Vrije Universiteit Brussel

Similarity on Graphs & Hypergraphs

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Content

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Preordered & partially ordered sets

Definition: Directed sets

Let L be a preordered set. A subset D of L is directed provided it is nonempty and every finite subset of D has an upper bound in D.

Definition: Lower sets

$$\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$$

X is an *lower set* if $X = \downarrow X$.

Definition: Dcpo

A poset is complete with respect to directed sets if every directed set has supremum. A **dcpo** is a **d**irected **c**omplete **po**set.

The "Way Below"-relation

Definition

Let L be a poset. x is way below y, in symbols $x \ll y$, iff for all dircted subsets $D \subseteq L$, for which $\sup D$ exists, the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$.

$$\twoheaddownarrow x = \{u \in L : u \ll x\}$$

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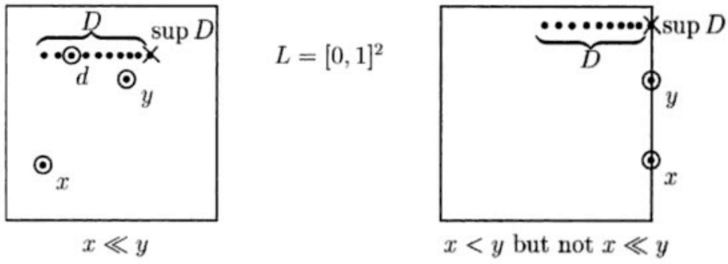
Example



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Example

Let X be a topological space and $\mathcal{O}(X)$ the complete lattice of open sets in X. Suppose $U, V \in \mathcal{O}(X)$ and define the order $\leq$ as the inclusion relation $\subseteq$.

$$\begin{aligned} U \ll V \Leftrightarrow & \forall \mathcal{G} \text{ open cover of } V, \\ & \exists \mathcal{G}' \subseteq \mathcal{G} \text{ finite cover of } U. \end{aligned}$$

Domains

Definition: Continuous posets

A poset L is called *continuous* if $\downarrow x$ is directed with supremum x for all $x \in L$.

Definition: Domain

A dcpo which is continuous as a poset will be called a *domain*.

Example

Let M be a set and $L = 2^M$, then 2^M is a continuous lattice and thus a domain.

$\mathcal{S}$-limits on complete semilattices

Definition: Lower limit

Let L be a complete semilattice. For any net $(x_j)_{j\in J}$ we write

$$\liminf_j x_j = \sup_j \inf_{i\geq j} x_i,$$

and call $\liminf_j x_j$ the *lower limit* or *liminf* of the net.

Definition: $\mathcal{S}$-limit

Let $\mathcal{S}$ denote the class of the pairs $((x_j)_{j\in J}, x)$ such that $x \leq \liminf_j x_j$ $(\mathcal{S} = \{((x_j)_{j\in J}, x) | x \leq \liminf_j x_j\})$. For each such pair we say that x is an $\mathcal{S} - \mathit{limit}$ of $(x_j)_{j\in J}$ and we write briefly $x \equiv_{\mathcal{S}} \lim x_j$.

The Scott Topology

Theorem: The general relation between convergence & topology

Let $\mathcal{L}$ be a class of pairs $((x_j)_{j\in J}, x)$ consisting of a net and an element of L, then

$$\mathcal{O}(\mathcal{L}) = \{U \subseteq L \mid \text{ if } ((x_j)_{j\in J}, x) \in \mathcal{L} \text{ and } x \in U \text{ then eventually } x_j \in U\}$$

is a topology.

Definition: The Scott topology

Take $\mathcal{L} = \mathcal{S}$, then $\mathcal{O}(\mathcal{S})$ is the *Scott topology*.

The Scott topology

Theorem

Let L be dcpo and $U \subset L$. Then $U \in \mathcal{O}(\mathcal{S})$ iff the following two conditions are satisfied:

(i) $U = \uparrow U$;

(ii) $\sup D \in U$ *implies* $D \cap U \neq \emptyset$ *for all directed set* $D \subseteq L$.

