Contract theory, 'putting it all together'

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1/16/2022

First best

Risk aversion means the wage is flat. The participation constraint means that $\ln c(a) = a^3/3 \iff c(a) = e^{a^3/3}$.

The principal wants to maximize profits: $1 + a - c(a)$. The first order condition gives: $1 - a^2 e^{a^3/3} = 0$. I solve for this value using R.

```
marginal_profit <- function(a){
  1- exp((a^3)/3) * (a^2)
}

first_best_effort <- uniroot(marginal_profit, c(0,1))$root

first_best_wage <- exp(first_best_effort^3/3)
```

1st best effort level: 0.8893947.

1st best wage: 1.2642898.

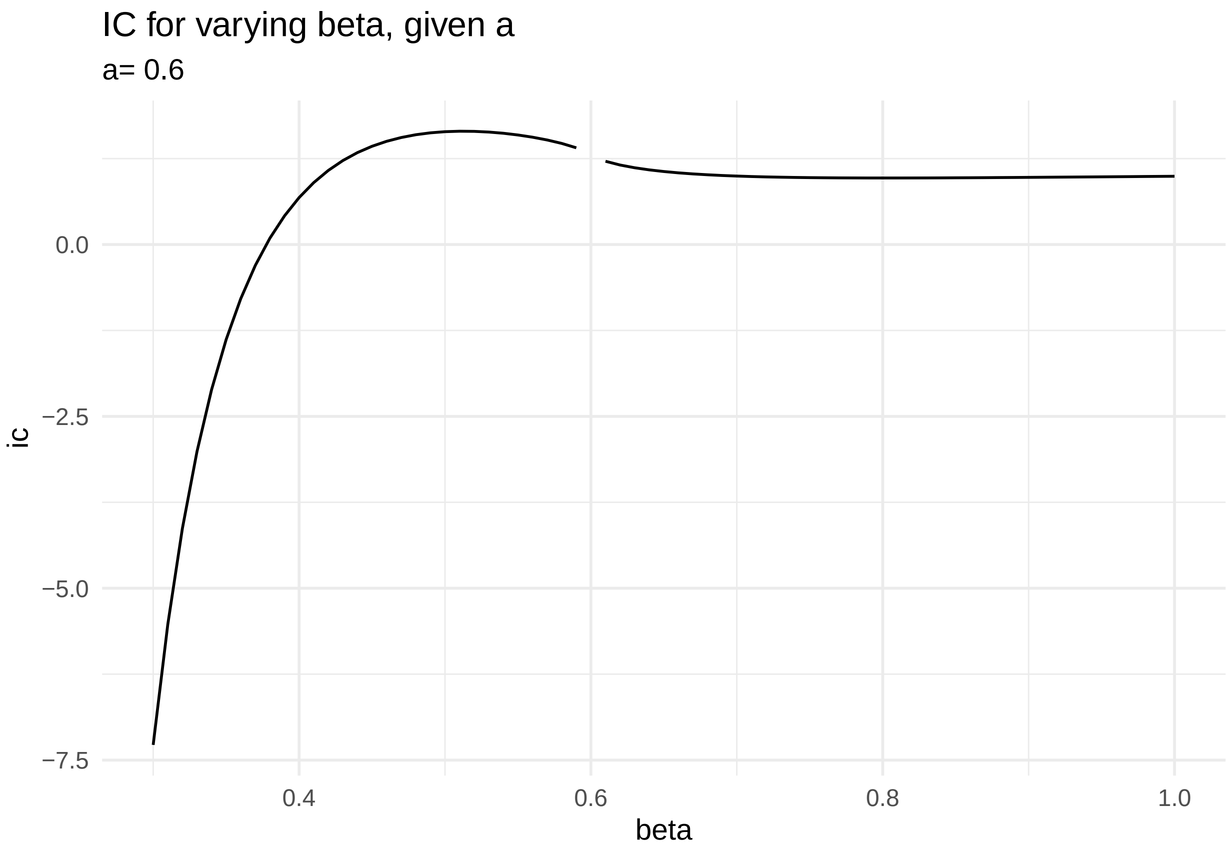
Determine the minimum feasible bonus given u(c)

To solve this exercise, I make use of the analytical expressions provided in the hints to solve first for $\tilde{\beta}$ and then for w.

```
pc <- function(w, beta_t, a){
  pc <- log(w) + log(1-beta_t/a) - exp(-(1-a/beta_t))*expint(1-a/beta_t) - a^3/3
}

ic <- function(beta_t, a){
  numerator <- 1- (1-a/beta_t) * exp(-(1-a/beta_t)) * expint(1-a/beta_t)
  denominator <- a

  ic <- numerator/denominator - a^2
}
```

Before starting to solve for beta tilde given a, let's look at how the IC behaves around a given value for effort.

```
## Warning in expint(1 - a/beta_t): NaNs produced
```

