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Recent advances in the domain wall six-vertex model

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And also:

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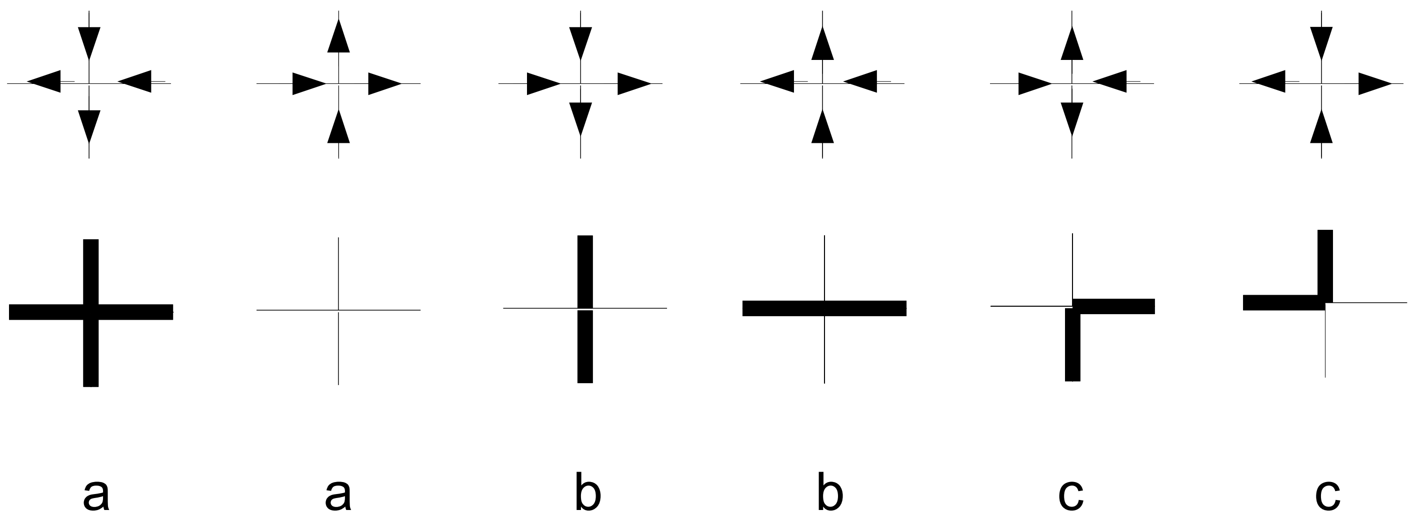
A. Sportiello (Univ. Milano & Paris XIII)
P. Zinn-Justin (UPMC, Paris)

The six-vertex model with Domain Wall BC

[Korepin'82]



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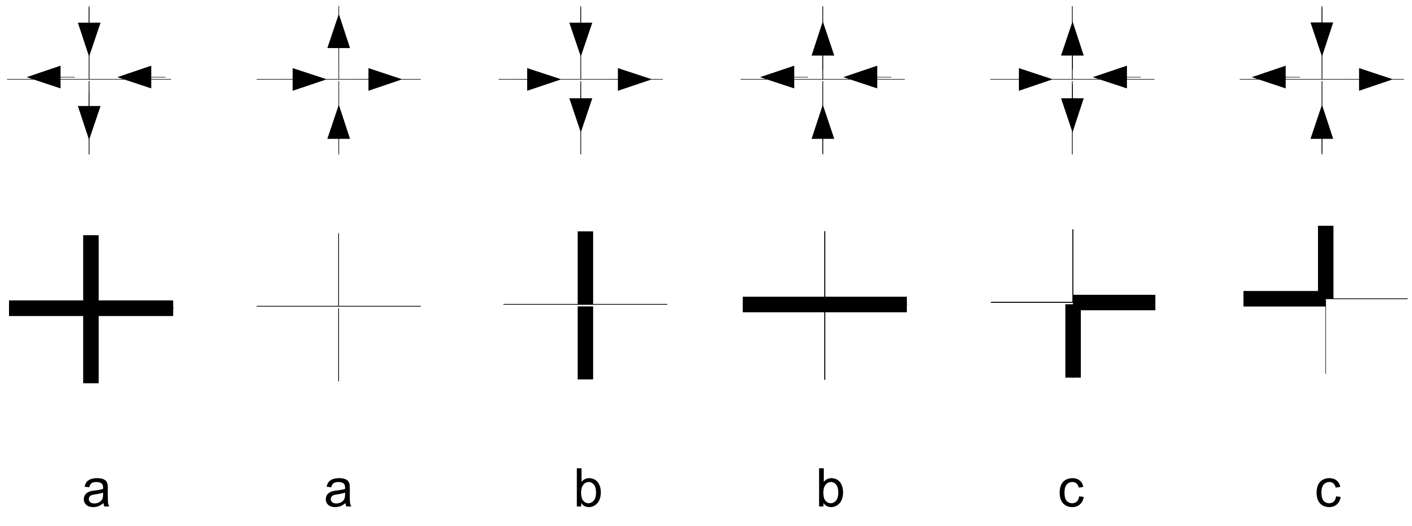

$$a = \sin(\lambda+\eta)$$
$$b = \sin(\lambda-\eta)$$
$$c = \sin 2\eta$$

$$\Delta = \frac{a^2+b^2-c^2}{2ab}$$

$$t = \frac{b}{a}$$

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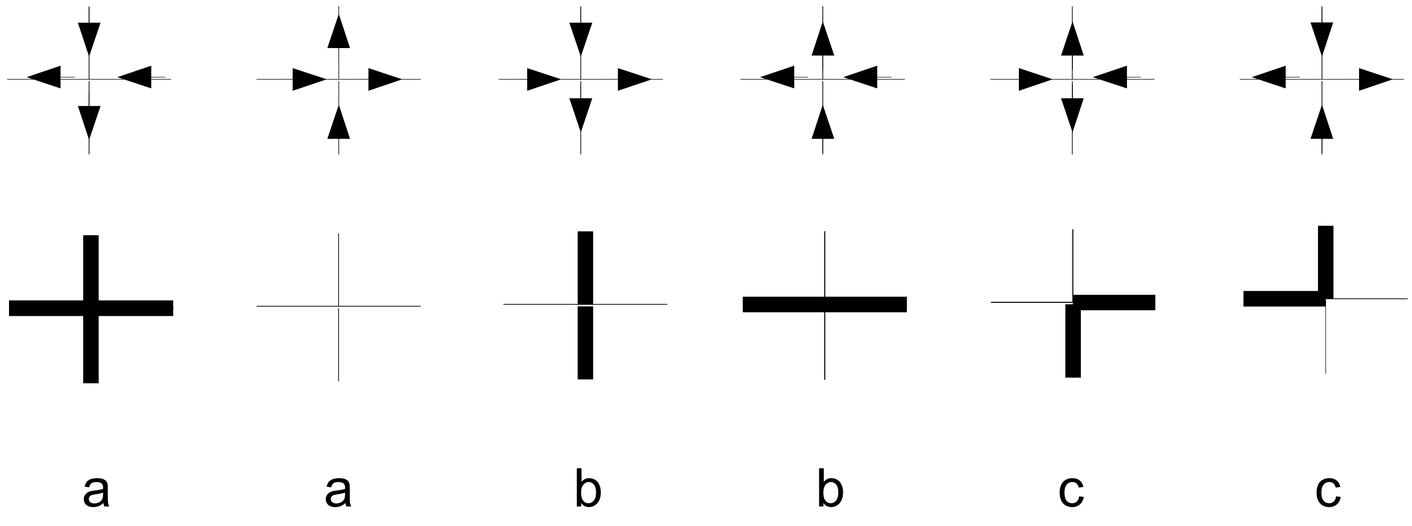

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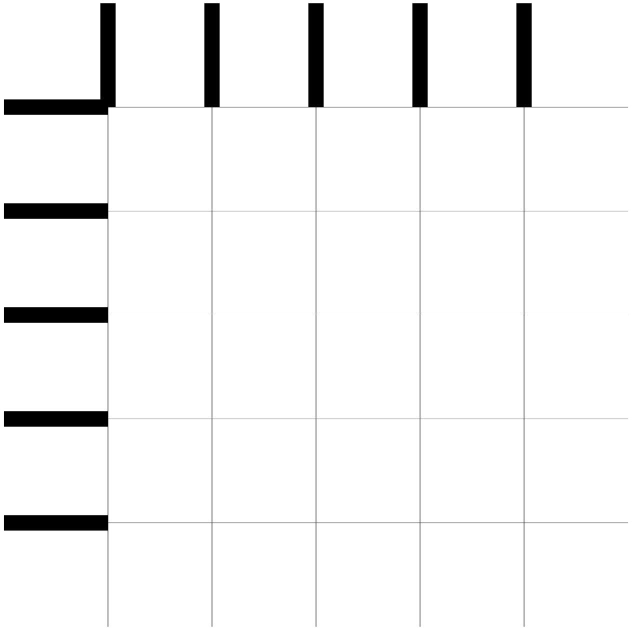
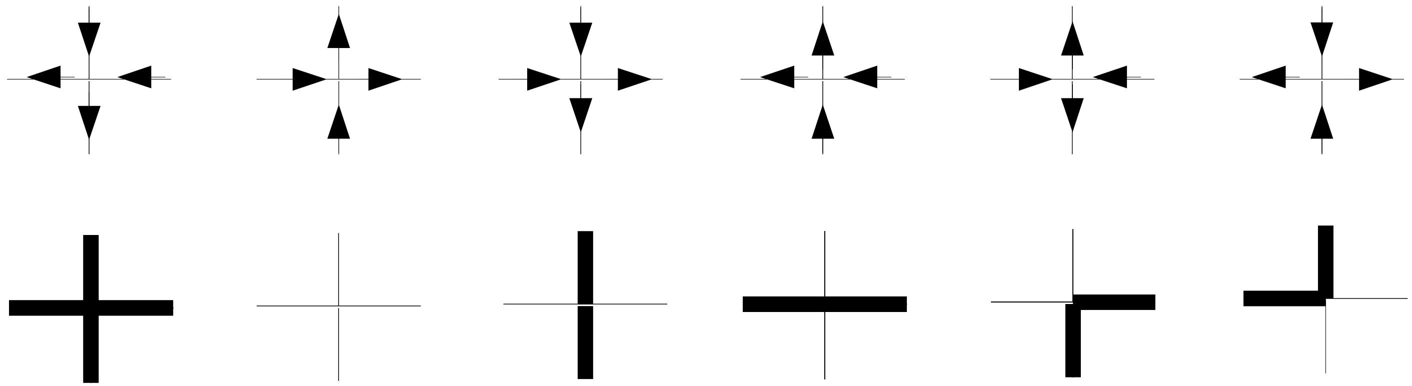


