Vine copulas for capital requirements: a probability equivalent level analysis

Master’s Degree in Mathematical Engineering

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Where are we?

Quantitative Risk Management aims to measure and manage risk, i.e the probability of something bad happening. In this work the focus is on:

- **Risk measures** to measure risk
- **Vine copulas** to measure dependencies
- **Probability equivalent levels** to manage risk

Between one Risk and another

Risks are almost countless, those where the following work would have greater opportunities are related to the financial context:

- **Market Risk:**
- **Liquidity Risk:**
- **Credit Risk:**
- **Systemic Risk:**

Over time, unified regulation over the financial system was introduced, with the aim to ensure for example that banks (Basel III) and insurance/reinsurance companies (Solvency II Directive) maintain an appropriate level of capital, sufficient to cover potential losses.

How do we measure Risk?

Risk measures are functional that maps a continuous random variable R_t^P to a real number:

$$\xi : R_t^P \to \mathbb{R} \tag{1}$$

where t $\geq$ 0 is a time instant, R_t^P is a continuous measurable function that represent, e.g. the value of a portfolio P at time t. Two risk measures studied in this work:

Value-at-risk (VaR)

$$VaR_\alpha^{P,t} := sup\left\{r|F_{R_t^P}(r) \leq \alpha\right\} = \mathcal{Q}_{R_t^P}(\alpha) \tag{2}$$

Expected Shortfall (ES)

$$ES_\alpha^{P,t} := \mathbb{E}\left[R_t^P|R_t^P \leq VaR_\alpha^{P,t}\right] = \frac{1}{\alpha}\int_0^\alpha rf_{R_t^P}(r)\,dr \tag{3}$$

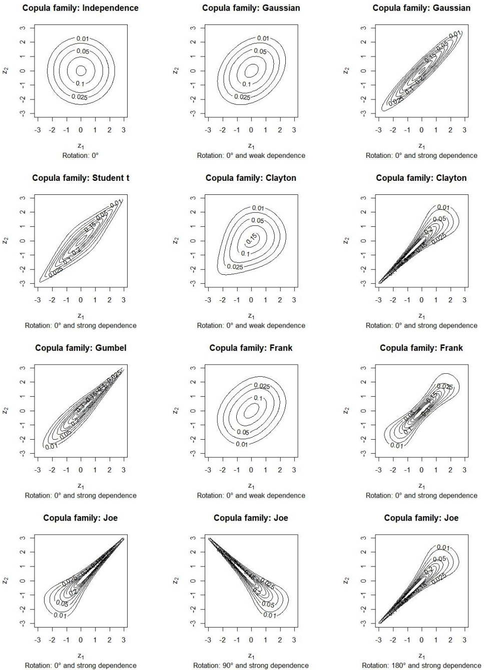
What about dependencies?

1 Introduction

Copulas are functions used to **factorise n-dimensional multivariate cumulative distribution** $F_X(x_1, \dots, x_n)$ (CDF) into:

- a set of univariate marginal CDFs $\{F_i(x_i)\}_{i=1}^{n}$
- a function C called copula that maps the n univariate marginals from $[0, 1]^n$ into $[0, 1]$.

Under certain requirements of coherency it is possible to write $\mathbf{F_X(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))}$ $\forall x = (x_1, \dots, x_n)$.



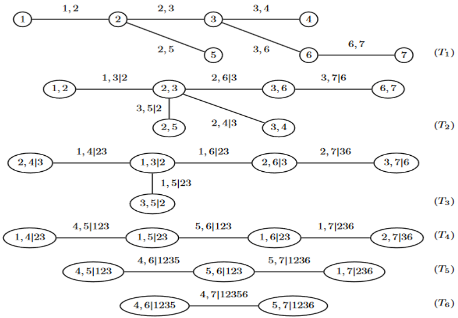
Vine structure: embracing the diversity of copula families

The idea behind Vine copulas is to further **decompose the dependence** among n variables by studying the dependence **among the $\frac{\mathbf{n(n-1)}}{\mathbf{2}}$ possible pairings of variables.**