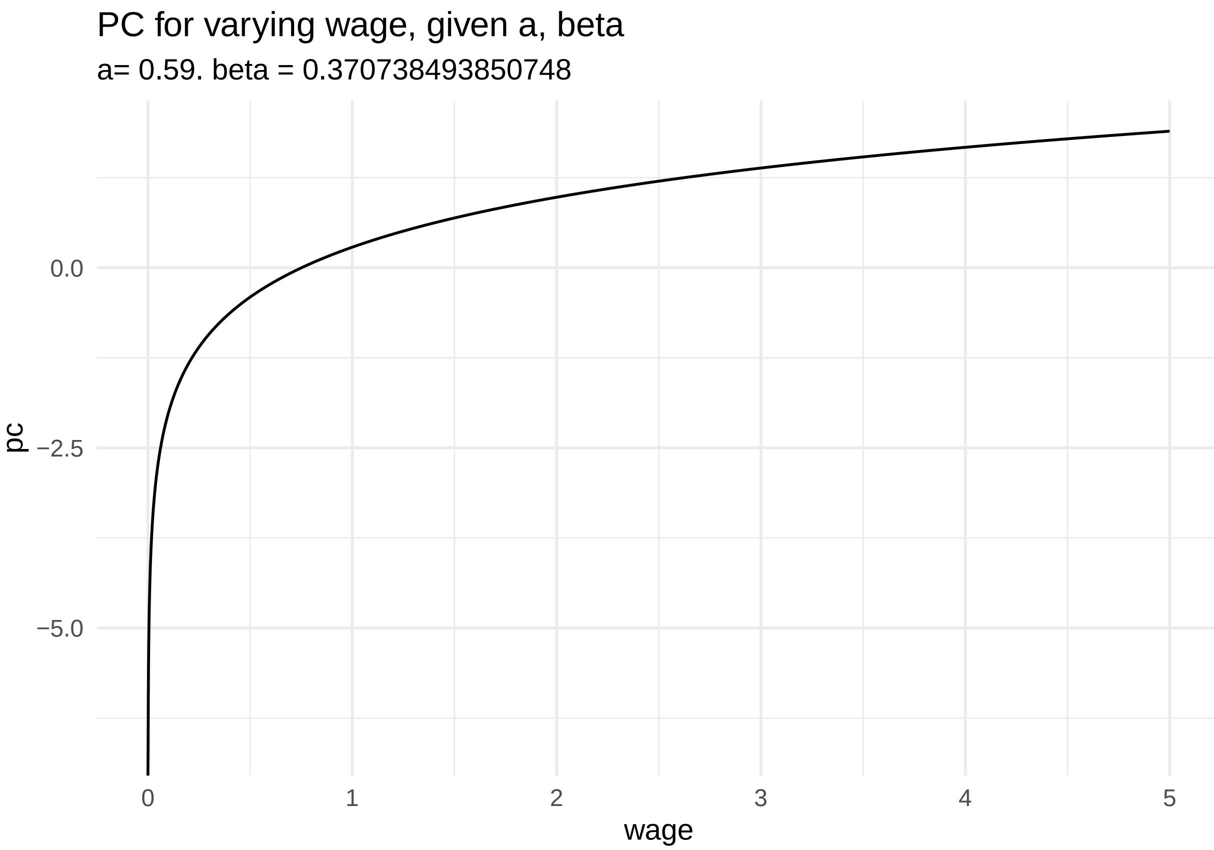


The function is seemingly not well defined near the effort level which is due to the fact that EI(0) is not well defined. Use this information to constrain the search interval. As the descent in the left tail is steep, it also makes sense to limit the root finding to positive values.

```
# Seemingly not well defined near a. Limit search from above based on a.

solve_for_b <- function(a){
  interval_min <- 0.05
  interval_max <- a - 0.01
  root <- uniroot(f = ic, interval = c(interval_min, interval_max), a = a)
}

a_vals <- seq(10,100)/100

df_ic <- do.call(rbind, lapply(a_vals, solve_for_b)) %>% cbind(a_vals)
```

Having found beta tilde given the effort, we can now solve for the wage level given beta tilde and the effort.

I begin by plotting the relationship for a given pair of effort and beta tilde.

