



The Scott Topology

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Preordered & partially ordered sets

Definition: Directed sets

Let L be a preordered set. A subset D of L is directed provided it is nonempty and every finite subset of D has an upper bound in D .

Definition: Lower sets

$$\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$$

X is an *lower set* if $X = \downarrow X$.

Definition: Dcpo

A poset is complete with respect to directed sets if every directed set has supremum. A **dcpo** is a **directed complete poset**.



The “Way Below”-relation

Definition

Let L be a poset. x is way below y , in symbols $x \ll y$, iff for all directed subsets $D \subseteq L$, for which $\sup D$ exists, the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$.

$$\downarrow x = \{u \in L : u \ll x\}$$



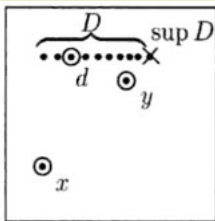
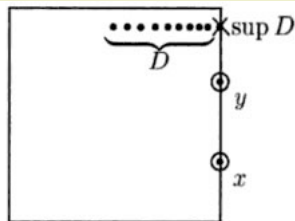


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Example


 $x \ll y$
 $L = [0, 1]^2$

 $x < y$ but not $x \ll y$



The “Way Below”-relation

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Example

Let X be a topological space and $\mathcal{O}(X)$ the complete lattice of open sets in X . Suppose $U, V \in \mathcal{O}(X)$ and define the order \leq as the inclusion relation \subseteq .

$$U \ll V \Leftrightarrow \forall \mathcal{G} \text{ open cover of } V, \\ \exists \mathcal{G}' \subseteq \mathcal{G} \text{ finite cover of } U.$$



Domains

Definition: Continuous posets

A poset L is called *continuous* if $\downarrow x$ is directed with supremum x for all $x \in L$.

Definition: Domain

A dcpo which is continuous as a poset will be called a *domain*.

Example

Let M be a set and $L = 2^M$, then 2^M is a continuous lattice and thus a domain.





\mathcal{S} -limits on complete semilattices

Definition: Lower limit

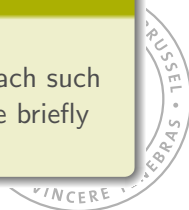
Let L be a complete semilattice. For any net $(x_j)_{j \in J}$ we write

$$\liminf_j x_j = \sup_j \inf_{i \geq j} x_i,$$

and call $\liminf_j x_j$ the *lower limit* or *liminf* of the net.

Definition: \mathcal{S} -limit

Let \mathcal{S} denote the class of the pairs $((x_j)_{j \in J}, x)$ such that $x \leq \liminf_j x_j$ ($\mathcal{S} = \{((x_j)_{j \in J}, x) \mid x \leq \liminf_j x_j\}$). For each such pair we say that x is an \mathcal{S} -*limit* of $(x_j)_{j \in J}$ and we write briefly $x \equiv_{\mathcal{S}} \lim x_j$.





The Scott Topology

Theorem: The general relation between convergence & topology

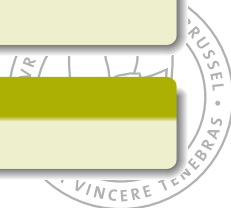
Let \mathcal{L} be a class of pairs $((x_j)_{j \in J}, x)$ consisting of a net and an element of L , then

$$\mathcal{O}(\mathcal{L}) = \{U \subseteq L \mid \text{if } ((x_j)_{j \in J}, x) \in \mathcal{L} \text{ and } x \in U \text{ then eventually } x_j \in U\}$$

is a topology.

Definition: The Scott topology

Take $\mathcal{L} = \mathcal{S}$, then $\mathcal{O}(\mathcal{S})$ is the *Scott topology*.



The Scott topology

Theorem

Let L be dcpo and $U \subset L$. Then $U \in \mathcal{O}(S)$ iff the following two conditions are satisfied:

- (i) $U = \uparrow U$;
- (ii) $\sup D \in U$ implies $D \cap U \neq \emptyset$ for all directed set $D \subseteq L$.





The Scott topology

Example: the Sierpinski topology

On the chain $L = \{0, 1\}$, the *Scott topology* equals the *Sierpinski topology*.

Example: the unit interval

If L is the unit interval: $L = [0, 1]$, then L is a dcpo. Any Scott open set has the form $]x, 1]$ if $0 \leq x \leq 1$ or $[0, 1]$.





The Scott topology on domains

Definition: Topological

If \mathcal{S} is precisely the class of convergent nets for the Scott topology, then we say that \mathcal{S} is *topological*.

Theorem

\mathcal{S} -convergence is topological $\Leftrightarrow L$ is a domain





Definition

Theorem

For a function $f : S \rightarrow T$, with S, T dcpo's, the following conditions are equivalent:

- 1. f is continuous with respect to the Scott topologies, that is*

$$f^{-1}(U) \in \sigma(S), \forall U \in \sigma(T),$$

- 2. f preserves suprema of directed sets, that is, f is order preserving and $f(\sup D) = \sup f(D)$, for all directed subsets D of S .*

Definition: Scott-continuous

A function $f : S \rightarrow T$ between dcpo's is *Scott-continuous* iff it satisfies the equivalent conditions in the previous theorem.

Kleene Fixed-Point theorem

Theorem: Kleene Fixed-Point theorem

Let L be a dcpo with a least element \perp , then it has the following properties:

- (i) **Existence:** Every Scott-continuous self-map $f : L \rightarrow L$ has a least fixed-point, notated by $\text{LFP}(f)$.
- (ii) **Construction:** The least fixed-point can be approximated by the recursively defined *Kleene chain*:

$$x_0 = \perp, \quad x_{n+1} = f(x_n) = f^{n+1}(\perp)$$

in the sense that

$$\text{LFP}(f) = \sup_n x_n = \sup_n f^n(\perp).$$



Kleene Fixed-Point theorem

Example: The factorial function

The definition of factorial as the function that maps $n \in \mathbb{N}$ to $f(n)$ = if $n = 0$ then 1 else $n.f(n-1)$ is obtained as the least fixed point of the higher-order function F , mapping any function f to the function f' defined by $f'(n)$ = if $n = 0$ then 1 else $n.f(n-1)$.



Definitions

Category: $CONT$

The category whose objects are *continuous lattices* and whose morphisms are *Scott-continuous maps* will be denoted by $CONT$.

Definition: Σ -functor

Let L be a continuous lattice, we call $\Sigma : CONT \rightarrow TOP$ the functor that associates to L its Scott topology $\Sigma(L)$.





Definitions

Definition: Specialization order

The relation

$$x \leq y \iff x \in \overline{\{y\}}$$

is called the *specialization order*.

Definition: Ω -functor

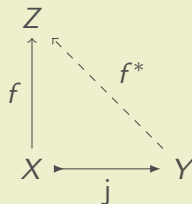
We denote by $\Omega : TOP \rightarrow POSET$ the functor which associates with a space X the poset $\Omega X = (X, \leq)$, where \leq is the specialization order, and with $\Omega f = f$.





Definition: Injective spaces

A T_0 -space Z is called *injective* iff every continuous map $f : X \rightarrow Z$ extends continuously to any space Y containing X as a subspace.



Category: *INJ*

The category *INJ* is the full subcategory of *TOP* consisting of *injective spaces* and all continuous maps.



Theorem

- (i) *If L is a continuous lattice, then $\Sigma L = (L, \sigma(L))$ is an injective space and $\Omega \Sigma L = L$.*
- (ii) *If X is an injective T_0 -space, then $\Omega X = (X, \leq)$ is a continuous lattice (with respect to the specialization order) and $\Sigma \Omega X = X$.*





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