Given a portfolio $X = (X_1, X_2, X_3)$, where X_i is the i-th asset, its density can be written as:

$$\begin{aligned} f_{123}(x_1, x_2, x_3) &= f_3(x_3) f_{2|3}(x_2|x_3) f_{1|23}(x_1|x_2, x_3) \\ &= f_3(x_3) c_{23}(F_2(x_2), F_3(x_3)) f_2(x_2) \cdot \\ &\cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) c_{12}(F_1(x_1), F_2(x_2)) f_1(x_1) \end{aligned} \tag{4}$$

Formally, **(F,$\mathcal{V}$,B)** is an R-vine copula specification if $F = (F_1, ..., F_n)$ is a **vector of continuous invertible distribution functions**, B is the **set of bivariate copulas families**, called pair-copulas, and $\mathcal{V}$ is a **n-dimensional R-vine**, a set of trees with specific properties.



$$M = \begin{bmatrix} 7 & & & & & & \\ 4 & 4 & & & & & \\ 5 & 6 & 6 & & & & \\ 1 & 5 & 5 & 5 & & & \\ 2 & 1 & 1 & 1 & 1 & & \\ 3 & 2 & 2 & 3 & 3 & 3 & \\ 6 & 3 & 3 & 2 & 2 & 2 & 2 \end{bmatrix}$$

An set of trees $\mathcal{V} = (T_1, \ldots, T_{n-1})$ is called a regular vine if:

- T_1 is a tree with nodes $N_1 = 1, \ldots, n$ and a set of edges denoted by E_1.
- For $i = 2, \ldots, n-1,$ T_i is a tree with nodes $N_i = E_{i-1}$ and edge set E_i.
- For $i = 2, \ldots, n-1$ and $\{a, b\} \in E_i$ with a=$\{a_1, a_2\}$ and b=$\{b_1, b_2\}$ it must hold that $\#(a \cap b) = 1$ (proximity condition), where $\#$ denotes the cardinality of a set.

Inference of R-vine copulas specifications

- **Density** is obtained in a recursively manner starting from the formulas:

$$
\begin{aligned}
f_{1\ldots n}(x) &= \prod_{k=1}^{n} f_k(x_k) \prod_{i=1}^{n-1} \prod_{e \in E_i} c_{C_{e,a},C_{e,b}|D_e}\left(F_{Ce,a|D_e}(x_{C_{e,a}}|x_{D_e}) F_{Ce,b|D_e}(x_{C_{e,b}}|x_{D_e})\right) \\
F_{Ce,a|D_e}(x_{C_{e,a}}|x_{D_e}) &= \frac{\partial C_{Ca|D_a}\left(F_{C_{a,a_1}|D_a}(x_{C_{a,a_1}}|x_{D_a}), F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}}|x_{D_a})\right)}{\partial F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}}|x_{D_a})} \\
&:= h\left(F_{C_{a,a_1}|D_a}(x_{C_{a,a_1}}|x_{D_a}), F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}}|x_{D_a})\right)
\end{aligned}
\tag{5}
$$

- **Simulation** is done applying $x_j = F^{-1}_{j|12\ldots j-1}(u_j|x_1, \ldots, x_{j-1})\, j = 1, \ldots n$, where $u_j \sim U[0, 1]$ $\forall j$, while $F^{-1}_{j|12\ldots j-1}$ is computed using the inverse of h-functions.
- **Structure selection is done sequentially:** pair-copulas parameters of the first tree are first estimated, then variables are mapped into copula scale and used to estimate pair-copulas parameters of the subsequent tree. The process is repeated for all trees. Kendall tau is used for weighing edges to find maximum spanning trees.

