Problem set: Firms and Inventories

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1 Solving the stationary firm inventory problem

1.1 Preamble

In this problem set we are going to solve the firm inventory problem in Kahn and Thomas (2007). For ease we will solve the stationary problem without aggregate shocks.[1]

The solution will also focus on introducing two techniques:

- Approximating value functions with splines
- Finding the *optimal choice* using off grid search

Details of both of these techniques are provided in the lecture 1 slides.

I will provide a sketch walkthrough of the method. You should see the lecture notes and the original paper for further details. *If something is missing/unclear check the paper.*

1.1.1 Splines

We will approximate both the $\mathbb{E}_\xi V^0$ and V^1 value functions with splines at $N_s = 25$ grid points.

It is faster (quite dramatically so) to split up the fitting of a polynomial spline to a function and to evaluate a point on that spline. Rather than fitting the spline everytime you evauate a point.

Use the Matlab command B `= spline(`x`,`V`)` to fit the spline and Vf `= ppval(`B`,`x^*`)` to evaluate the point x^*. Where x is a set of nodes, V is the value function evaluated on those nodes and x^* is an off grid point to be evaluated

1.1.2 Golden search

Golden search is just a method for finding an extremum of a function within a fixed interval. It does this by narrowing the range of values within the interval. By updating

[1] So there is no need to solve the aggregate law of motion with the Krusell & Smith method

the size of the interval width with the *golden ratio* it does this in a maximally efficient way. See: `en.wikipedia.org/wiki/Golden-section_search` or lecture 1 slides.

I provide a short Matlab code that undertakes this operation `goldsearch.m` but you will need to adapt it and integrate it into your code. *Note:* in the firm problem you should constrain search over the relevant region e.g. $m \in [0, s_1]$

1.2 Calibration and parameters

Following the paper

- Final good production: $G(m, n) = m_m^\theta n_n^\theta$
- Intermediate good production: $zF(k, l) = \bar{z}k^\alpha l^{1-\alpha}$
- Household utility: $u(c, 1 - n^h) = \log c + \eta(1 - n^h)$.
- Parameters $\beta = 0.9840, \eta = 2.1280, \alpha = 0.3739, \theta_m = 0.4991, \theta_n = 0.3275, \delta = 0.0173, \bar{\xi} = 0.2198, \bar{z} = 1.0032, \sigma = 0.012$
- Inventories grid: $N_s = 25$ with bounds $\mathcal{S} \in [0, 2.5]$.
- Use log grid spacing:

$$\mathcal{S} = [0, \psi^{0\{\texttt{linspace(}\ \log(0.1042/\psi^1)/\log(\psi^0),\ \log(2.5)/\log(\psi^0),\ N_s-1))\}}] \tag{1}$$

- With $\psi^0 = 10, \psi^1 = 25$.[2]

Numerical parameters

- Number of periods without adjustment to check $J^{max} = 10$
- precision of golden search choice: $\epsilon_{gs} = 1e^{-10}$
- precision of value function: $\epsilon_{vfi} = 1e^{-6}$
- beginning of period value of inventories below which to use all remaining stocks. set $m = s$: $\epsilon_0 = 1e^{-8}$
- precision of market clearing: $\epsilon_p = 1e^{-8}$
- Initial bounds for $p^* \in [3.2, 3.3]$

[2]Note: This is just log-spacing with more points located at in the lower part of the interval. Normal log-spacing used previously will also work but this will help you replicate the exact results in the paper.

1.3 Known equilibrium objects

We can immediately compute a number of equibrium objects from the household and intermediate goods problem. In particular wages:

$$\omega = \frac{\eta}{p^*} \tag{2}$$

The intermediate good price:

$$q = (p^*)^{\alpha-1}\bar{z}^{-1}\Big(\frac{(1-\beta(1-\delta)}{\beta\alpha}\Big)^{\alpha}\Big(\frac{\eta}{1-\alpha}\Big)^{1-\alpha} \tag{3}$$

which is a function of p

1.4 Inner most loop: Firm choices

Tip: you will need to find s^* and m^* and the associated policies and values many times. Write solution to these problems as separate functions to be called in your main program.

