Vrije Universiteit Brussel

Game Theory: Distributed Selfish Load Balancing on Networks

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- Coordination mechanisms

Definition

A strategic game $\langle N, (A_i), \succeq_i \rangle$ consists of:

- a finite set N (the set of **players**),
- for each player $i \in N$ a nonempty set A_i (the **set of actions** available to player i),
- for each player $i \in N$ a preference relation $\succeq_i$ on $A = \times_{j \in N} A_j$ (the **preference relation** of player i).

Remark

The preference relation $\succeq_i$ of player i in a strategic game can be represented by a **payoff function** or **utility function** $u_i : A \to \mathbb{R}$, in the sense that $u_i(a) \geq u_i(b)$ whenever $a \succeq_i b$.

Pure and mixed strategy profiles

Pure strategy profile

$$a = (a_1, ..., a_n) \in A, a_i \in A_i$$

Mixed strategy profile

$$\alpha = (\alpha_i)_{i \in N} \in \Delta(A), \alpha_i(a_j) = \mathbb{P}[A_i = a_j]$$

Now,

$$\mathbb{P}[\alpha = a] = \prod_{i \in N} \alpha_i(a_i)$$

The expected pay off for player i under a mixed strategy profile α:

$$U_i(\alpha) = \sum_{a \in A} \left(\prod_{j \in N} \alpha_j(a_j) \right) u_i(a)$$

Nash equilibria

Pure Nash equilibrum

A pure strategy profile $a^* \in A$ is a **pure Nash Equilibrum** if for each player $i \in N$:

$$u_i(a^*_{-i}, a^*_i)) \geq u_i(a^*_{-i}, a_i) \quad \forall a_i \in A_i$$

Mixed Nash equilibrum

A mixed strategy profile α^* is a **mixed Nash Equilibrum** if for each player $i \in N$:

$$U_i(\alpha^*_{-i}, \alpha^*_i)) \geq U_i(\alpha^*_{-i}, \alpha_i) \quad \forall \alpha_i$$

Theorem of Nash

Theorem

Every finite strategic game has a mixed Nash equilibrum.

Lemma: Brouwer fixed point theorem

Let X be a **non-empty**, convex and compact set. If $f : X \rightarrow X$ is continuous, then there must exist $x \in X$ such that $f(x) = x$.

Proof.

$\Delta(A_i)$ is the set of mixed strategy profiles of a player i. Note that $(\alpha_i(a_1), ..., \alpha_i(a_k))$ with $a_j \in A_i$ (the pure actions of player i are the elements in $\Delta(A_i)$.

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$\Delta(A_i)$ is the set of mixed strategy profiles of a player i. Note that $(\alpha_i(a_1), \ldots, \alpha_i(a_k))$ with $a_j \in A_i$ (the pure actions of player i are the elements in $\Delta(A_i)$.

- The set $\Delta(A_i)$ is **non-empty** by definition of a strategic game.

Theorem of Nash

Lemma: Brouwer fixed point theorem

Let X be a non-empty, **convex** and compact set. If $f : X \to X$ is continuous, then there must exist $x \in X$ such that $f(x) = x$.

Proof.

- To proof that the set $\Delta(A_i)$ is **convex**, take $\vec{x} = (\alpha_i^x(a_1), ..., \alpha_i^x(a_k))$ and $\vec{y} = (\alpha_i^y(a_1), ..., \alpha_i^y(a_k))$ then $\vec{z} = \theta\vec{x} + (1 - \theta)\vec{y}$ for some $\theta \in [0, 1]$ is in $\Delta(A_i)$ because $\vec{z}$ is also a mixed strategy for player i (the sum of the components of $\vec{z}$ is 1).

Theorem of Nash

Lemma: Brouwer fixed point theorem

Let X be a non-empty, convex and **compact** set. If $f : X \to X$ is continuous, then there must exist $x \in X$ such that $f(x) = x$.

Proof.

- The **compactness** in $\mathbb{R}^k$ can be shown by proving that the set is closed and bounded. The set is bounded because $0 \leq \alpha_i(a_j) \leq 1$. To proof closeness in $\mathbb{R}^k$, we'll proof that the limit of every convergent sequence in $\Delta(A_i)$ is an element of $\Delta(A_i)$. Consider a convergent sequence in $\Delta(A_i) : ((\alpha_i^n(a_1), ..., \alpha_i^n(a_k))_n \to (\alpha_i^*(a_1), ..., \alpha_i^*(a_k))$.