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2 Monte Carlo Simulation

Is Sequential Estimation efficient? Monte Carlo reveals all

Approach: simulate data knowing parameter and family matrices, apply structure selection algorithm to derive a plausible R-vine copula specification, evaluate the accuracy of the model in reconstructing the initial structure by means of 3 quantities:

- **General τ difference**: based on the definition of Kendall Tau
  $\rho_\tau(X, \hat{X}) = P\{(X_1 - \hat{X}_1)(X_2 - \hat{X}_2) > 0\} - P\{(X_1 - \hat{X}_1)(X_2 - \hat{X}_2) < 0\} =$
  $= \mathbb{E}[sign((X_1 - \hat{X}_1)(X_2 - \hat{X}_2))]$
- **Lower τ difference**: based on the definition of lower exceedance Kendall Tau
  $\tau^{lower}(X_i, \hat{X}_i) := \tau(X_i, \hat{X}_i | X_i \leq 0.2, \hat{X}_i \leq 0.2) \; i = 1, 2$
- **Upper τ difference**: based on the definition of upper exceedance Kendall tau
  $\tau^{upper}(X_i, \hat{X}_i) := \tau(X_i, \hat{X}_i | X_i \geq 0.8, \hat{X}_i \geq 0.8) \; i = 1, 2$

Navigating Monte Carlo’s Scenarios

1) Fixed BB1 pair-copulas, no rotations

Two-parameter copulas used to capture more than one type of dependence, since dependency coefficients in tails are different depending on the tail being considered.

2) BB1 pair-copulas, random rotations

Families chosen randomly between BB1 copula without rotation or with 90°, 180°, 270° rotations.

3) Mixed pair-copulas, few families

Parameters chosen according to a prespecified Kendall tau and family matrix. Allowed families are: Gaussian, Student-t, Gumbel, 180° rotated Gumbel, Frank.

4) Mixed pair-copulas, all families

All implemented families and their rotations randomly selected: Gaussian, Student-t, Clayton, Gumbel, Frank, Joe, BB1, BB6, BB7, BB8, Tawn type 1, Tawn type 2.

Evidence of model validity: results from Simulation study

N_1 is the number of simulated samples.

N_2 is the number of repetition to mitigate estimate variability and prevent biased results.

Scenario	Dimensions	General τ diff.	Lower τ diff.	Upper τ diff.
1	$\{N_1 = 500, N_2 = 100\}$	0.011	0.039	0.057
1	$\{N_1 = 500, N_2 = 1000\}$	0.012	0.040	0.061
1	$\{N_1 = 1000, N_2 = 1000\}$	0.009	0.033	0.056
2	$\{N_1 = 500, N_2 = 100\}$	0.012	0.020	0.031
2	$\{N_1 = 500, N_2 = 1000\}$	0.013	0.029	0.023
2	$\{N_1 = 1000, N_2 = 1000\}$	0.012	0.030	0.050
3	$\{N_1 = 500, N_2 = 100\}$	0.022	0.057	0.063
3	$\{N_1 = 500, N_2 = 1000\}$	0.020	0.058	0.067
3	$\{N_1 = 1000, N_2 = 1000\}$	0.015	0.046	0.051
4	$\{N_1 = 500, N_2 = 100\}$	0.081	0.076	0.080
4	$\{N_1 = 500, N_2 = 1000\}$	0.036	0.043	0.057
4	$\{N_1 = 1000, N_2 = 1000\}$	0.017	0.029	0.116