$N \times N$

The six-vertex model with Domain Wall BC

[Korepin'82]



a a b b c c

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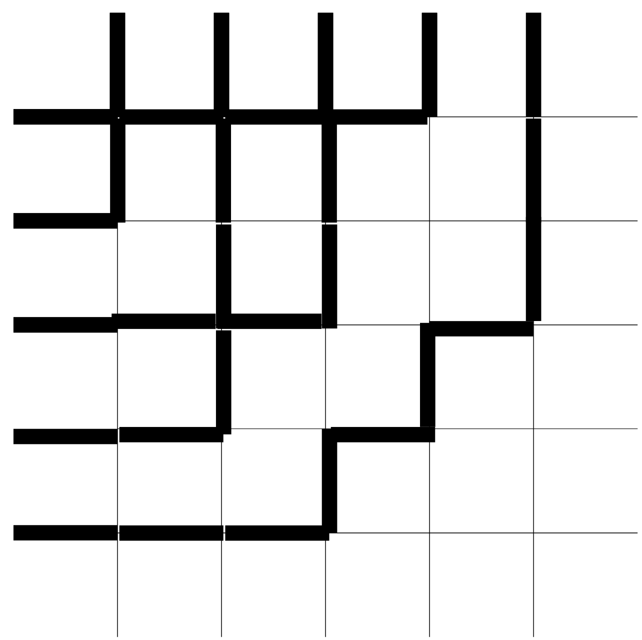


$N \times N$

$$\Delta = \frac{a^2+b^2-c^2}{2ab}$$

$$t = \frac{b}{a}$$

Combinatorics

$$\Delta = 0, \ \pm\frac{1}{2}$$

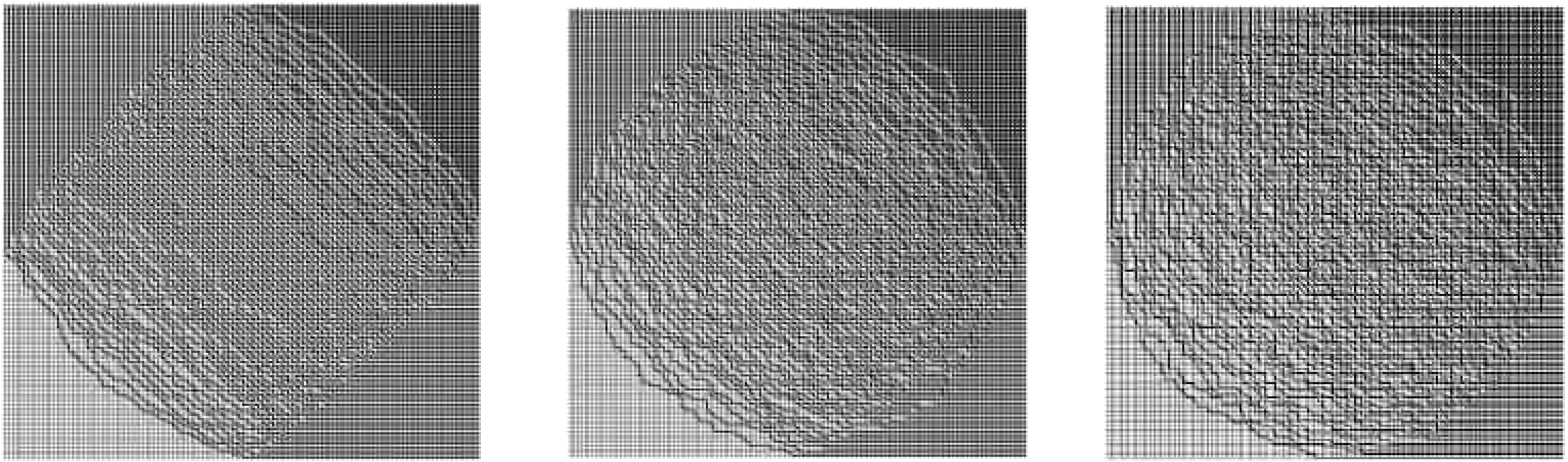
- Alternating Sign Matrices
- Razumov-Stroganov conjecture (now a theorem)
- domino tilings of Aztec Diamond
- plane partitions
- ...

[see, e.g., various other talks in this conference]

Spatial phase separation, limits shapes, arctic curves

- 6VM with DWBC generalizes domino tiling of Aztec diamond, where Arctic Circle theorem holds [Jockush-Propp-Shor'98]
- Free energy per site in thermodynamic limit: DWBC $\neq$ periodic BC [Korepin Zinn-Justin'00][Zinn-Justin'00] [Bleher-Fokin-Liechty'05-'09]
- Numerics [Eloranta'99][Syljuasen-Zvonarev'04] [Allison-Reshetikhin'05][Wieland'08]

Domain Wall six vertex model: numerical results



$\Delta = -3$ $\qquad$ $\Delta = -0.92$ $\qquad$ $\Delta = 0$ (free fermions)

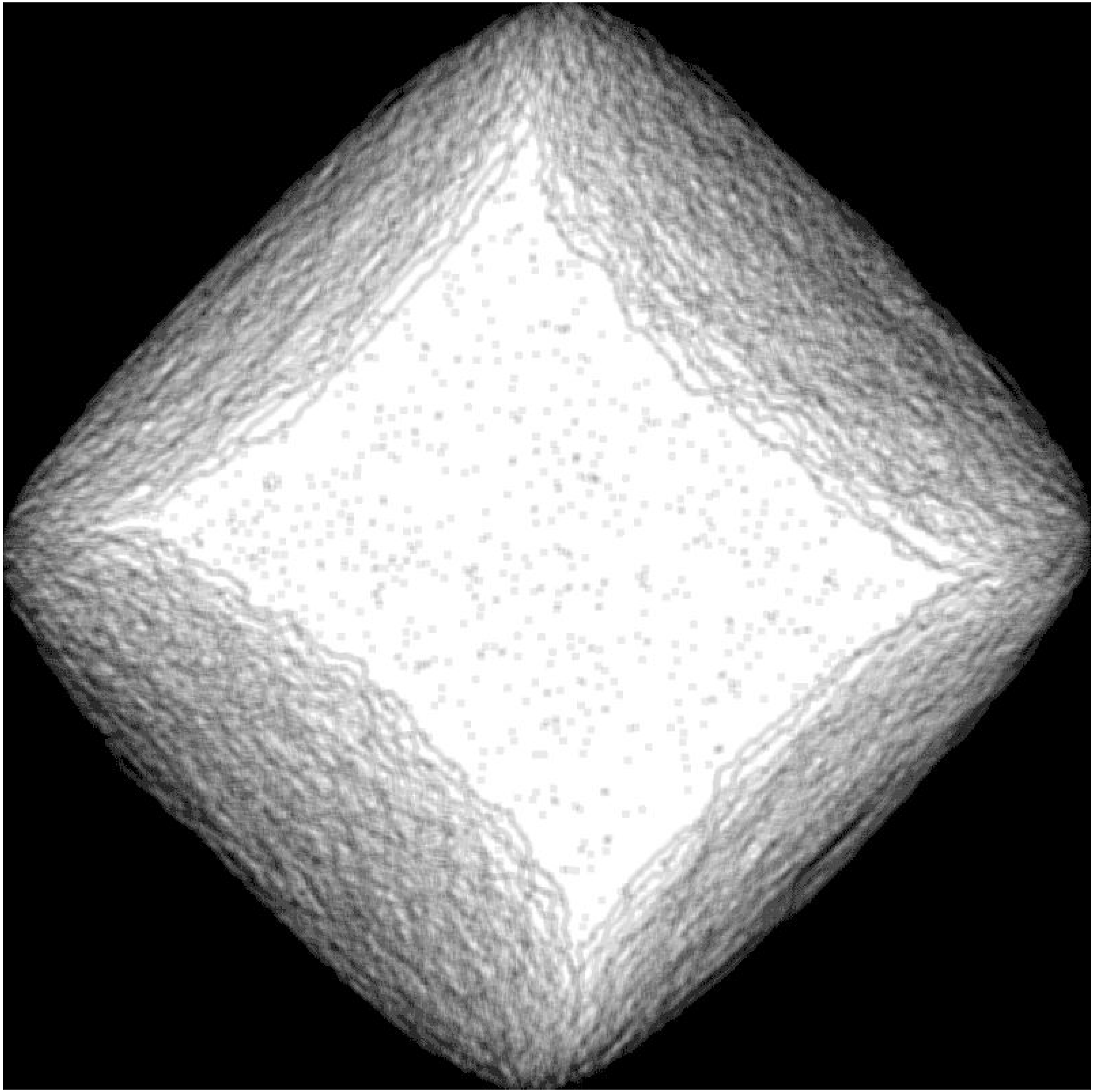
$N = 225$

[Allison-Reshetikhin'05]

