



Game Theory: Distributed Selfish Load Balancing on Networks

Bachelor Thesis I

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January 2013



Abstract for non-mathematicians

Explaining the subject of my Bachelor thesis to those who do not study either Mathematics or Computer Science isn't an easy task, but I can give an idea of the problem studied by giving an example. Imagine, for example, that there are only two roads from Paris to Brussels: road A & road B. If you take road A you'll always drive 5 hours, independent from the usage of road A by other drivers (*agents* is the more game theoretical word). If you take road B, the time you need depends of the number of other agents using road B: you'll drive $5\frac{x}{100}$ hours, with x the number of agents, but also on this road the maximum time spend is 5 hours, so mathematically that becomes $\min(5, 5\frac{x}{100})$ hours. Because every agent acts selfish and rational, the result under this circumstances and with only this information will be that every agent will take at any time road B: with road A they drive always 5 hours, with road B there is a chance to get to Brussels in less time. But consider now the following situation: exact 100 agents decide at the same time to go to Brussels, that means that all of them will take road B and so everyone will drive 5 hours. It would be much better that 50 agents would take road A (and thus drive 5 hours) and the other 50 take road B (and drive $5\frac{50}{100} = 2.5$ hours), that would reduce the average time to 3.75 hours! This kind of problems where the selfishness of agents reduce the power (load) of a general system take place in a wide range of real life problems: not only in traffic, but also in economics, politics and, that's more my interest, also in Computer Science: in a distributed system computers interact on a selfish base because there isn't a central computer that organizes the network. It's extremely interesting to study which information these kind of agents (computers) need to know to reduce their selfishness and reach the optimal power of the distributed network.

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1 Introduction

Whenever a set of task should be executed on a set of resources, the problem of load balancing evidently arouses. We need to balance the load among the resources in order to exploit the available resources efficiently and fair.

1.1 Makespan scheduling on uniformly related machines

One of the most fundamental load balancing problems is *makespan scheduling on uniformly related machines*. This problem is defined by m machines with speeds s_1, \dots, s_m and n tasks with weights w_1, \dots, w_n . Let $[n] = \{1, \dots, n\}$ be to set of tasks and $[m]$ the set of machines. Now, the problem is to find an assignment function $A : [n] \rightarrow [m]$ of the tasks to the machines that is as balanced as possible. The load of machine $j \in [m]$ under A is defined as:

$$\ell_j = \sum_{\substack{i \in [n] \\ j = A(i)}} \frac{w_i}{s_j}$$

The *makespan* is defined as

$$\max_{j \in [m]} (\ell_j)$$

Now, of course, the objective is to minimize the makespan. When there is a central machine, it ins't that hard to design algorithms that compute the mapping A and minimizes the makespan. Suppose, however, that there is not a central machine that can enforce an efficient mapping of the tasks to the machines (e.g. in P2P Networks). This naturally leads to the following game theoretic setting in which we will be able to analyze what happens to the makespan if there is no global authority but selfish agents aiming at maximizing their individual benefit decide about the assignment of tasks. The understand the problem and it's solution, we first give the most important game theoretical results you'll need.

1.2 Import results from Game Theory

Important note: This section doesn't aim to give the reader an introduction in Game Theory. Instead, it gives only the relevant results that the reader must know for understanding the rest of this paper.

1.2.1 Congestion games

Definition 1.1. [JOARU1994] A strategic game $\langle N, (A_i), \succeq_i \rangle$ consists of:

- a finite set N (the set of **players**)
- for each player $i \in N$ a nonempty set A_i (the **set of actions** available to player i)
- for each player $i \in N$ a preference relation \succeq_i on $A = \times_{j \in N} A_j$ (the **preference relation** of player i)

Remark 1.2. Under a wide range of circumstances the preference relation \succeq_i of player i in a strategic game can be represented by a **payoff function** $u_i : A \rightarrow \mathbb{R}$, in the sense that $u_i(a) \geq u_i(b)$ whenever $a \succeq_i b$. Sometimes this function is also called a utility function. A strategic game is then often denoted as $G = \langle N, (A_i), (u_i) \rangle$