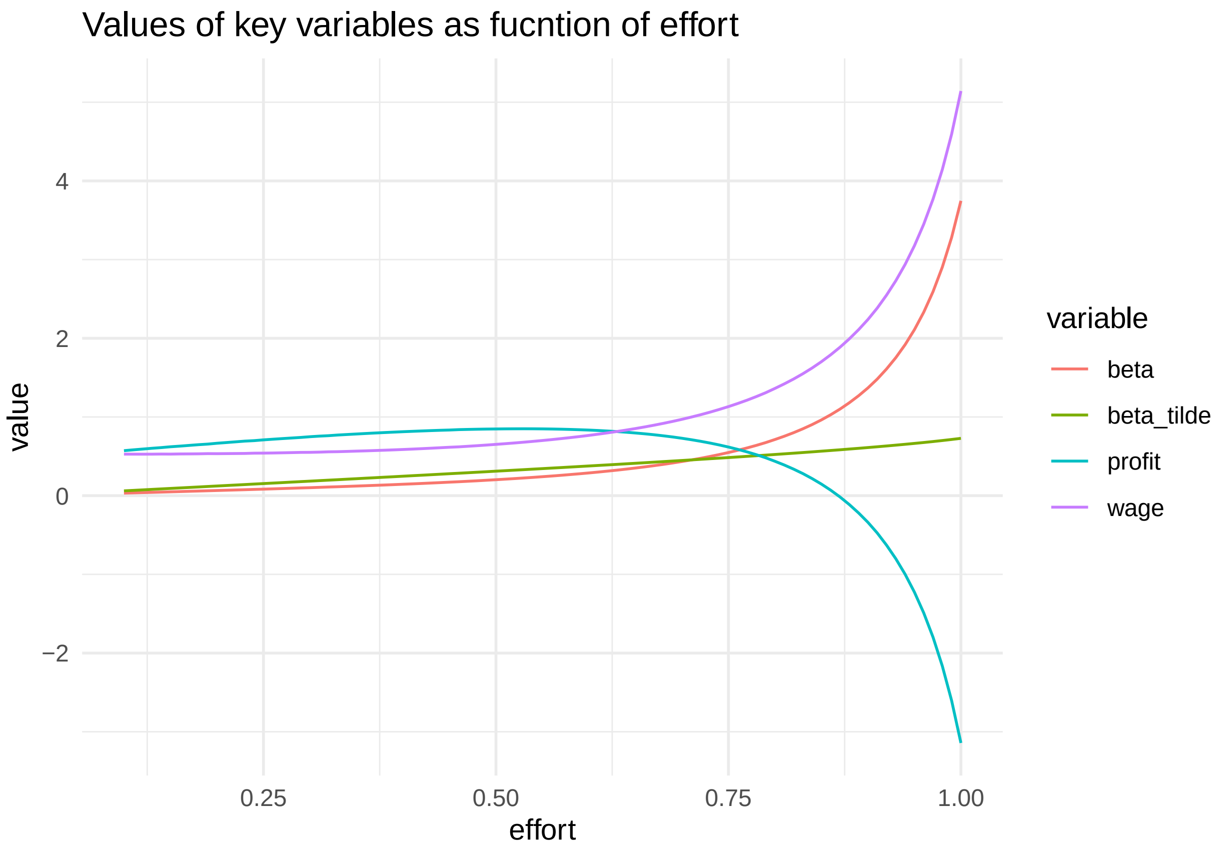


Next, solve for the wage. Computationally, I solve again for the beta tilde, given the effort, and then the wage given these two values.

```
solve_for_w <- function(a_val){
  # Find beta, given a
  beta <- solve_for_b(a_val)$root
  # Set interval limits (based on arbitrary guess)
  interval_min <- 0
  interval_max <- 10
  # Search over PC for a wage that makes the PC binding.
  root <- uniroot(f = pc, interval = c(interval_min, interval_max), beta = beta, a = a_val)
  # Output the desired values
  data_row <- cbind(beta_tilde = beta, wage = root$root, effort = a_val)
}

df <- do.call(rbind, lapply(a_vals, solve_for_w))
```

Now we are ready to plot the relationships.

```
df <- as.data.frame(df) %>%
  mutate(beta = beta_tilde * wage,
         profit = 1 + effort - wage) # Note that the expected incentive payout is 0 so beta doesn't aff
```



Solve for optimal effort

Since the profits have already been calculated and stored in a data frame, finding the optimal effort level is simple.

```
optimal_contract <- df %>% filter(profit == max(df$profit))
optimal_contract
```

```
##   beta_tilde      wage effort      beta    profit
## 1  0.3309003 0.6798645   0.53 0.2249673 0.8501355
```

As the principal must now compensate the agent for the agent's risk aversion, we find that the optimal effort is lower than under first best conditions.

Plot the agent's objective function given the equilibrium compensation

For this exercise, we want to plot the agent's expected utility given the optimal contract and check that it is concave and finds a maximum for the effort level stipulated by the contract.

Calculate the agent's expected utility as:

$$\mathbb{E}^a[\log(wage^* + \beta^* L(x|a))] = \int_0^\infty \log(wage^* + \beta^* L(x|a)) f(x|a) dx$$

For some values of a and confirm that the expectation is maximized with a^*. Note that the exponential distribution has support for $x \geq 0$.

```r
Lx <- function(a, x){
  # Continuous action, continuous distribution likelihood ratio
  L = (x-a)/a^2
}

expected_utility <- function(x, a, wage_opt, beta_opt){
  u <- log(wage_opt + beta_opt * Lx(a, x)) * dexp(x, rate = 1/a)
}

evaluate_exp_utility <- function(effort){
  eu <- integrate(expected_utility, lower = 0, upper = Inf,
                  a = effort, wage_opt = optimal_contract$wage,
                  beta_opt = optimal_contract$beta)
  eu <- eu$value - (effort^3)/3
  data_row <- cbind(exp_utility = eu, effort)
}


df.agent <- do.call(rbind, lapply(seq(40,90)/100, evaluate_exp_utility))
```