$N = 1000$

$\Delta = -3$

$t = 0.5$



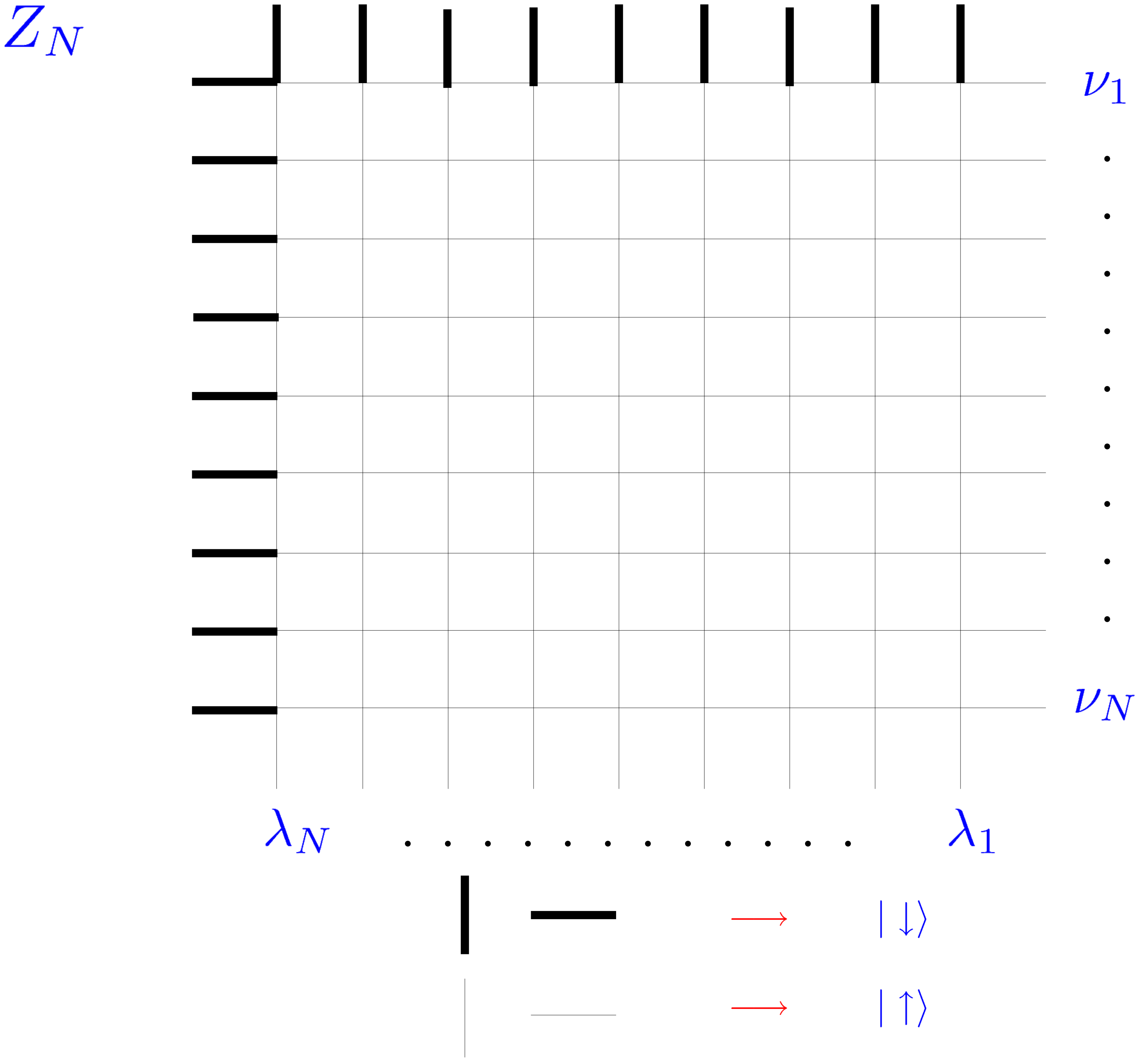
[Allison-Reshetikhin'05]

Spatial phase separation

- Artic Curves
- Fluctuations
- Limit shapes (in the heigth function interpretation of the 6VM with DWBC)
- Spatial modulation of polarization in the disordered region

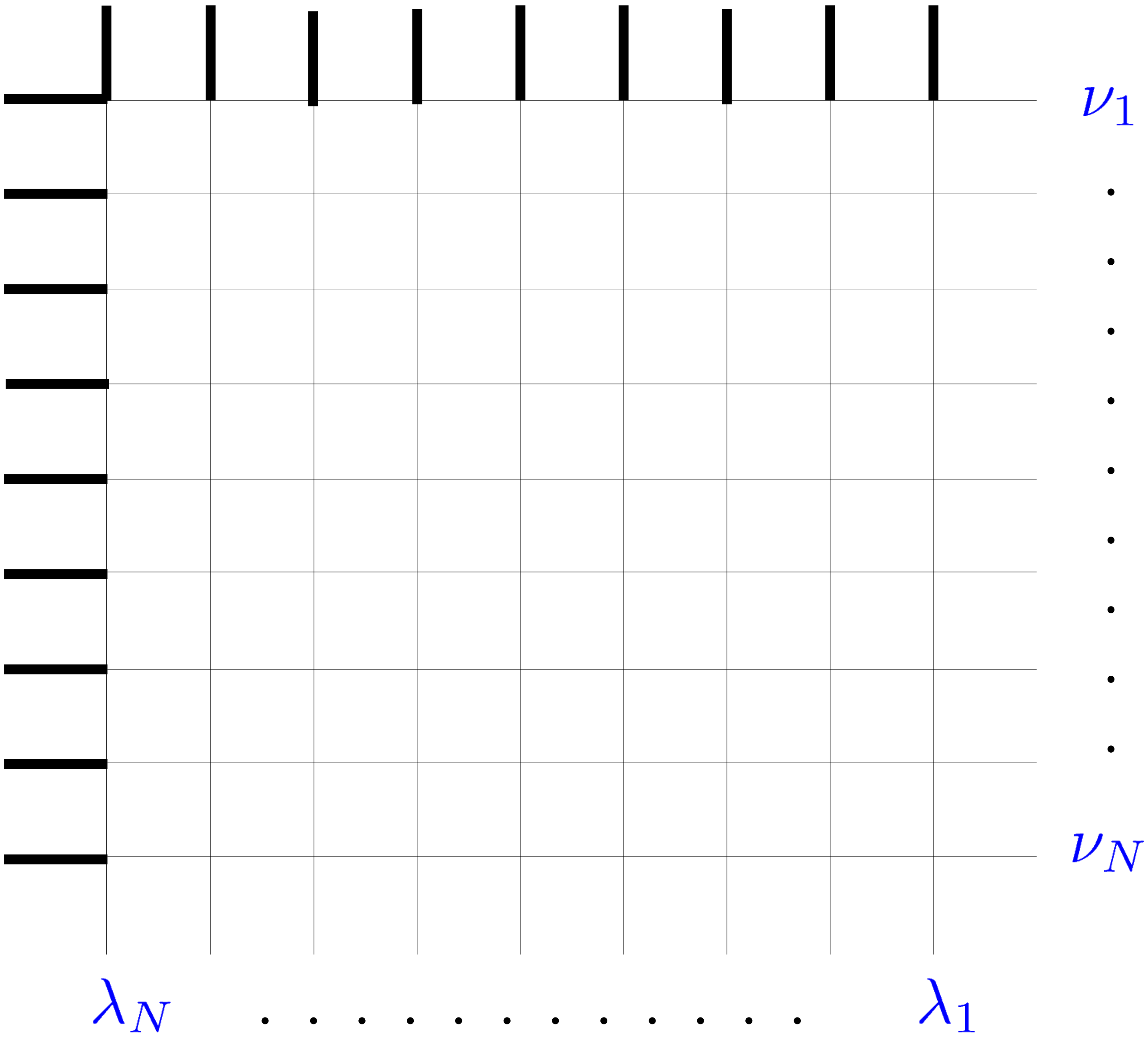
Our program:

- Compute some useful correlation function, (our ultimate goal is polarization).
  NB: not so easy: translation invariance is broken by BC.
- Study its behaviour in the scaling limit $N \to \infty$, $a \to 0$, Na fixed
  (a = lattice spacing)



