Quantitative Macro II — Homework 2

This is a *research project*, not a homework. The goal is for you to learn something both about quantitative macroeconomics and computation. The findings should be written up in a research report where you clearly communicate the economics environment, the question being answered and the results. Remember, ignorance and failure to communicate are observationally equivalent. Further all of the code used to generate the findings are to be included with the report. There are no ”right” answers. The bonus questions are potentially very difficult.

The research project is due November 28, 2022. All projects must be submitted online via Athena.

1 Incomplete Markets Models with Aggregate Risk: Krusell & Smith (1998) vs Boppart, Krusell & Mitman (2018) and Auclert et al (2021)

Consider the Krusell-Smith economy from QMM1.

There is an economy with a continuum of measure 1 of households. Each household has preferences:

$$u(c) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

The household faces the budget constraint:

$$c + k' = w((1-\tau)\tilde{L}\epsilon + (1-\epsilon)\mu) + (1 + r - \delta)k$$

where k is the level of capital, ϵ is the idiosyncratic employment status, w is the wage and r is the real interest rate (all prices are functions of the aggregate state). The agent can't borrow $k' \geq 0$. As in Kieran's question, let $\gamma = 2$ and $\beta = 0.99$.

- Use the average of the markov transition matrix for employment from Kieran's part:
$$\Pi(\epsilon, \epsilon') = \begin{bmatrix} 0.47 & 0.53 \\ 0.04 & 0.96 \end{bmatrix}$$
so that the steady state unemployment rate is 0.0702
- Normalize $\tilde{L} = 1/u$, such that the steady state labor endowment is 1

- Set the unemployment replacement rate $\mu = 0.15$
- The production function in the economy is given by:

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

  where K is aggregate capital and L is aggregate labor. Capital depreciates at rate $\delta = 0.025$, and the capital share $\alpha = 0.36$.

- Assume that $\log A_t$ follows an AR(1) process

$$\log A' = \rho \log A + \eta'$$

  where η' is iid normal random variable with mean $-\sigma^2/2$ and variance σ^2. Let $\rho_A = 0.95$ and $\sigma_A = 0.007$.

Assume that agents can't borrow $a_{t+1} \geq 0$.

1. First, solve for the steady state of the economy. Report the interest rate r and the aggragete capital stock K.
2. Solve for the equilibrium by linearizing the model using the method of BKM—solving for the non-linear MIT shock. Report the variance of C, I, Y and their correlations with A.
3. Compare the impulse responses and dynamic properties of the model solved using BKM with the second moments from the Krusell-Smith model solved in Kieran's HW.
4. Compute the impulse response using the Sequence Space Jacobian Method of Auclert et al (2021) and compare the results to the other two solution methods. Plot the Jacobian of the household problem and the fake news matrix used in computing the overall Jacobian of the model. In order to do this, recall that you need to solve for the expectation (/prediction) vectors, and the policy functions. Compare the policy functions computed under this method that uses superposition (i.e. exploits only the time to a shock) vs the fully computed policy functions from part 2 of the problem.
5. Solve for the non-linear transitions from part 2 but with larger shocks, $\eta = 0.35, 0.7$. Do you detect non-linearities?

6. **Bonus:** The Krusell-Smith version of the model has cyclical employment risk (the markov transition matrix changes with the aggregate state). Think about how you could incorporate that cyclical risk into the BKM or Auclert et al method, incorporate it, solve the model, and see how it compares to your answers in the previous part.

2 Basic Arellano (2008) model

- Take the simplest Arellano model: $y \in \{y_1, \cdots, y_N\}$, transition matrix Π. Default subject to costs:

$$V^d(y) = u(y - \tau(y)) + \beta \mathbb{E}_{y'|y}\left[V^d(y')\right] \tag{1}$$

- Value function without default

$$\begin{aligned} V(b, y) \quad &= \quad \max_{b'} u(c) + \beta \mathbb{E}_{y'|y}\left[V^o(b', y')\right] \\ &\text{s.t.} \quad c + b = y + Q(b', y) \end{aligned} \tag{2}$$

- Value V^o including the option to default at the beginning of period

$$V^o(b, y) = \max_{p \in \{0,1\}} pV(b, y) + (1 - p)V^d(y) \tag{3}$$

  where $p = 1$ denotes the decision to repay and $p = 0$ denotes the decision to default.

