the important research directions on the $P3P$ problem is its multisolution phenomenon. Fischler and Bolles [2] presented some examples of multisolutions of the P3P problem. In 1986, Wolfe [7] pointed out that the six permutations of the three control points combined with four-solution possibility can produce 24 possible camera-triangle configurations consistent with a single perspective view [6], [7]. Yuan [6] gave a necessary condition for the existence of the solution for first time. In 1991, Wolfe and Jones [12] gave a geometric explanation to this multisolution phenomenon in the image plane under the assumption of "canonical view."

In 1997, Su et al. [5] applied Wu-Ritt's zero decomposition method to find the main solution branch and some nondegenerate branches for the P3P problem. But a

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Gao 2003

Complete Solution Classification for the Perspective-Three-Point Problem

A Novel Parametrization of the Perspective-Three-Point Problem for a Direct Computation of Absolute Camera Position and Orientation

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Abstract

The Perspective-Three-Point (P3P) problem aims at determining the position and orientation of the camera in the world reference frame from three 2D-3D point correspondences. This problem is known to provide up to four solutions that can then be disambiguated using a fourth point. All existing solutions attempt to first solve for the position of the points in the camera reference frame, and then compute the position and orientation of the camera in the world frame, which alignes the two point sets. In contrast, in this paper we propose a novel closed-form solution to the P3P problem, which computes the aligning transformation directly in a single stage, without the intermediate derivation of the points in the camera frame. This is made possible by introducing intermediate camera and world reference frames, and expressing their relative position and orientation using only two parameters. The projection of a world point into the parametrized camera pose then leads to two conditions and finally a quartic equation for finding up to four solutions for the parameter pair. A subsequent backsubstitution directly leads to the corresponding camera poses with respect to the world reference frame. We show that the proposed algorithm offers accuracy and precision comparable to a popular, standard, state-of-the-art approach but at much lower computational cost (15 times faster). Furthermore, it provides improved numerical stability and is less affected by degenerate configurations of the selected world points. The superior computational efficiency is particularly suitable for any RANSAC-outlier-rejection step, which is always recommended before applying PnP or non-linear optimization of the final solution.

1. Introduction

The Perspective-n-Point (PnP) problem is originated from camera calibration [1, 10, 17, 28]. Also known as pose estimation, it aims at retrieving the position and orientation of the camera with respect to a scene object from n corresponding 3D points. This problem has found many applications in computer animation [30], computer vision [16], augmented reality, automation, image analysis, automated cartography [10], photogrammetry [1, 24], robotics [35], and model-based machine vision systems [34]. In 1981, Fischler and Bolles [10] summarized the problem as follows: *Given the relative spatial locations of n control points, and given the angle to every pair of control points P_i from an additional point called the center of perspective C, find the lengths of the line segments joining C to each of the control points*. The next step then consists of retrieving the orientation and translation of the camera with respect to the object reference frame.

The Direct Linear Transformation was first developed by photogrammetrists [31] as a solution to the PnP problem—when the 3D points are in a general configuration—and then introduced in the computer vision community [7, 16]. When the points are coplanar, the homography transformation can be exploited [16] instead.

In this paper, we address the particular case of PnP for $n = 3$. This problem is also known as Perspective-Three-Point (P3P) problem. The P3P is the smallest subset of control points that yields a finite number of solutions. When the intrinsic camera parameters are known and we have $n \geq 4$ points, the solution is generally unique.

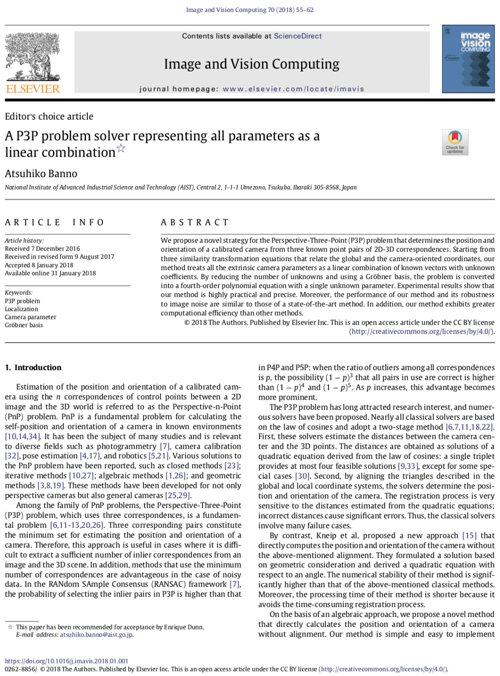
The P3P problem was first investigated in 1841 by Grunert [14] and in 1903 by Finsterwalder [8], who noticed that for a calibrated camera there can be up to four solutions, which can then be disambiguated using a fourth point. In the literature, there exist many solutions to this problem, which can be classified into iterative, non-iterative, linear, and non-linear ones. In 1991, Haralick et al. [15] reviewed the major direct solutions up to 1991, including the six algorithms given by Grunert (1841) [14], Finsterwalder (1903)—as summarized by Finsterwalder and Scheufele in [8]—, Merritt (1949) [25], Fischler and Bolles (1981) [10], Hung et al. (1985) [20], Linnainmaa et al. (1988) [23], and Grafarend et al. (1989) [13], respectively. They also gave the analytical solution for the P3P problem with re-