We will need to solve for the optimal output price p^* that satisfies market clearing. This is done in the final loop. All other sections are conditional on the current guess of p.

1.4.1 Optimal inventory level (s^*)

Given an initial guess for V^1 solve the problem of an adjusting firm to find the optimal inventory target level s^*

This is the solution to :

$$\max_{s_1} -pqs_1 + V^1(s_1) \tag{4}$$

Also store the value of adjusting

$$V^a = -pqs^* + V^1(s^*) \tag{5}$$

1.4.2 Optimal sub-period production $m(s_1)$

Given an initial guess[3] for $\mathbb{E}_\xi V^0$ solve for m^* and n^* for each $s \in \mathcal{S}$

$$V^1(s_1) = \max_{m,n} p[G(m,n) - \sigma(s_1 - m) - \omega n] + \beta\int_{\underline{\xi}}^{\bar{\xi}} V^0(s_1 - m, \xi)H(d\xi) \tag{6}$$

or given our guess $\mathbb{E}_\xi V^0$:

$$V^1(s_1) = \max_{m,n} p[G(m,n) - \sigma(s_1 - m) - \omega n] + \beta\mathbb{E}_\xi V^0(s_1 - m) \tag{7}$$

[3] A good initial guess is $\mathbb{E}_\xi V^0 = p^{1/(1-\theta_m)}(1-\theta_n)(\frac{\theta_n}{\eta})^{\theta_n/(1-\theta_n)}\mathcal{S}^{\theta_m/(1-\theta_n)}$

Notice the choice of n is a static problem so really we can replace n with $n(m)$ where $n(m)$ satisfies $\omega = G_n(m,n)$ or:

$$n(m) = (\theta_n p(m^{\theta_m})/\eta)^{1/(1-\theta_n)} \tag{8}$$

Check this for yourself!

This gives us the maximised firm value function.

$$V^1(s_1)^* = p[G(m^*, n(m^*)) - \sigma(s_1 - m^*) - \omega n(m^*)] + \beta \mathbb{E}_\xi V^0(s_1 - m) \tag{9}$$

And we also need to check if the firm finds it optimal to choose $m = s_1$ exhausting inventories completely and paying no storage costs (the corner solution). After finding the optimal solution check the firm would not prefer to set $m = s_1$, by constructing the value function of the firm for this choice and comparing it to $V^1(s_1)^*$:

$$V^{m=s}(s_1) = p[G(s_1, n(s_1)) - \omega n(s_1)] + \beta \mathbb{E}_\xi V^0(0) \tag{10}$$

1.5 Inner loop (II): Iterate to solve the firm's value functions

Having solved for the optimal choice ($m(s)^*$ and V^a) we need to check for convergence. We already have V^1 but we need to create $\mathbb{E}_\xi V^0(s_1 - m)$ from:

$$V^0(s,\xi) = pqs + \max\big\{ -p\omega\xi + V^a, -pqs + V^1(s)\big\} \tag{11}$$

We need to find the threshold cost draw ξ^T that makes a firm indifferent between adjusting and not-adjusting the inventories holdings *and* calculate the share of firms adjusting:

$$-p\omega\tilde{\xi}(s) + V^a = -pqs + V^1(s) \tag{12}$$

Check in bounds $\xi^T = \min(\max(\underline{\xi}, \tilde{\xi}), \bar{\xi})$. The share adjusting is $Pr(\xi \leq \xi^T)$ which is straightforward for uniform H. Let $\mathcal{H}(s) = Pr(\xi \leq \xi^T)$.

We can now integrate out ξ by using the share of firms adjusting and not adjusting to form the maximised firm value function. Note we also need to calculate the expected costs for firms that adjust. Again straightforward for a uniform distribution:

$$\mathbb{E}_\xi V^0(s) = \mathcal{H}(s)(pqs + V^a) - p\omega \int_{\underline{\xi}}^{\xi^T} \xi H(d\xi) + (1 - \mathcal{H}(s))V^1(s) \tag{13}$$

I use the updated $V^1(s)$ here.

Iterate until: $\max\big\{ \max\{|TV^1 - V^1|\}, \max\{|T\mathbb{E}V^0 - \mathbb{E}V^0|\}\big\} < \epsilon_{vfi}$

1.6 Inner loop (III): Inventories sequence

Once we have found the firm value functions we need to find the actual choices made by firms.