- Debt pricing

$$Q(b', y) = \frac{b'}{R}\mathbb{E}_{y'|y}\left[p\left(b', y'\right)\right] = \frac{b'}{R}\mathbb{P}_{y'|y}\left[V(b', y') \geq V^d(y')\right] \tag{4}$$

- Assume: that income is iid 2 state, $y \in \{y_L, y_H\} = \{0.2, 1.2\}$, probabilities $(\pi_L, \pi_H) = (0.2, 0.8)$, no default cost $\tau(y) = 0$. Let $u(c) = -c^{-1}$, $\beta = 0.8$ and $R = 1.1$. Solve for the equilibrium policy functions and price schedule using value function iteration. Define the grid for borrowin with 150 points from 0 to 2. Plot the four value functions and identify the default thresholds. Plot the debt laffer curve, $Q(b)$, as a function of b. Discuss the economics of the problem and the optimal behavior of the sovereign.

3 Adding EV shocks to Arellano (2008)

- Modify the environment as follows: choice of default now depends on realization of EV shocks. Modify (3) to

$$V^o(b,y) = \mathbb{E}_{\epsilon,\epsilon^d}\left[\max_{p\in\{0,1\}} p\left(V(b,y)+\epsilon\right) + (1-p)\left(V^d(y)+\epsilon^d\right)\right]$$

where ϵ, ϵ^d have extreme value distribution with parameter α. This results in an ex-ante probability of choosing to continue equal to

$$\begin{aligned} p\left(b,y\right) &= \frac{e^{\alpha V(b,y)}}{e^{\alpha V(b,y)}+e^{\alpha V^d(y)}} \\ &= \frac{1}{1+e^{\alpha\left(V^d(y)-V(b,y)\right)}} \end{aligned}$$

- Accordingly, we modify (4) to read

$$Q(b',y) = \frac{b'}{R}\mathbb{E}_{y'|y}\left[p\left(b',y'\right)\right] = \frac{b'}{R}\mathbb{E}_{y'|y}\left[\frac{e^{\alpha V(b',y')}}{e^{\alpha V(b',y')}+e^{\alpha V^d(y')}}\right] \quad (5)$$

and use the closed form expression for V^o,

$$\begin{aligned} V^o(b,y) &= \frac{\gamma}{\alpha}+\frac{1}{\alpha}\log\left(e^{\alpha V(b,y)}+e^{\alpha V^d(y)}\right) \quad (6) \\ &= \frac{\gamma}{\alpha}+\frac{1}{\alpha}\log\left(\left(e^{\alpha\left(V(b,y)-V^d(y)\right)}+1\right)e^{\alpha V^d(y)}\right) \\ &= V^d\left(y\right)+\frac{\gamma}{\alpha}+\frac{1}{\alpha}\log\left(1+e^{\alpha\left(V(b,y)-V^d(y)\right)}\right) \\ &= V(b,y)+\frac{\gamma}{\alpha}+\frac{1}{\alpha}\log\left(1+e^{\alpha\left(V^d(y)-V(b,y)\right)}\right) \end{aligned}$$

- Using the same parameters as above, and setting $\alpha = 1$, solve for the equilibrium policy functions and price schedule using value function iteration. Make the same plots as for the previous problem. Compare the value functions and Laffer curve with and without EV shocks. Note: $\gamma$is Euler's constant, i.e. $\sum_{k=1}^{n}\frac{1}{k}\sim\gamma+\ln\left(n\right)$, which is available in Matlab as `eulergamma`.

4 Solving Arellano (2008) with EV shocks using EGM

- Consider first the consumption policy for both problems produced by value function iteration, as well as the value function

- **Conjecture**: the value function is continuous and semi-differentiable. The envelope
$$V'(b,y) = -u'(c(b,y)) \tag{7}$$
holds everywhere, meaning that if V' is not differentiable at point b^* then its left and its right derivatives coincide with those of u',
$$\lim_{b\to(b^*)^+} V'(b,y) = -\lim_{b\to(b^*)^+} u'(c(b,y))$$
$$\lim_{b\to(b^*)^-} V'(b,y) = -\lim_{b\to(b^*)^-} u'(c(b,y))$$
The points b^* are points where c itself is discontinuous.