The Scott topology

Example: the Sierpinski topology

On the chain $L = \{0, 1\}$, the *Scott topology* equals the *Sierpinski topology*.

Example: the unit interval

If L is the unit interval: $L = [0, 1]$, then L is a dcpo. Any Scott open set has the form $]x, 1]$ if $0 \leq x \leq 1$ or $[0, 1]$.

The Scott topology on domains

Definition: Topological

If $\mathcal{S}$ is precisely the class of convergent nets for the Scott topology, then we say that $\mathcal{S}$ is *topological*.

Theorem

$$\mathcal{S}\text{-convergence is topological} \Leftrightarrow L \text{ is a domain}$$

Definition

Theorem

For a function $f : S \to T$, with S, T dcpo's, the following conditions are equivalent:

1. *f is continuous with respect to the Scott topologies, that is*

$$f^{-1}(U) \in \sigma(S),\ \forall U \in \sigma(T),$$

2. *f preserves suprema of directed sets, that is, f is order preserving and $f(\sup D) = \sup f(D)$, for all directed subsets D of S.*

Definition: Scott-continuous

A function $f : S \to T$ between dcpo's is *Scott-continuous* iff it satisfies the equivalent conditions in the previous theorem.

Kleene Fixed-Point theorem

Theorem: Kleene Fixed-Point theorem

Let L be a dcpo with a least element $\perp$, then it has the following properties:

(i) **Existence**: Every Scott-continuous self-map $f : L \to L$ has a least fixed-point, notated by $\mathrm{LFP}(f)$.

(ii) **Construction**: The least fixed-point can be approximated by the recursively defined *Kleene chain*:

$$x_0 = \perp,\ x_{n+1} = f(x_n) = f^{n+1}(\perp)$$

in the sense that

$$\mathrm{LFP}(f) = \sup_n x_n = \sup_n f^n(\perp).$$

Kleene Fixed-Point theorem

Example: The factorial function

The definition of factorial as the function that maps $n \in \mathbb{N}$ to `f(n) = if` n `= 0 then 1 else n.`$f(n-1)$ is obtained as the least fixed point of the higher-order function F, mapping any function f to the function f' defined by `f(`n`) = if` n `= 0 then 1 else n.`f`(n−1)`.

Definitions

Category: *CONT*

The category whose objects are *continuous lattices* and whose morhpisms are *Scott-continuous maps* will be denoted by $CONT$.

Definition: Σ-functor

Let L be a continuous lattice, we call $\Sigma : CONT \to TOP$ the functor that associates to L its Scott topology $\Sigma(L)$.

Definitions

Definition: Specialization order

The relation

$$x \leq y \Longleftrightarrow x \in \overline{\{y\}}$$

is called the *specialization order*.

Definition: Ω-functor

We denote by $\Omega : TOP \to POSET$ the functor which associates with a space X the poset $\Omega X = (X, \leq)$, where $\leq$ is the the specialization order, and with $\Omega f = f$.

Definition: Injective spaces

A T_0-space Z is called *injective* iff every continuous map $f : X \to Z$ extends continuously to any space Y containing X as a subspace.

$$\begin{tikzcd}
Z & \\
X \arrow[u, "f"] \arrow[r, tail, "j"'] & Y \arrow[ul, dashed, "f^*"']
\end{tikzcd}$$

Category: *INJ*

The category *INJ* is the full subcategory of *TOP* consisting of *injective spaces* and all continuous maps.

Theorem

(i) *If L is a continuous lattice, then $\Sigma L = (L, \sigma(L))$ is an injective space and $\Omega\Sigma L = L$.*

(ii) *If X is an injective T_0-space, then $\Omega X = (X, \leq)$ is a continuous lattice (with respect to the specialization order) and $\Sigma\Omega X = X$.*

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