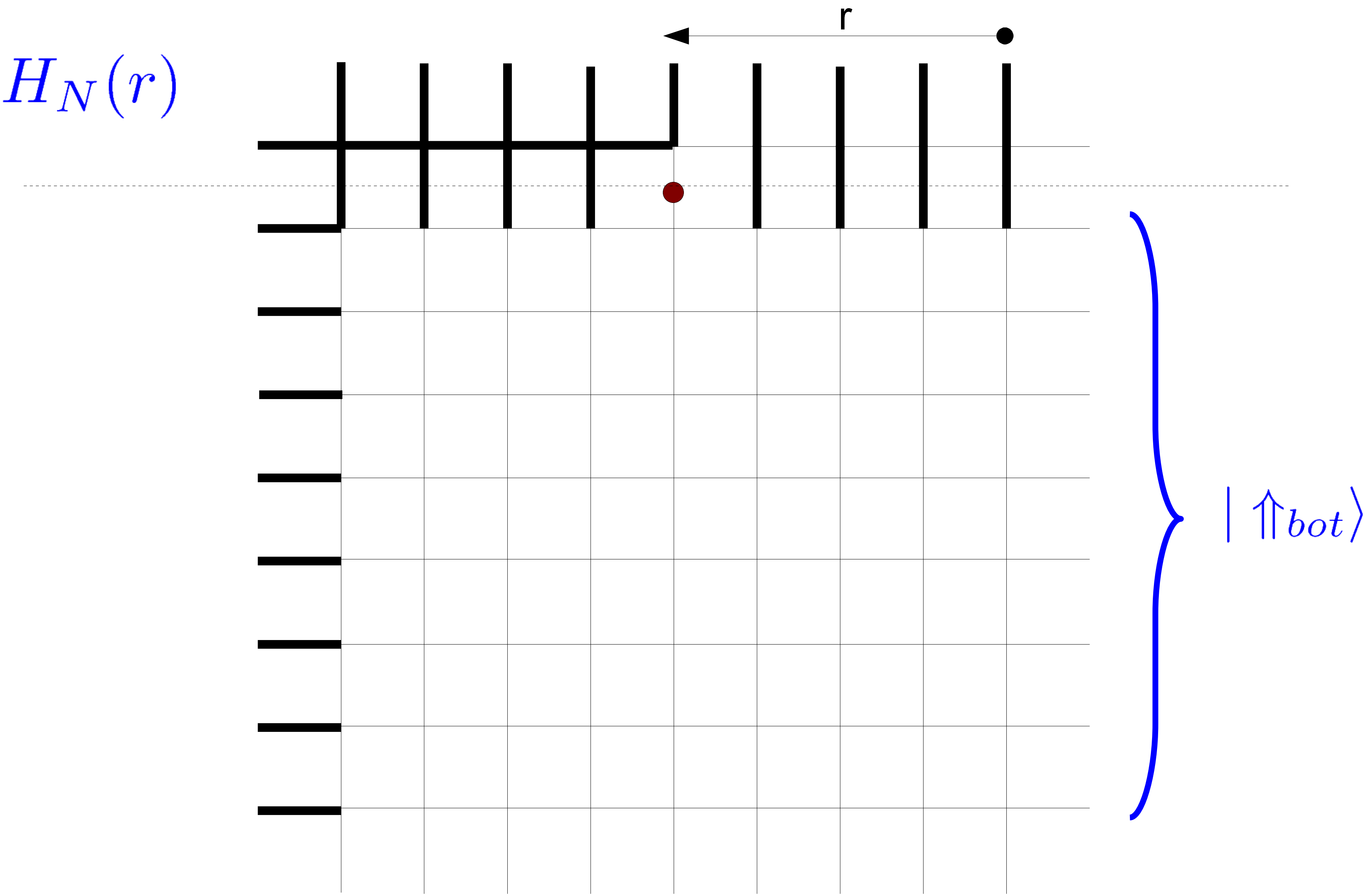
Z_N
ν_1
ν_N
λ_N
λ_1
$\longrightarrow \; |\downarrow\rangle$
$\longrightarrow \; |\uparrow\rangle$

Z_N



$$Z_N = \langle \Downarrow | B(\lambda_N) \ldots B(\lambda_1) | \Uparrow \rangle$$

IK determinant formula [Korepin'82][Izergin'87]



$$H_N(r) = (\text{simple prefactor})$$
$$\times \langle \Downarrow_{bot} | B(\lambda_N) \ldots B(\lambda_{r+1}) A(\lambda_r) B(\lambda_{r-1}) \ldots B(\lambda_1) | \Uparrow_{bot} \rangle$$

[Bogoliubov-Pronko-Zvonarev'02]

Recall, from RTT relation, $A(\lambda)B(\lambda') = f(\lambda,\lambda')B(\lambda')A(\lambda) + g(\lambda,\lambda')B(\lambda)A(\lambda')$

Thus: $\langle \Downarrow_{bot} |B(\lambda_N)\dots B(\lambda_{r+1})A(\lambda_r)B(\lambda_{r-1})\dots B(\lambda_1)| \Uparrow_{bot}\rangle =$

$$= \sum_{\alpha=1}^{r} \phi(\lambda_\alpha;\{\lambda\})\ \langle \Downarrow_{bot} |B(\lambda_N)\dots B(\lambda_{\alpha+1})B(\lambda_{\alpha-1})\dots B(\lambda_1)| \Uparrow_{bot}\rangle$$

⇨ $H_N^{(r)}$ expressed as linear combination of `partition function'

⇨ I-K type determinant representation [Bogoliubov-Pronko-Zvonarev'02]

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⇨ $H_N^{(r)}$ expressed as linear combination of `partition function'

⇨ I-K type determinant representation [Bogoliubov-Pronko-Zvonarev'02]

This can be inverted. Indeed define generating function: $h_N(z) = \sum_{r=1}^{N} H_N^{(r)} z^{r-1}$

Define further, for $s = 1,\dots,N$, the totally symmetric polynomials:

$$h_{N,s}(z_1,\dots,z_s) = \frac{1}{\prod_{1\le j<k\le s}(z_k - z_j)} \det_{1\le j,k\le s}\left[h_{N-s+k}(z_j)(z_j-1)^{k-1} z_j^{s-k}\right]$$

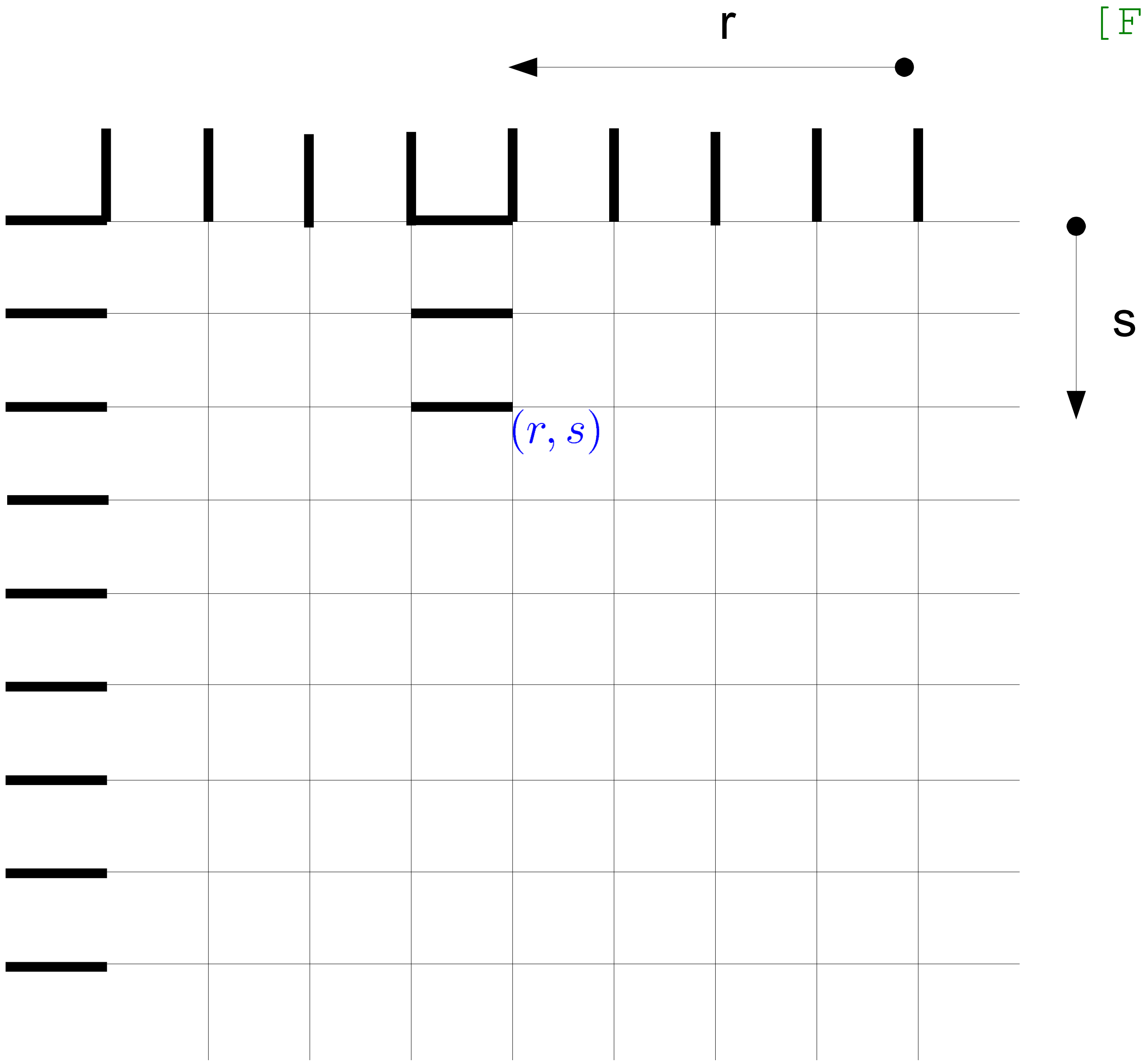
The following holds [FC-AP'08]:

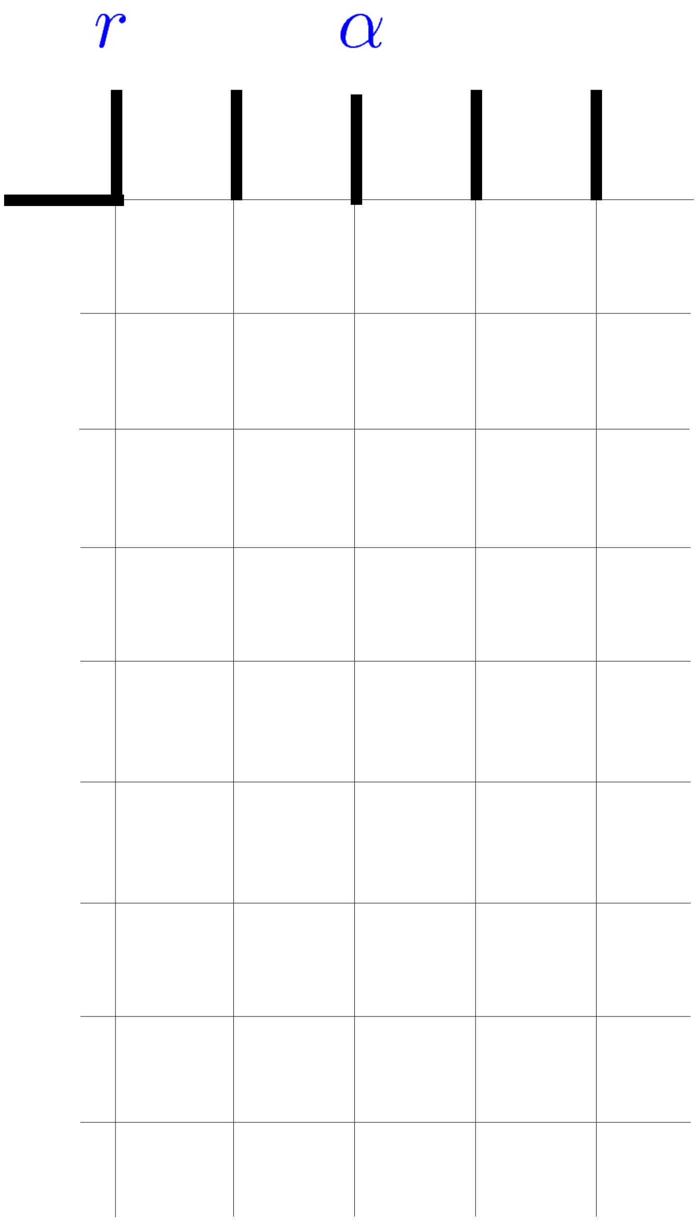
$$Z_N(\lambda_1,\dots,\lambda_s,\lambda,\dots,\lambda;0,\dots,0) = Z_N^{hom}\, h_{N,s}(z_1,\dots,z_s)$$

where: $z_j := \dfrac{a(\lambda)}{b(\lambda)}\,\dfrac{b(\lambda_j)}{a(\lambda_j)}$

$F_N(r, s)$ Emptiness Formation Probability (EFP)

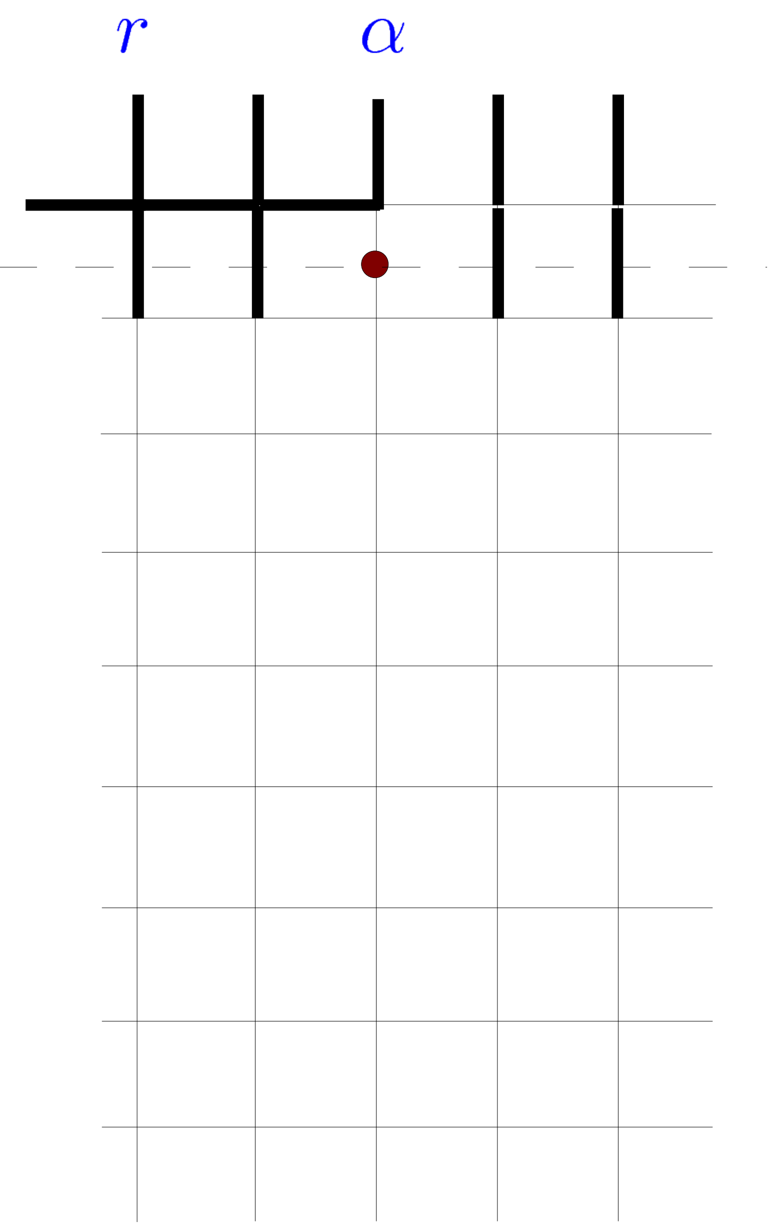
[FC-AP,'08]





$$v \in \prod_{k=2}^{N} \mathcal{H}_k$$

$$v := \langle \downarrow_1 | B(\lambda_r) \dots B(\lambda_1) | \Uparrow \rangle =$$



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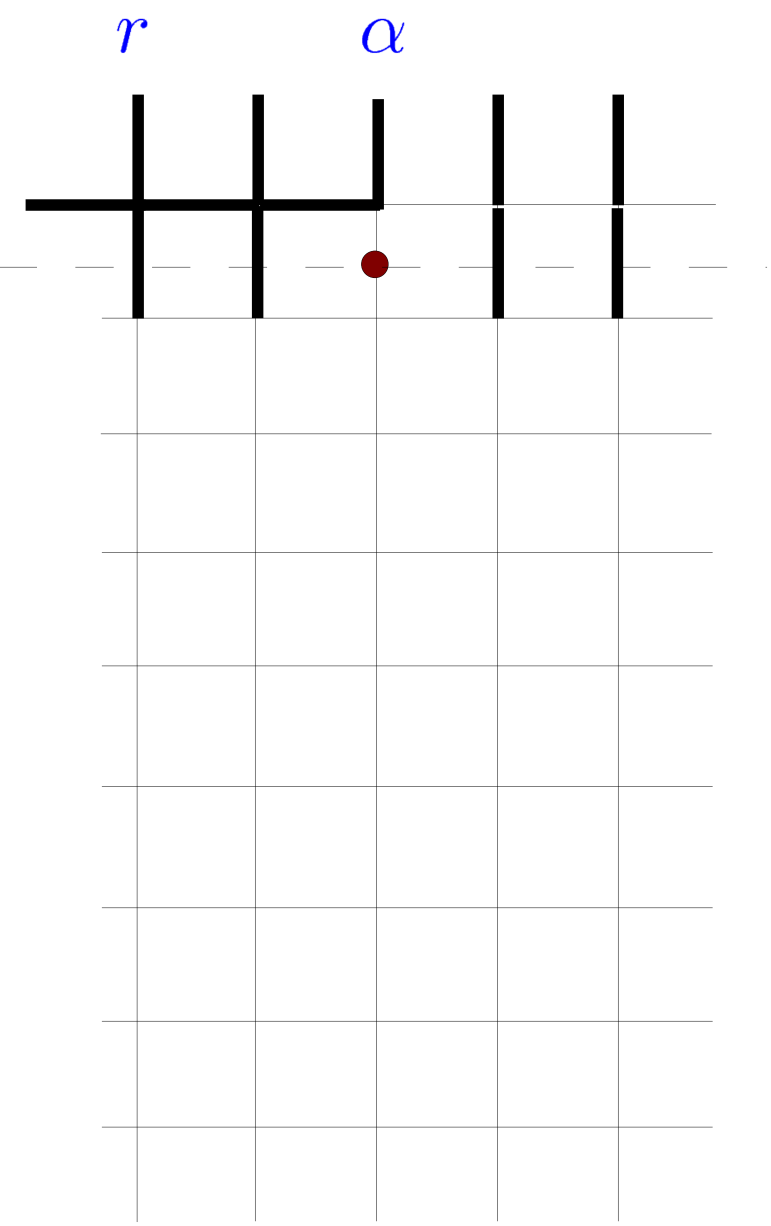
$$v := \langle \downarrow_1 | B(\lambda_r) \dots B(\lambda_1) | \Uparrow \rangle =$$