Definition 1.3. [YMAN2003] A congestion model $(N, M, (A_i)_{i \in N}, (c_j)_{j \in M})$ is defined as follows:

- a finite set N of players
- a finite set M of facilities
- For $i \in N$, A_i denotes the set of strategies of player i , where each $a_i \in A_i$ is a non-empty subset of the facilities.
- For $j \in M$, $c_j \in \mathbb{R}^n$ denotes the vector of costs where $c_j(k)$ is the cost related to each user of facility j , if there are exactly k players using that facility.

Definition 1.4. [JOARU1994, YMAN2003] A congestion game associated with a congestion model is a strategic game with:

- a finite set N of players,
- for each player $i \in N$, there is a nonempty set of actions (strategies) A_i ,
- the preference relation \succeq_i for each player i is defined by a payoff function: Let $A = \times_{i \in N} A_i$ be the set of outcomes. For any $\vec{a} \in A$ and for any $j \in M$, let $n_j(\vec{a})$ be the number of players using facility j , assuming \vec{a} is the current action profile ($n_j(\vec{a}) = |\{i | i \in A_i\}|$). Then the payoff function for player i becomes: $u_i(\vec{a}) = \sum_{j \in a_i} c_j(n_j(\vec{a}))$.

1.2.2 Nash Equilibra in congestion games

One of the most fundamental concepts in game theory is that of Nash equilibrium. This notion captures a steady state of the play of a strategic game in which no player can improve his cost by unilaterally changing his strategy.

Definition 1.5. [THAR2011] A strategy profile $\vec{a} \in A$ is a **pure Nash Equilibrium** if for all $i \in N$:

$$u_i(\vec{a}) \leq u_i(a_i^*, \vec{a}_{-i}) \quad \forall a_i^* \in A_i$$

Remark 1.6. The notation (a_i^*, a_{-i}^*) for a strategy profile \vec{a}^* is a slight abuse of notation that is quite common in Game Theory, meaning that $a_i^* \in A_i$ and $a_{-i}^* \in A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n$. It's important to realize that players have no exact order in a strategy profile, so they can always be re-ordered.

Although there is not enough space to give a deep explanation of the concept of a Nash equilibrium, it's important to know that a Nash equilibrium is not necessary an optimal solution of a strategic game. It's only a profile in which no player will benefit from changing his strategy on it's own.

We can prove that a finite congestion game has a Nash Equilibrium. The proof of this statement use the notion of a potential function. We will first define a potential function for congestion games and proof that it holds the right properties. With this result we can then finally proof the existence of a pure Nash Equilibrium in congestion games.

Definition 1.7. Consider a function $\Phi : A \rightarrow \mathbb{R}$ defined on the space of pure strategy profiles of a (strategic) game G . If player i switches unilaterally from a_i to a_i^* , taking us from profile \vec{a} to profile $\vec{a}^* = (a_i^*, a_{-i}^*)$ and the following property holds:

$$\Phi(\vec{a}) - \Phi(\vec{a}^*) = u_i(\vec{a}) - u_i(\vec{a}^*)$$

then the function Φ is called a **potential function**.

Lemma 1.8. The function

$$\begin{aligned} \Phi : A &\rightarrow \mathbb{R} \\ \vec{a} &\mapsto \sum_{j=1}^m \sum_{k=1}^{n_j(\vec{a})} c_j(k) \end{aligned}$$

is a potential function for congestion games.

Proof. [ROSENTHAL73, ETE2007](Rosenthal 1973) In remark 1.6 we already mentioned that players can be re-ordered. In particular, re-index player i as player n and vice versa.