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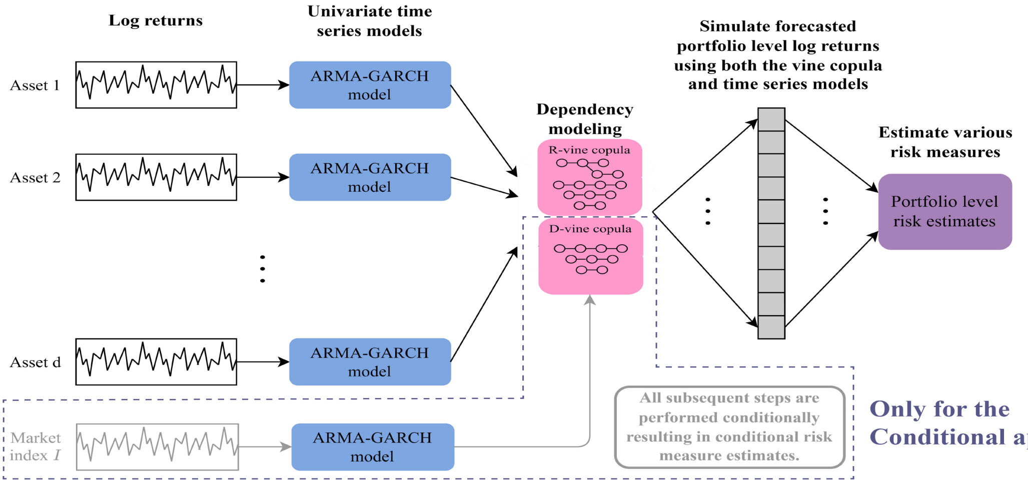
Vine copula based risk measure estimation

3 Probability Equilavent Level Analysis



Only for the Conditional approach

What are probability equivalent levels?

Conditional risk measures can be better defined as:

Conditional Value-at-risk (CoVaR)

$$\begin{aligned} CoVaR_{v,u}[\Omega|I] &= VaR_v[\Omega|I = VaR_u[I]] = \\ &= F^{-1}_{\Omega|I=VaR_u(X)}(v) \end{aligned}$$

u $\in (0,1)$ is the confidence level for market index I, while v $\in (0,1)$ is the confidence level of portfolio Ω.

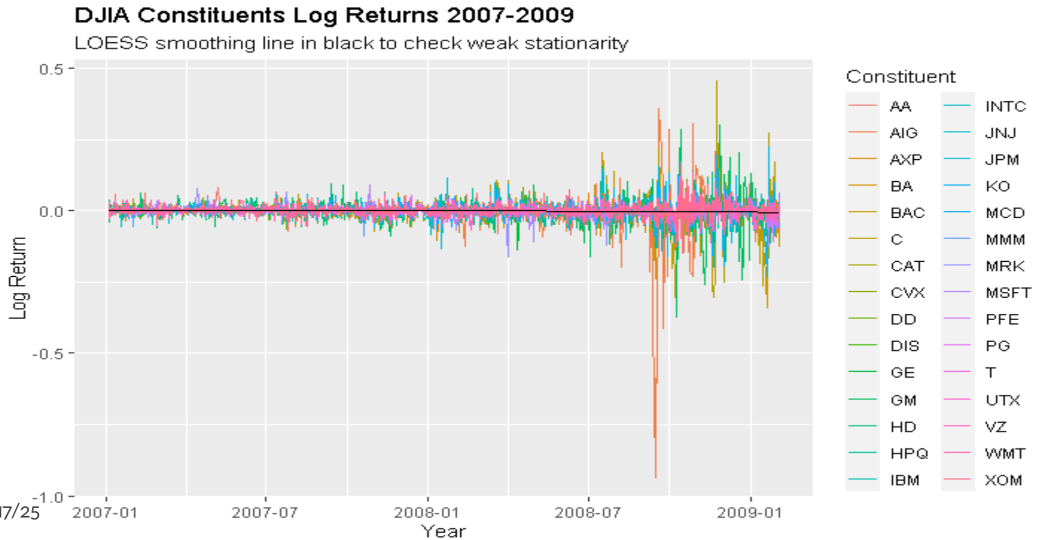
Conditional Expected Shortfall (CoES)

$$\begin{aligned} CoES_{v,u}[\Omega|I] &= ES_v[\Omega|I = VaR_u[I]] = \\ &= \mathbb{E}[\Omega|\Omega \leq VaR_v(\Omega), I = VaR_u[I]] \end{aligned}$$

nothing but the expected shortfall given that the market index I is at level u.