$$= \sum_{\alpha=1}^{r} \prod_{\beta=\alpha+1}^{r} a(\lambda_\beta, \nu_1) \cdot c \cdot \prod_{\beta=1}^{\alpha-1} b(\lambda_\beta, \nu_1)$$

$$\times B_{bot}(\lambda_n) \dots B_{bot}(\lambda_{\alpha+1}) A(\lambda_\alpha) B_{bot}(\lambda_{\alpha-1}) \dots$$
$$\dots B_{bot}(\lambda_1) | \Uparrow_{bot} \rangle$$



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recall, from RTT relation,

$$A(\lambda)B(\lambda') = f(\lambda,\lambda')B(\lambda')A(\lambda) + g(\lambda,\lambda')B(\lambda)A(\lambda')$$

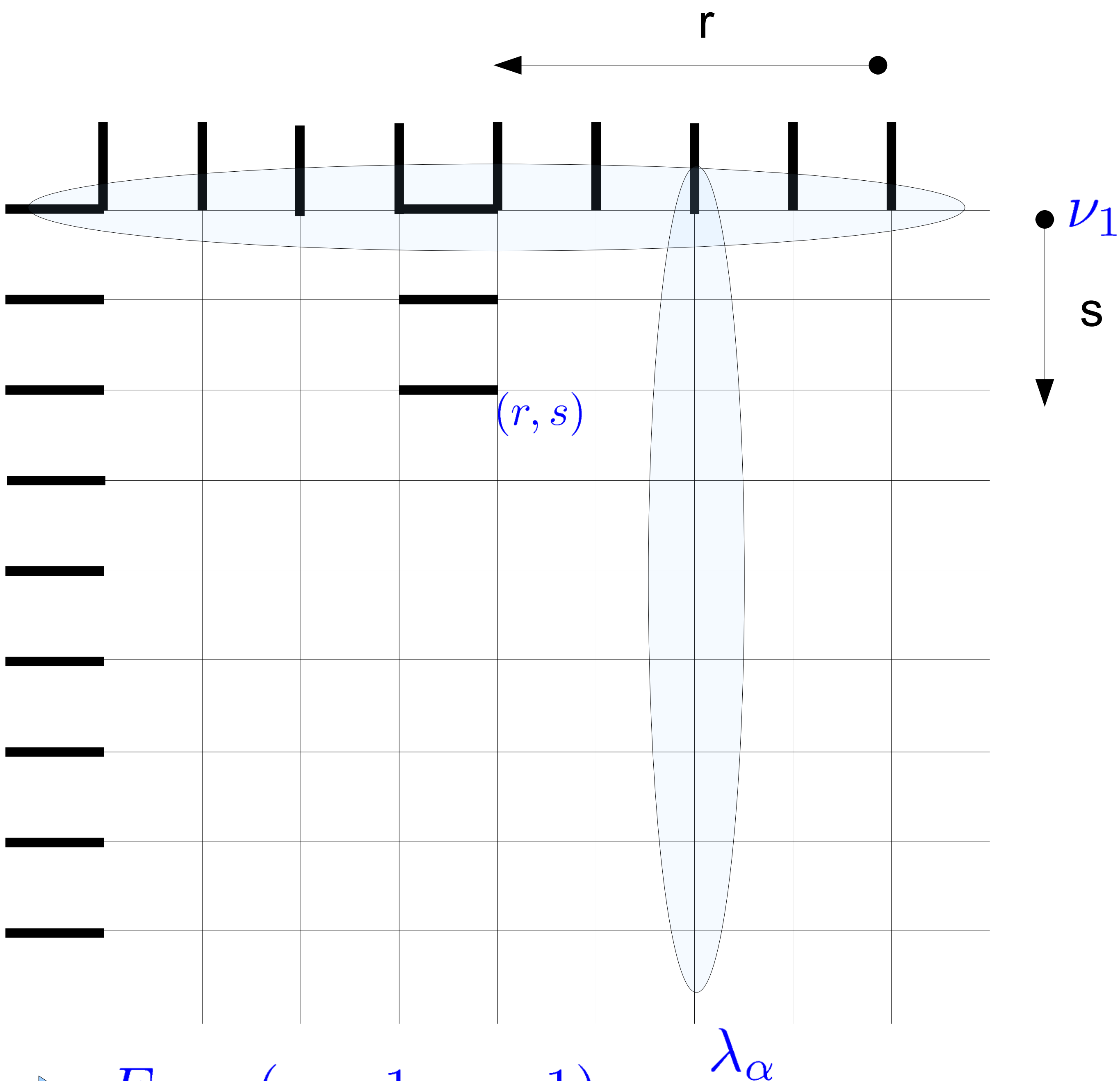
Thus:

$$v = c \sum_{\alpha=1}^{r} \prod_{\substack{\beta=1 \\ \beta\neq\alpha}} b(\lambda_\beta, \nu_1) \prod_{\substack{\beta=1 \\ \beta\neq\alpha}} f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^{N} a(\lambda_\alpha, \nu_k)$$

$$\times B_{bot}(\lambda_n) \dots B_{bot}(\lambda_{\alpha+1}) B_{bot}(\lambda_{\alpha-1}) \dots B_{bot}(\lambda_1) | \Uparrow_{bot} \rangle$$

$\Rightarrow$ recursion relation between products of B's and of B_{bot}'s.

$F_N(r,s)$ Emptiness Formation Probability (EFP)



$$F_N(r,s) \Rightarrow F_{N-1}(r-1,s-1)$$

Emptiness Formation Probability [FC-AP'08]

- Solve previous recurrence relation
- Perform homogeneous limit: determinantal representation is obtained
- Use standard technology from theory of Orthogonal polynomials

The following Multiple Integral Representation is obtained ($r, s = 1, 2, \ldots, N$):

$$F_N(r,s) = \left(-\frac{1}{2\pi i}\right)^s \oint_{C_0} \cdots \oint_{C_0} d^s z \, h_{N,s}(z_1, \ldots, z_s) \prod_{j=1}^{s} \frac{1}{z_j^r (z_j - 1)^{s-j+1}} \times \prod_{1 \le j < k \le s} \frac{(t^2 z_j - 2\Delta t z_j - 1)(z_k - 1)(z_j - z_k)}{t^2 z_j z_k - 2\Delta t z_j + 1}.$$

where C_0 are simple anticlockwise contours, enclosing $z = 0$ and no other singularity.