Notice: Because all firms are identical and there are no shocks all reset to the same s^* use the same m in production and start with the same s_1 next period. [4]

Therefore we can solve sequentially from s^* in the periods since last adjustment space. We must do this until all firms have adjusted or exhasted inventories, which will occur in finite number of periods. These points will <u>not</u> be on the grid $\mathcal{S}$.

1. We already know the level of s^* chosen by firms that adjust their inventories stock. Beginning at s^* solve the optimal sub period problem (1.4.2) and share of firms adjusting implied by equation (12).

2. Start each new period with the implied end of period inventories $s' = s_1 - m(s_1)$

3. If the starting inventory stock is sufficiently small: $s_j < \epsilon_0$ assume the firm uses all remaining inventories this period $m = s_j$.

4. We need to record the share of firms adjusting at each point in the sequence: $\mathcal{H}(s)$ and other outputs relevant for calculating the economy's aggregates.

5. Finally, check whether all firms up to J^{max} have used all inventories. If not increase J^{max}.[5] If prior to this $(\tilde{J}^{max})$ we only need to calculate the distribution up to this point $(\tilde{J}^{max}) \leq J^{max}$

6. We should now have a sequence of beginning of period inventory stock $\hat{\mathcal{S}} \equiv \{s_j\}_{j=0}^{J^{max}}$ and associated policies, where $s_0 = s^*$.

1.7 Inner loop (I): Compute the final good distribution

Next compute the share of firms at each point on this distibution $\mu(s)$ given the transition function $\mathcal{H}(s)$. We can do this in the usual way or by finding the eigenvector associated with the unit eigenvalue.

Alternatively, the transition matrix here allows for a simpler computation. We can calculate the probabilty that a firm has not adjusted after j periods. Because of the ergodic distribution, this can be reinterpreted as a distribution over states:

1. Start with a measure of 1 firms at the adjustment inventory level $s^* - m(s^*)$.

2. Solve forward to find the mass of firms that haven't adjusted from $j = 1, ..., J^{max}$

[4] The cost drawn only affects *whether* a firm adjusts and not its other policies.

[5] This should not be the case in the correct solution

3. At J^{max} we need to capture the mass of firms that will still adjust at some period in the future. So set $\mu(s_{J^{max}}) = \mu(s_{J^{max}})/\mathcal{H}(s_{J^{max}})$.[6]

4. Rescale the distribution so the total measure of firms equals 1.[7]

1.8 Outer loop: Market clearing

Now we need to check market clearing and update our guess of the output price p. Remember that in equilibrium $p = U_1(C, 1 - N)$

1. Use the distribution μ to find total intermediate goods demand X.

2. Invert the intermediate goods production to find the implied capital stock K and L.

$$X = \bar{z}F(K, L) \tag{14}$$

3. Calculate total household consumption:

$$C = \sum_{j=0}^{J^{max}} \big[G(m(s_j), n(s_j)) - \sigma(s_j - m(s_j))\big]\mu(s_j) - \delta K \tag{15}$$

4. Given household preferences $p = u'(C) = 1/C$.

   - Update the guess for p
   - If $p < 1/C$ then $p^* > p$. If $p > 1/C$ then $p^* < p$

5. Use bi-section to find p^*.

1.9 Questions

You can check that your code works by comparing the distribution of final goods firms with **Table 2**. Don't worry if the numbers are a little out, its most like a numerical feature of the computation.

1. Now set the parameter $\bar{\xi} = 0.333$ to simulate the *poor inventory management* environment of pre-1984. Check if you need to increase the max number of inventory states J^{max}.

2. Recalculate the distribution. How does it change? What happens to the target inventory level? Submit this table with your code.

3. (Without calculating) What do you think would happen to the volatility of GDP under this parameter change?

[6]This is essentially an infinite sum of the future probabilities of adjustment conditional on arriving at period J^{max}. Notice that as firms at this point have exhausted all their inventories they have same adjustment policy for all time $\mathcal{H}(0)$.

[7]Note: we are making use of the homogeneity of the transition function