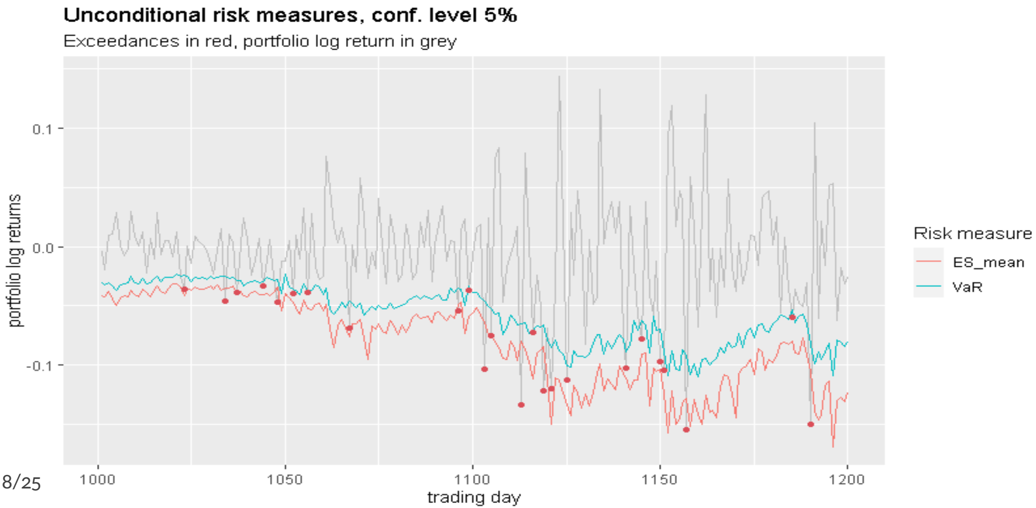
$\mathbf{u_v}$ is a probability equivalent level of CoVaR-VaR at the risk level $\mathbf{v}$ ($\mathbf{PELCoV_v}$) for Ω if $CoVaR_{v,u_v}[\Omega|I] := VaR_v[\Omega|I = VaR_{u_v}[I]] \equiv VaR_v[\Omega]$. Same for CoES, ES and so $\mathbf{PELCoES_v}$.

Application to Dow Jones Industrial Average Costituents



Portfolio Ω_1 test set predictions

$\boldsymbol{\Omega_1}$= {Alcoa (AA), American Express (AXP), Boeing (BA), Bank of America (BAC)},
$\boldsymbol{I_1}$={Citigroup (C), Caterpillar (CAT)}



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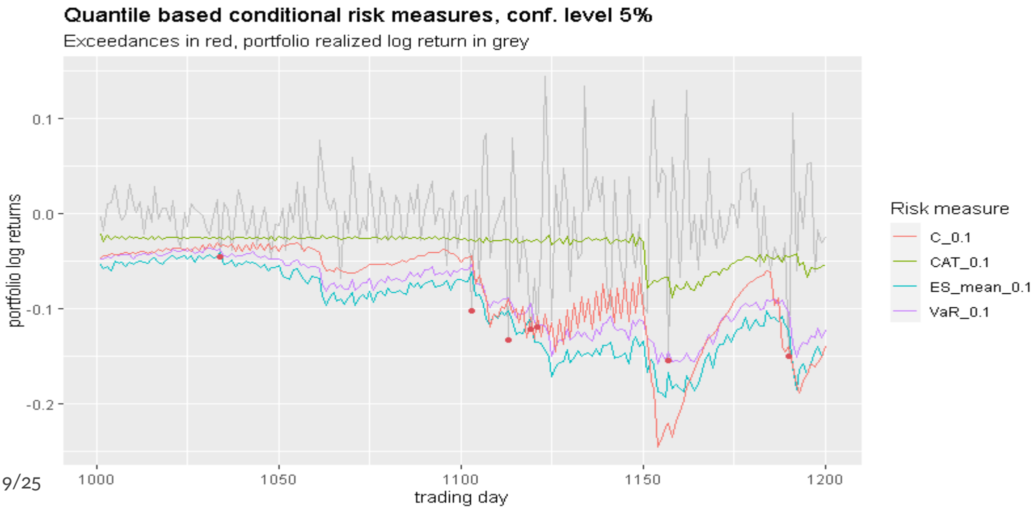