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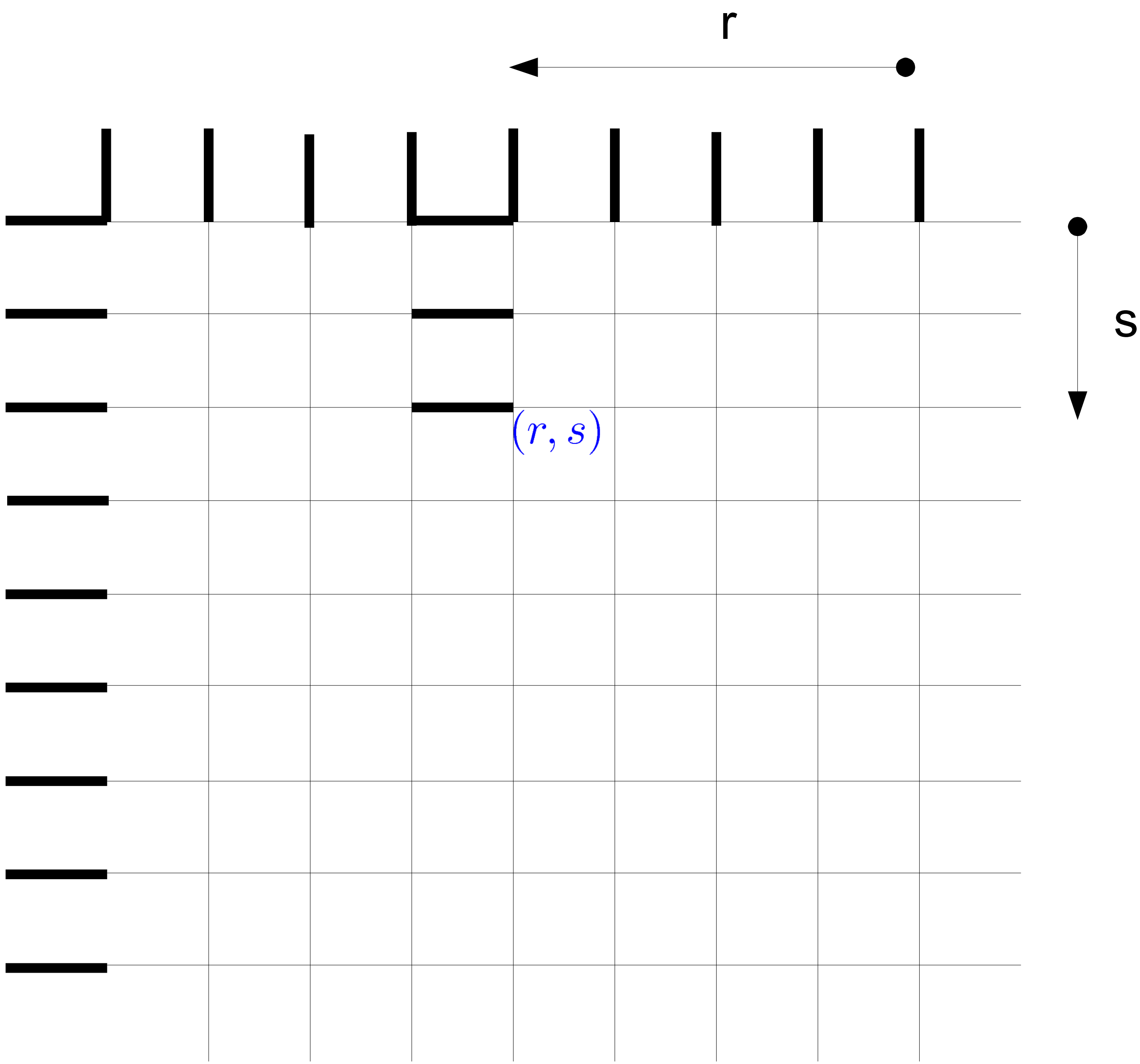
Symmetrizing the integrand, and using a ice identity by [Kitanine-Maillet-Slavnov-Terras'02], we equivalently have :

$$F_N(r,s) \propto \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t) z_j + 1]^{s-1}}{z_j^r (z_j - 1)^s} \prod_{\substack{j,k=1 \\ j \ne k}}^{s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1}$$

$$\times h_{N,s}(z_1, \ldots, z_s) h_{s,s}(u_1, \ldots, u_s).$$

where $\quad u_j := \dfrac{1 - z_j}{(t^2 - 2\Delta t) z_j + 1}.$

$F_N(r,s)$ Emptiness Formation Probability (EFP)



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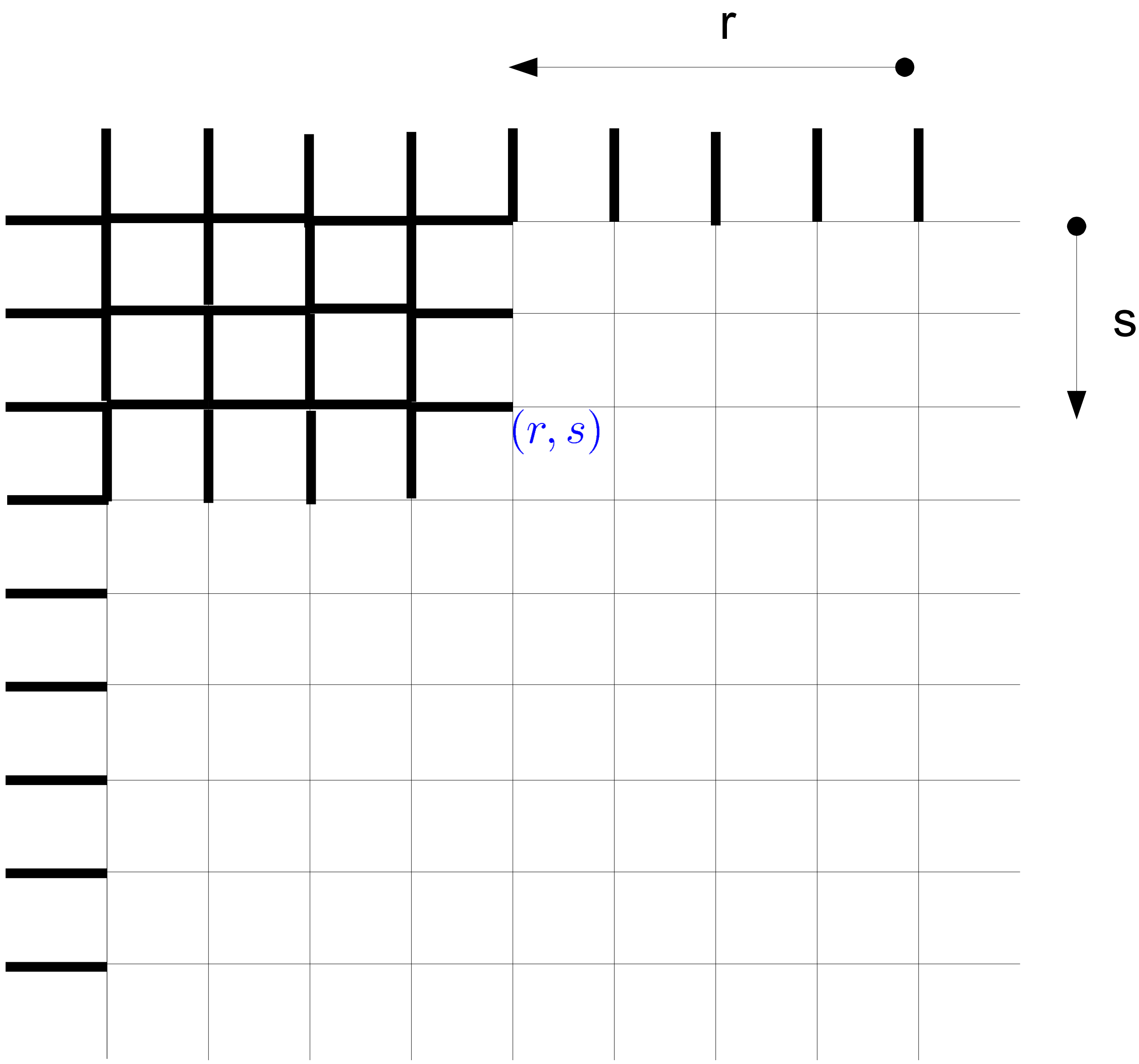

- Stepwise behaviour in correspondence of the Arctic curve
- Ability to discriminate only the top-left portion of the curve

We want now to study the behaviour of previous Multiple Integral Representation in the scaling limit:

$$N, r, s \to \infty \qquad \frac{r}{N} = x \qquad \frac{s}{N} = y \qquad x, y \in [0, 1]$$

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Using Random Matrix Model techniques, it can be shown [FC-AP'10] that EFP has a stepwise behaviour, occurring for x, y such that equation

$$\frac{y}{z-1} - \frac{x}{z} - \frac{yt^2}{t^2 z - 2\Delta t + 1} + \lim_{N\to\infty} \frac{1}{N} \partial_z \ln h_N(z) = 0 \, .$$

has two coinciding real roots $z \in [1, +\infty)$.

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- NB: the result depends ONLY on the BOUNDARY correlation function!
- We thus need to evaluate the large N behaviour of $h_N(z)$ for generic Boltzmann weights (disorder regime [FC-AP'10]; antiferroelectric regime [FC-AP-PZJ'10])
- It appears that in disordered regime Arctic curve is: algebraic for rational η, transcendent for irrational η [FC-AP'10] [FC-AP-VN'11]
- This differs from the free fermion case, were only algebraic curves appear, for any choice of lattice and BC [Kenyon-Okounkov-Sheffield,'03-'05]

In both cases we get the parametric form of the Arctic curve, with parameter $\zeta \in [0, \zeta_{max}]$

$$
\begin{aligned}
x &= \frac{1}{\Phi(\zeta+\lambda-\eta, 2\eta)\Psi(\zeta, 2\eta) - \Psi(\zeta+\lambda-\eta, 2\eta)\Phi(\zeta, 2\eta)} \\
&\qquad \times \left\{ \left[\Psi(\zeta, \lambda-\eta) - \gamma^2 \Psi(\gamma\zeta, \gamma(\lambda-\eta))\right] \Phi(\zeta, 2\eta) \right. \\
&\qquad\qquad \left. - \left[\Phi(\zeta, \lambda-\eta) - \gamma \Phi(\gamma\zeta, \gamma(\lambda-\eta))\right] \Psi(\zeta, 2\eta) \right\}, \\
y &= \frac{1}{\Phi(\zeta+\lambda-\eta, 2\eta)\Psi(\zeta, 2\eta) - \Psi(\zeta+\lambda-\eta, 2\eta)\Phi(\zeta, 2\eta)} \\
&\qquad \times \left\{ \left[\Psi(\zeta, \lambda-\eta) - \gamma^2 \Psi(\gamma\zeta, \gamma(\lambda-\eta))\right] \Phi(\zeta+\lambda-\eta, 2\eta) \right. \\
&\qquad\qquad \left. - \left[\Phi(\zeta, \lambda-\eta) - \gamma \Phi(\gamma\zeta, \gamma(\lambda-\eta))\right] \Psi(\zeta+\lambda-\eta, 2\eta) \right\}.
\end{aligned}
$$

where

$$
\Phi(\mu) := \frac{\sin(2\eta)}{\sin(\mu+\eta)\sin(\mu-\eta)},
$$

$$
\Psi(\zeta) := \cot\zeta - \cot(\zeta+\lambda-\eta) - \gamma\cot\gamma\zeta + \gamma\cot\gamma(\zeta+\lambda-\eta),
$$

