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Breakdown-free gmres for mean-variance efficient portfolios

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Problem Overview

Solution of large linear systems of equations

Given Ax=b with $A \in \mathbb{R}^{nxn}$, b and x $\in \mathbb{R}^n$, how to solve the system?

Gmres

Iterative projecton method which seeks, for example at step m, an approximate solution x_m from the affine Krilov space $x_0 + K_m$ of dimension m. Projecton condition is the Galerkin condition (same of choosing x_m as minimizer of norm of the residual).

$x_m = x_0 + V_m y$ with y $\in \mathbb{R}^n$, V_m basis of
$K_m = K_m(A, r_0) = span(r_0, Ar_0, ..., A^{m-1} r_0)$ and $r0 = \frac{b}{||b||}$.

Breakdown of gmres

Gmres is based on the relations:

- $AV_k = V_k H_k + f_k e_k^T$ with H_k upper Hessenberg matrix.
- $AV_k = V_{k+1}\hat{H}_k$ with $V_{k+1} = \left[V_k, \frac{f_k}{||f_k||}\right]$, $\hat{H}_k = \begin{bmatrix} H_k \\ ||f_k||e_k^T \end{bmatrix}$.
- At each iteration x_m satisfies
  $r_m = b - Ax_m \perp AK_m \equiv \min_{x_m \in K_m} ||b - Ax_m|| = \min_{y \in \mathbb{R}^m} ||\beta e_1 - \hat{H}_m y||$
  with $\beta = ||r_0||$.

What happens if A is singular? H_k is also singular and the Arnoldi process (used to generate the columns of V_k) stops.

Breakdown of gmres

In general, when the Arnoldi process breaks down, it happens that $Av_N \notin K_N$, f_N=0 and dim(K_{N+1})=N. Assuming that it happens at iteration N$\leq$n there are 2 possible cases:

- If dim(AK_N)=N $\Rightarrow$ rank(H_n)=N $\Rightarrow$ span(b) + AK_N = K_{N+1} but dim(K_{N+1})=N $\Rightarrow$ b $\in AK_N$ so Ax_N= b. This is called a bening breakdown, since the solution is found when the process stops.
- If dim(AK_N)<N $\Rightarrow$ rank(H_n)<N $\Rightarrow$ A$x_N \neq$ b because b $\notin AK_N$. This is called hard breakdown and it is the main problem of gmres when used to solve linear systems with singular matrices.

Solution for hard breakdowns

Theorem for consistent systems

Given A singular and Ax=b consistent. If applying gmres with initial solution x_0=0 brings a hard breakdown at step N, then the solution of the linear system at that step is:

- x=$\hat{x}$+u if $A^N b \neq 0$, with $\hat{x} \in K_{N-1}$ and u $\in$ ker(A^p) (p is the index of A, that it the order of largest Jordan block of A associated with eigenvalue 0).
- x $\in$ ker(A^{N+1}) if A^Nb=0.

Solution for hard breakdowns

Theorem for inconsistent systems

Given A singular, with index p and Ax=b inconsistent. If applying gmres with initial solution x_0=0 brings a hard breakdown at step N and $A^N b \neq 0$, then any least-square solution of Ax_N=b can be written as x=$\hat{x}$+u$-A^D$r where $\hat{x} \in K_{N-1}$, u $\in$ ker(A^p)=ker(A^D), r=b-Ax $\in$ ker(A^T) and A^D is the Drazin inverse of A.

(if A have a representation A=C$\begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix} C^{-1}$ with C $\in \mathbb{C}^{nxn}$, then

A^D:=C $\begin{bmatrix} 0 & 0 \\ 0 & J_1^{-1} \end{bmatrix} C^{-1}$).

Breakdown-free gmres

Idea of the algorithm in few steps

1. Start with the classic gmres until a hard breakdown in the arnoldi decomposition occurs.
2. If an hard breakdown occurs at step N consider K_N=span$(v_1, v_2, ..., v_N)$. Replace v_N with a vector orthogonal to all $v_1, v_2, ..., v_N$. Create a matrix U_1=$[u_1]$ where u_1 is a column vector equal to the old v_N.
3. Compute a generalized Arnoldi decomposition to continue with iterations k=N,N+1,N+2,... requiring V_k columns to be orthonormal and orthogonal to U_1. This in translated in the new relation $Av_k - V_k(V_k^T Av_k) - U_1(U_1^T Av_k) = f_k$.
4. If another hard breakdown occurs repeat point 2 and 3 enlarging matrix U.
5. To satisfy the Galerkin condition solve the usual least-square problem replacing the matrix $\hat{H}_k$ with $\tilde{H}_k := \begin{bmatrix} \bar{H}_k \\ G_k \end{bmatrix}$ where G_k contains all terms $U_{p+1}^T Av_k$ as p and k vary.

Minimum variance portfolio application

- **Context**: portfolio optimization in finance. Derivation of efficient frontier is based on availability of expected return of each stock and covariance matrix of stock returns. Minimum vairance portfolio obtained solving the system of equations $\begin{cases} Ax = 1 \\ x^T\mathbf{1} = 1 \end{cases}$ (A is the covariance matrix, x portfolio weights). The solution is $x = \frac{A^{-1}\mathbf{1}}{\mathbf{1}A^{-1}\mathbf{1}}$.
- **Dataset**: Historical stock data for DIJA 30 companies between 2017-2018.
- **Objective**: find minimum variance portfolio using bfgmres, since covariance matrices are usually highly singular. Once we have this portfolio it can be used to compare different portfolios or methods.