Unveiling PELCoV (and PELCoES) for Ω_1: Graphs vs. Data

$\boldsymbol{\Omega_1}$= {Alcoa (AA), American Express (AXP), Boeing (BA), Bank of America (BAC)},
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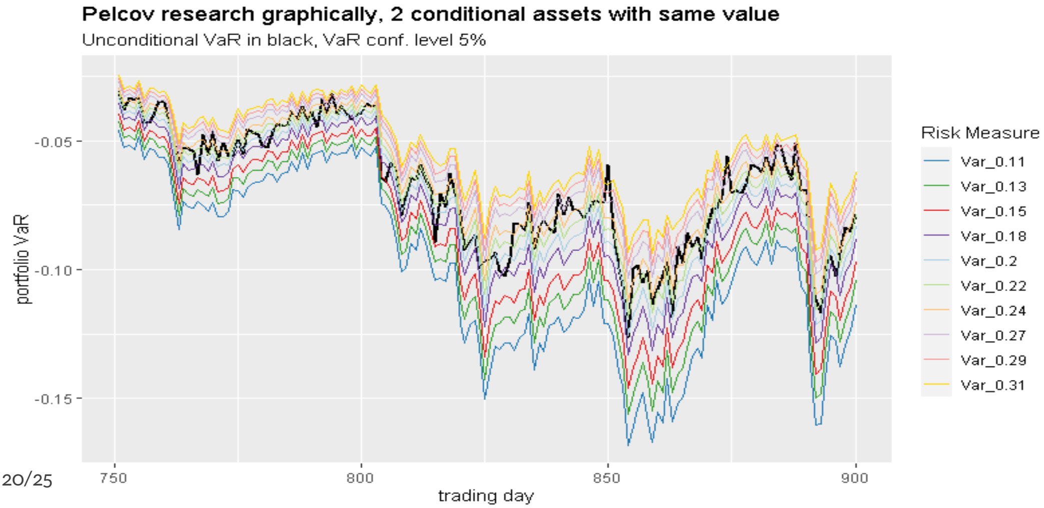
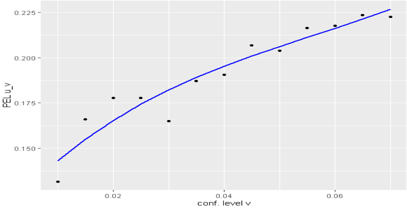


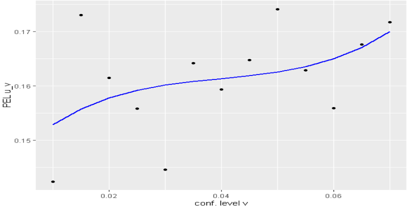
Probability equivalent levels for $\{\Omega_1, I_1\}$

$\boldsymbol{\Omega_1}$= {Alcoa (AA), American Express (AXP), Boeing (BA), Bank of America (BAC)},
$\boldsymbol{I_1}$={Citigroup (C), Caterpillar (CAT)}