Then, for $i' \in \{1, \dots, n\}$, define:

$$n_j^{i'}(s) = |\{i | j \in A_i \wedge i \in \{1, \dots, i'\}\}|$$

Now, by using the commutativity, you can exchange the order of summation and become:

$$\Phi(\vec{a}) = \sum_{i=1}^n \sum_{j \in a_i} c_j(n_j^i(\vec{a}))$$

Denote that $n_j^{(n)}(\vec{a}) = n_j(\vec{a})$, thus:

$$\sum_{j \in a_i} c_j(n_j^n(\vec{a})) = \sum_{j \in a_i} c_j(n_j(\vec{a})) = u_n(\vec{a})$$

So, if player n switches from strategy a_n to a_n^* , taking us from profile \vec{a} to profile $\vec{a}^* = (a_n^*, a_{-n}^*)$ then:

$$\Phi(\vec{a}) - \Phi(\vec{a}^*) = \left(\sum_{j \in a_1} c_j(n_j^1(\vec{a})) + \dots + u_n(\vec{a}) \right) - \left(\sum_{j \in a_1} c_j(n_j^1(\vec{a}^*)) + \dots + u_n(\vec{a}^*) \right)$$

By definition of n_j^i , this becomes (the n 'th player doesn't count for n_j^i with $i < n$):

$$\begin{aligned} &= \left(\sum_{j \in a_1} c_j(n_j^1(\vec{a})) + \dots + u_n(\vec{a}) \right) - \left(\sum_{j \in a_1} c_j(n_j^1(\vec{a})) + \dots + u_n(\vec{a}^*) \right) \\ &= u_n(\vec{a}) - u_n(\vec{a}^*) \end{aligned}$$

We switched player i with player n , so this holds for every player i . □

Theorem 1.9. [ROSENTHAL73](Rosenthal 1973) *Every congestion game has a pure Nash equilibrium.*

Proof. Start from a random strategy vector \vec{a} , and let repeatedly one player reduce its costs. That means, that at each step Φ is reduced identically. Since Φ can accept a finite amount of values (A is finite because it's the finite cartesian product of sets A_i ($i \in \{1, \dots, n\}$), and every A_i is finite because it's by definition a subset of the finite set of facilities M), this procedure will stop and reach a local minimum. At this point, no player can achieve any improvements, and we reach a pure *Nash Equilibrium*. □

Remark 1.10. *A strategy profile is called a **pure strategy profile** when in this strategy profile each player selects a single action with probability 1.*

1.2.3 Load balancing games

Now we have introduced congestion games and proofed that they always have a Nash Equilibrium, we take a look at a more specific kind of congestion games that are very important for this paper: *load balancing games*.

Definition 1.11. [ICAR2008] *A **load balancing game** is a congestion game where the strategies of players are singletons. So each $A_i = \{j\}$ ($i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$). In load balancing terminology, we use the terms *server* and *client* instead of the terms *resource* and *player*.*

2 References

References

- [BVOCK2007] B. Vöcking, *Selfish Load Balancing*, Chapter 20 in *Algorithmic Game Theory*, Cambridge University Press, December 2007.
- [JOARU1994] d J. Osborne and A. Rubinstein, *A course in Game Theory*, The MIT Press, 1994.

- [CW2007] C. Witteveen, *Intreerede: De Prijs van de Onafhankelijkheid*, TU Delft 2007.
- [YMAN2003] Y. Mansour, *Lecture 6: Congestion and potential games*, Computational Learning Theory, University of Tel Aviv, 2003.
- [JMAR] Jason R. Marden, *Lecture 12: Game Theory Course*, University of Colorado.
- [THAR2011] T. Harks, M. Klimm, R. H. Möhring, *Characterizing the Existence of Potential Functions in Weighted Congestion Games*, February 2011
- [ROSENTHAL73] R. W. Rosenthal, *A class of games possessing pure-strategy Nash equilibria*. International Journal of Game Theory, 2:65–67, 1973
- [ETE2007] K. Etessami, *Algorithmic Game Theory - Lecture 16 Best response dynamics and pure Nash Equilibria*, University of Edinburgh, 2007.
- [ICAR2008] I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, *Improving the Efficiency of Load Balancing Games through Taxes*, University of Patras, 2008.