Theorem of Nash

Lemma: Brouwer fixed point theorem

Let X be a non-empty, convex and **compact** set. If $f : X \to X$ is continuous, then there must exist $x \in X$ such that $f(x) = x$.

Proof.

$$\sum_{j=1}^{k} \alpha_i^*(a_j) = \sum_{j=1}^{k} \lim_{n\to\infty} \alpha_i^n(a_j) = \lim_{n\to\infty} \sum_{j=1}^{k} \alpha_i^n(a_j) = \lim_{n\to\infty} 1 = 1$$

This means that $(\alpha_i^*(a_1), ..., \alpha_i^*(a_k))$ is also a mixed strategy for player i, but by definition of $\Delta(A_i)$, this limit belongs to $\Delta(A_i)$. $\square$

Congestion Model

Definition

A **congestion model** $(N, M, (A_i)_{i \in N}, (c_j)_{j \in M})$ is defined as follows:

- a finite set N of **players**. Each player i has a **weight** (or demand) $w_i \in \mathbb{N}$,
- a finite set M of **facilities**.
- For $i \in N$, A_i denotes the set of **strategies** of player i, where each $a_i \in A_i$ is a non-empty **subset of the facilities**,
- For $j \in M$, c_j is a **cost function** $\mathbb{N} \to \mathbb{R}$, $c_j(k)$ denotes the cost related to the use of facility j under a certain load k;

Congestion Games

Definition: Congestion model as strategic game

- a finite set N of **players**,
- for each player $i \in N$, there is a nonempty set of **strategies** A_i,
- The preference relation $\succeq_i$ for each player i is defined by a **payoff function** $u_i : A \to \mathbb{R}$. For any $a \in A$ and for any $j \in M$, let $\ell_j(a)$ be *the expected load on facility* j, assuming a is the current pure strategy profile, so $\ell_j(a) = \sum_{\substack{i \in [n] \\ j \in a_i}} w_i$.

  Then the payoff function for player i becomes: $u_i(a) = \sum_{j \in a_i} c_j(\ell_j(a))$.

Theorem of Rosenthal

Theorem

Every congestion game has a pure Nash equilibrium.

Load balancing games

Definition

A **load balancing game** is congestion game based on a congestion model with:

- a finite set N of **tasks** (each task i has a weight w_i),
- for each player $i \in N$, there is a nonempty set of **machines**A_i with $A_i \subset M$. The elements of A_i are the possible machines on which task i can be executed.
- the preference relation $\succeq_i$ for each client i is defined by a **payoff function** $u_i : A \to \mathbb{R}$. For any $a \in A$ and for any $j \in M$, let $\ell_j(a)$ be *the expected load on machine* j, assuming a is the current pure strategy profile $\left(\ell_j(a) = \sum_{\substack{i\in[n] \\ j=a_i}} w_i\right)$.

  Then the payoff function for task i becomes: $u_i(a) = c_{a_i}(\ell_{a_i}(a))$.

Lineair cost functions

Payoff function: $u_i(a) = c_{a_i}(\ell_{a_i}(a))$

Take: $c_j(k) = \frac{k}{s_j}$, s_j: speed of machine j.

Pure strategies

The **payoff function**:

$$u_i(a) = c_{a_i}(\ell_{a_i}(a)) = \frac{\ell_{a_i}(a)}{s_{a_i}}, a \in A$$

The **makespan**:

$$\mathrm{cost}(a) = \max_{j \in [m]} c_j(\ell_j(a)) = \max_{j \in [m]} \frac{\ell_j(a)}{s_j}$$

Lineair cost functions

Payoff function: $u_i(a) = c_{a_i}(\ell_{a_i}(a))$

Take: $c_j(k) = \frac{k}{s_j}$, s_j: speed of machine j.