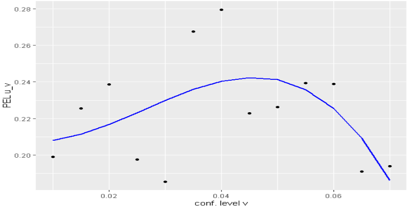
One conditional asset PELCoV



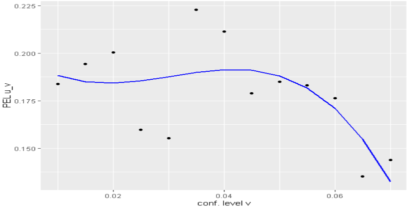
Two conditional assets PELCoV



One conditional asset PELCoES



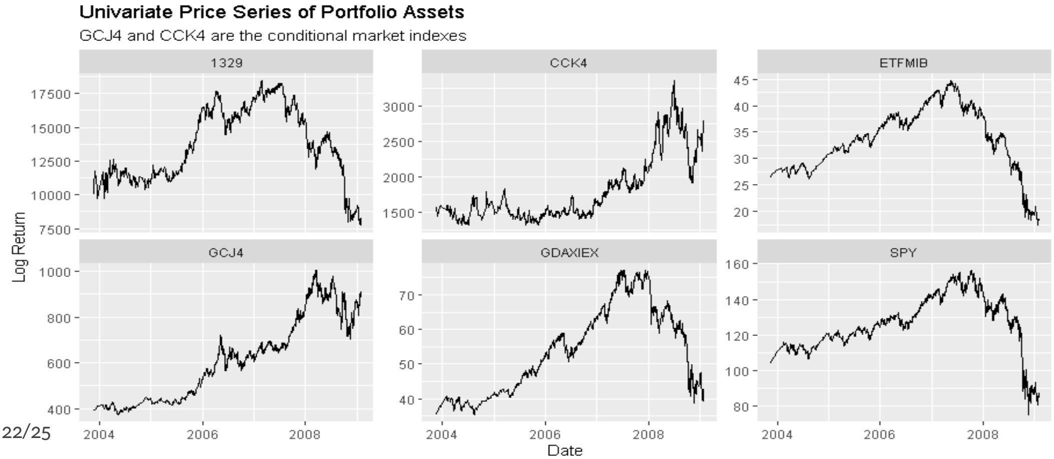
Two conditional assets PELCoES



What happens with negative correlated assets?

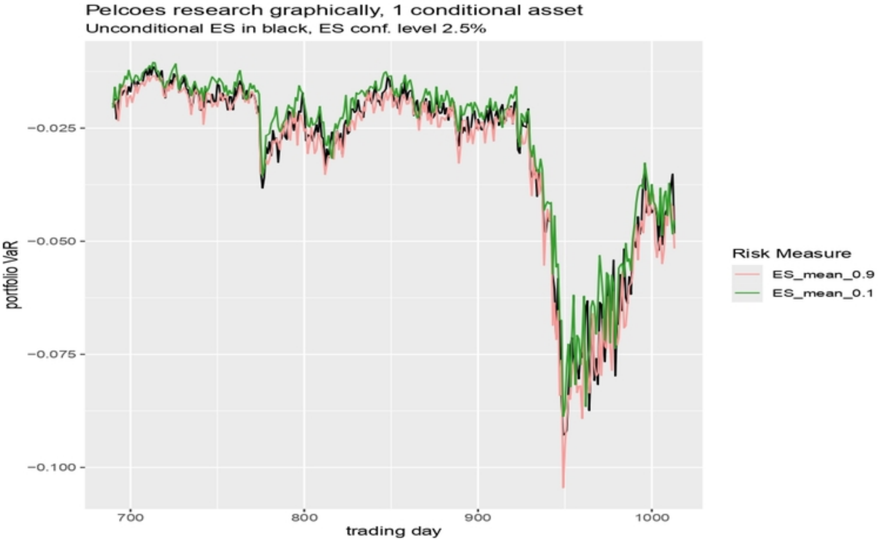
3 Probability Equilavent Level Analysis

As the market indexes conditioning level increases, should we expect conditional risk measures to increase as well? The answer is given studying $\{\Omega_2, I_2\}$.



Inverted Structure: A New Perspective of PELs

$\Omega_2 = \{1329, ETFMIB, GDAXIEX, SPY\}, I_2 = \{GCJ4, CCK4\}$ (stock market symbols)



Conclusions

- R-Vines are a class of flexible copulas that allow the construction of a **variety of dependency structures**, going beyond the adoption of a particular copula family.

- Conditional risk measures provide a **greater understanding of the different facets of systemic risk** and can be useful for adjustments in capital requirements based on the performance of specific financial entities.

- Existence of probability equivalent levels even in the multivariate case makes possible to identify when different risk measures are more or less conservative than others and possibly to **define critical alert thresholds for stressed market conditions.**

Vine copulas for capital requirements: a probability equivalent level analysis

Thank you for listening!

Any questions?