- Given the conjecture, the FOC for b' in (2), we have
$$\frac{dQ}{db}(b')c(b,y)^{-\sigma} = \beta\mathbb{E}_{y'|y}\left[\frac{dV^o(b',y')}{db'}\right]$$
use the expression for V^o in (6), and out conjectured envelope (7), then
$$\frac{dV^o(b,y)}{db} = \frac{e^{\alpha V(b,y)}}{e^{\alpha V(b,y)} + e^{\alpha V^d(y)}}\frac{dV(b,y)}{db} = -p(b,y)\,u'(c(b,y))$$
Combining we find
$$\frac{dQ}{db'}(b')c(b,y)^{-\sigma} = \beta\mathbb{E}_{y'|y}\left[p(b',y')\cdot\left(c'(b',y')\right)^{-\sigma}\right] \tag{8}$$
in other words the standard formulation of the Euler equation extends exactly to the case with default

- Now use the expression for Q in (5), to get

$$\begin{aligned}
\frac{dQ\left(b',y\right)}{db'} &= \frac{1}{R}\mathbb{E}_{y'|y}\left[\frac{1}{1+e^{\alpha\left(V^d(y')-V(b',y')\right)}}\right] \\
&\quad -\frac{b'}{R}\alpha\mathbb{E}_{y'|y}\left[\left(\frac{1}{1+e^{\alpha\left(V^d(y')-V(b',y')\right)}}\right)^2 e^{\alpha\left(V^d(y')-V(b',y')\right)}\left(-\frac{dV}{db'}\right)\right] \\
&= \frac{1}{R}\left(\mathbb{E}_{y'|y}\left[p\left(b',y'\right)+\alpha p\left(b',y'\right)\left(1-p\left(b',y'\right)\right)\frac{dV}{db}\right]\right) \\
&= \frac{1}{R}\left(\mathbb{E}_{y'|y}\left[p\left(b',y'\right)-\alpha p\left(b',y'\right)\left(1-p\left(b',y'\right)\right)u'\left(c\left(b',y'\right)\right)\right]\right) \\
&= \frac{1}{R}\left(\mathbb{E}_{y'|y}\left[p\left(b',y'\right)\left\{1-\alpha\left(1-p\left(b',y'\right)\right)u'\left(c\left(b',y'\right)\right)\right\}\right]\right)
\end{aligned}$$

where we used the conjectured envelope (7) one more time.

- The idea is to use the envelope condition:

$$\frac{dQ}{db}(b')c^{-\sigma} = \beta\mathbb{E}\left[p\left(b',y'\right)\cdot\left(c'\right)^{-\sigma}\right]$$

to solve out for the policy function $c\left(b,y\right)$, using

$$c\left(b,y\right)^{-\sigma} = \frac{\beta\mathbb{E}\left[p\left(b',y'\right)\cdot c\left(b',y'\right)^{-\sigma}\right]}{\frac{dQ}{db}(b')} \tag{9}$$

together with the budget constraint

$$c+b = y+Q(b',y)$$

- Implementation of the algorithm:
  - On grid for b', evaluate

$$c = \left(\frac{\beta\mathbb{E}\left[p\left(b',y'\right)\cdot c\left(b',y'\right)^{-\sigma}\right]}{\frac{dQ}{db}(b')}\right)^{-\frac{1}{\sigma}} \tag{10}$$

    back out the grid for b (today, given income y today) that solves

$$b = y+Q(b',y)-c$$

  - Flipping this around, for any given b there are possibly multiple allowable b'. In that case, pick the optimal b' by maximizing over the potential candidates.

- Solve the model using the same paramters as in Section 2, but now with EGM instead of VFI. Compare the policy functions, accuracy and speed performance of the two difference algorithms.

5 Bonus: Proving the conjecture

Prove that 7is true, or provide a counterexample, for u CRRA and generic a generic Markov chain for income, $y \in \{y_1, \cdots, y_N\}$, transition matrix Π.