Mixed strategies

The **expected payoff function**:

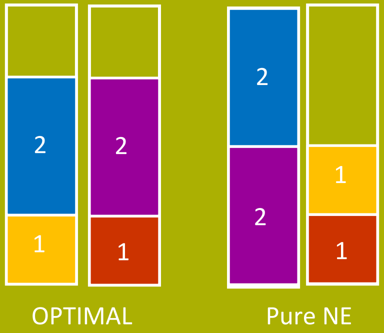
$$U_i^j(\alpha) = \frac{w_i + \sum_{k \neq i} w_k \alpha_k(j)}{s_j}$$

The **makespan**:

$$\mathrm{cost}(\alpha) = \mathbb{E}[\mathrm{cost}(a)] = \mathbb{E}\left[\max_{j \in [m]} \frac{\ell_j(a)}{s_j}\right]$$

A very easy example

Figure



- $\text{cost}(a_{\text{opt}}) = \max(3, 3) = 3$
- $\text{cost}(a_2) = \max(4, 4) = 4$

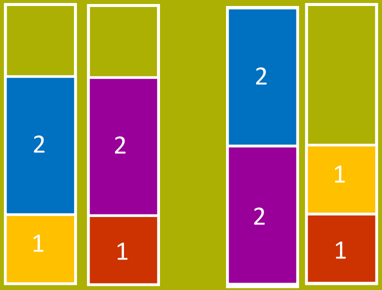
Definition

Definition

$$PoA(G) = \max_{\alpha \in \mathrm{Nash}(G)} \frac{\mathrm{cost}(\alpha)}{\mathrm{cost}(a_{\mathrm{opt}})}$$

A very easy example

Figure



OPTIMAL   Pure NE

- $PoA(G) = \frac{4}{3} = 1.33$

Bachmann-Landau notations

Definition: Big Oh

Big Oh is the set of all functions f that are bounded above by g asymptotically (up to constant factor).

$$O(g(n)) = \{f | \exists c, n_0 \geq 0 : \forall n \geq n_0 : 0 \leq f(n) \leq cg(n)\}$$

Definition: Asymptotical equality

Let f and g real functions, then f is asymptotically equal to g $\Leftrightarrow \lim_{x \to +\infty} \dfrac{f(x)}{g(x)} = 1$. Notation: $f \sim g$.

PoA in Pure Nash equilibria on uniformly related machines

Lemma

$\forall m \in \mathbb{R} : \Gamma^{-1}(m) \in O\left(\frac{\log m}{\log \log m}\right)$.

Proof.

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t$, $\Gamma^{-1}(m) = k$, then $k!$ is the greatest factorial smaller or equal to m. Because $m \sim k!$ and $k! \sim k^k$ we get:

$$\begin{aligned}
&\Rightarrow \quad m \sim k^k \\
&\Rightarrow \quad \log m \sim k \log(k) \\
&\Rightarrow \quad k \sim \frac{\log m}{\log(k)} \\
&\Rightarrow \quad k \sim \frac{\log m}{\log(\frac{\log m}{\log(k)})}
\end{aligned}$$

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$$\begin{aligned} &\Rightarrow \quad m \sim k^k \\ &\Rightarrow \quad \log m \sim k \log(k) \\ &\Rightarrow \quad k \sim \frac{\log m}{\log(k)} \\ &\Rightarrow \quad k \sim \frac{\log m}{\log(\frac{\log m}{\log(k)})} \end{aligned}$$

PoA in Pure Nash equilibria on uniformly related machines

Lemma

$\forall m \in \mathbb{R} : \Gamma^{-1}(m) \in O\left(\frac{\log m}{\log \log m}\right).$

Proof.

$$\Rightarrow \quad k \sim \frac{\log m}{\log \log m - \log \log(k)}$$

Because $m > k$:

$$\Rightarrow k \sim \frac{\log m}{\log \log m}$$

So that $\Gamma^{-1}(m) \in O\left(\frac{\log m}{\log \log m}\right)$ □

Summary

	Identical	Uniformly related
Pure NE	$2 - \frac{2}{m+1}$	$\Theta\left(\frac{\log m}{\log \log m}\right)$
Mixed NE	$\Theta\left(\frac{\log m}{\log \log m}\right)$	$\Theta\left(\frac{\log m}{\log \log \log m}\right)$

Policies

- **Shortest first**
- **Longest first**
- **Random order**
- **Round Robin**

Theorem

Under a longest-first policy, PoA for uniformly related machines is $\leq 2 - \frac{2}{m+1}$.

Theorem

Under a shortest-first policy, PoA for uniformly related machines is $\Theta(\log m)$

Taxation

Definition: Tax function

$\delta : M \times \mathbb{R} \to \mathbb{R}$

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