1

Kneip 2011

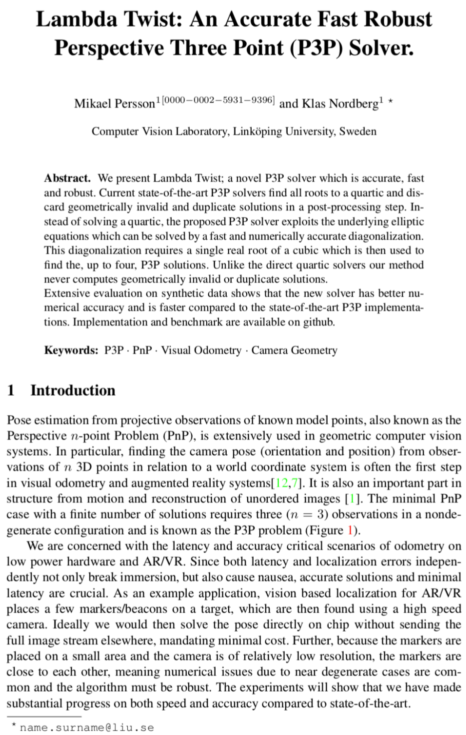
A novel parametrization of the perspective-three-point problem for a direct computation of absolute camera position and orientation

Recent Approaches



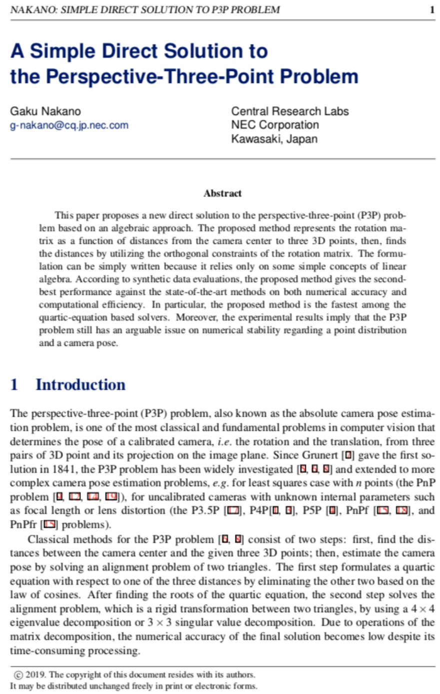
Banno
2018

A P3P problem solver representing all parameters as a linear combination



Persson
2019

Lambda Twist: An Accurate Fast Robust Perspective Three Point (P3P) Solver.



Nakano
2019

A Simple Direct Solution to the Perspective-Three-Point Problem

Orienting a calibrated camera by using > 3 control points

Spatial Resection Iterative Solution

Overview: Iterative Solution

- Over determined system with $I > 3$
- No direct solution but iterative LS
- Main steps
  - Build the system of observation equations
  - Measure image points $\boldsymbol{x}_i, \; i = 1, \ldots I$
  - Estimate initial solution $R, \boldsymbol{X}_o \rightarrow \boldsymbol{x}^{(0)}$
  - Adjustment
    - Linearizing
    - Estimate extrinsic parameter $\widehat{\boldsymbol{x}}$
    - Iterate until convergence

Summary

- P3P estimates the position and heading of a **calibrated camera** given control points
- Required **≥3 control points**
- **Direct solution**
  - Fast
  - Suited for outlier detection with RANSAC
- **Statistically optimal solution using iterative least squares**
  - Uses all available points
  - Assumes no outliers
  - Allows for accuracy assessments