(Disordered regime)

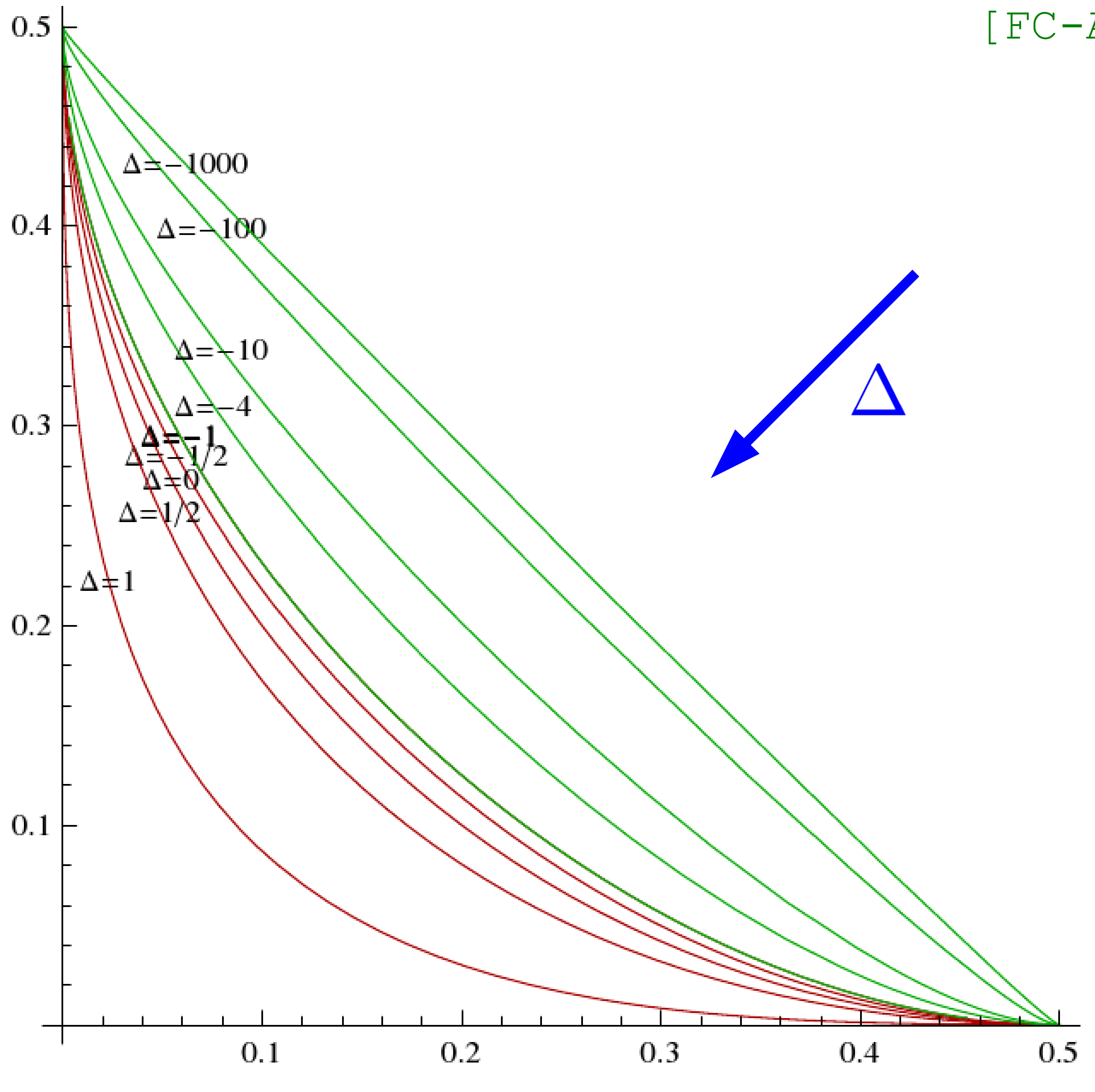
or

$$
\Phi(\mu) := \frac{\sinh(2\eta)}{\sinh(\eta-\mu)\sinh(\eta+\mu)},
$$

$$
\Psi(\zeta) := \cot\zeta - \coth(\eta-\lambda-\zeta) - \gamma\frac{\vartheta_1'(\gamma\zeta)}{\vartheta_1(\gamma\zeta)} + \gamma\frac{\vartheta_1'(\gamma(\zeta+\lambda-\eta))}{\vartheta_1(\gamma(\zeta+\lambda-\eta))},
$$

(Anti-ferroelectric regime)

[FC-AP,'09]
[FC-AP-PZJ,'10]



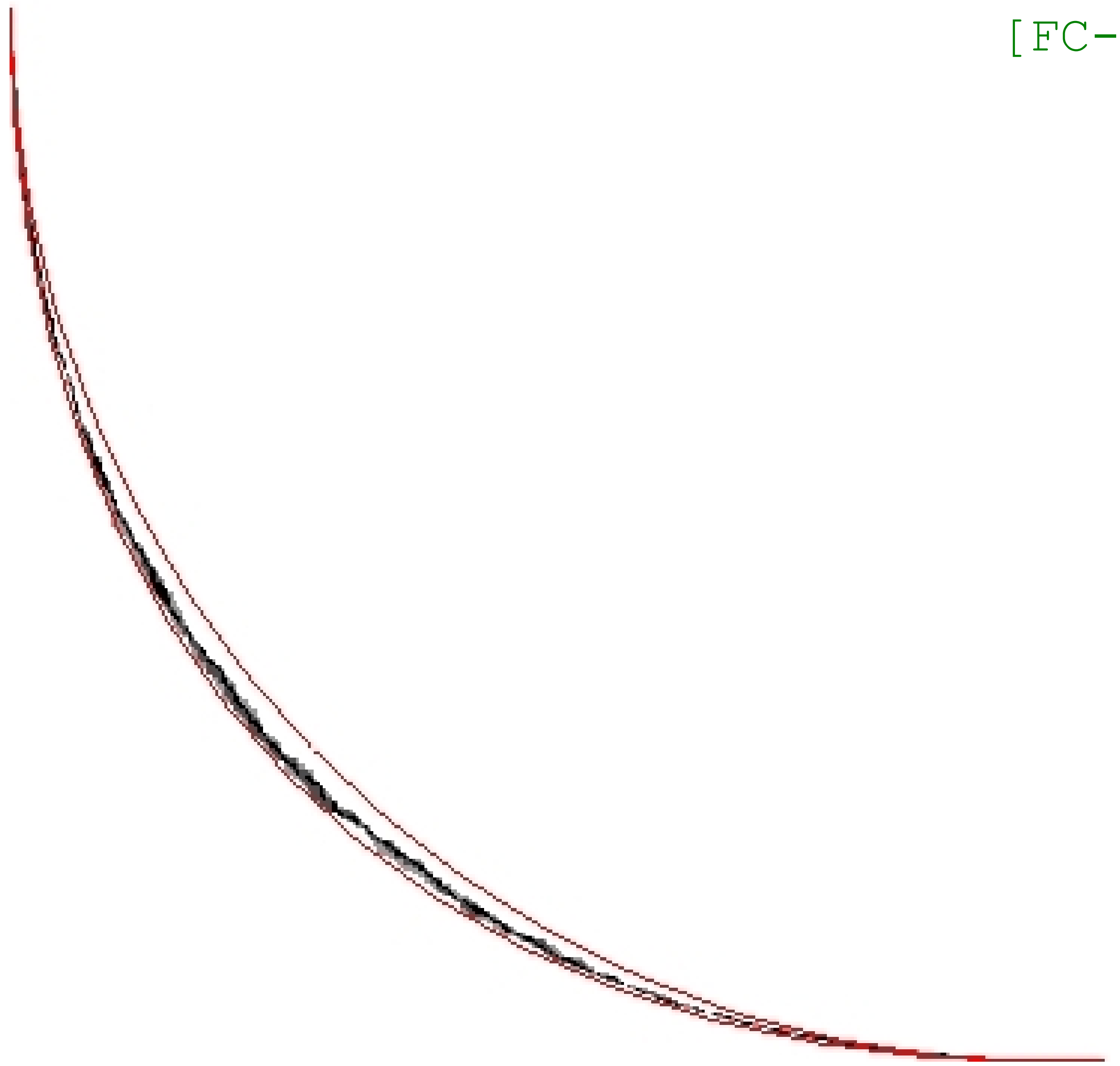
ASMs: N=1500

10 samples

$$\Delta = 1/2$$

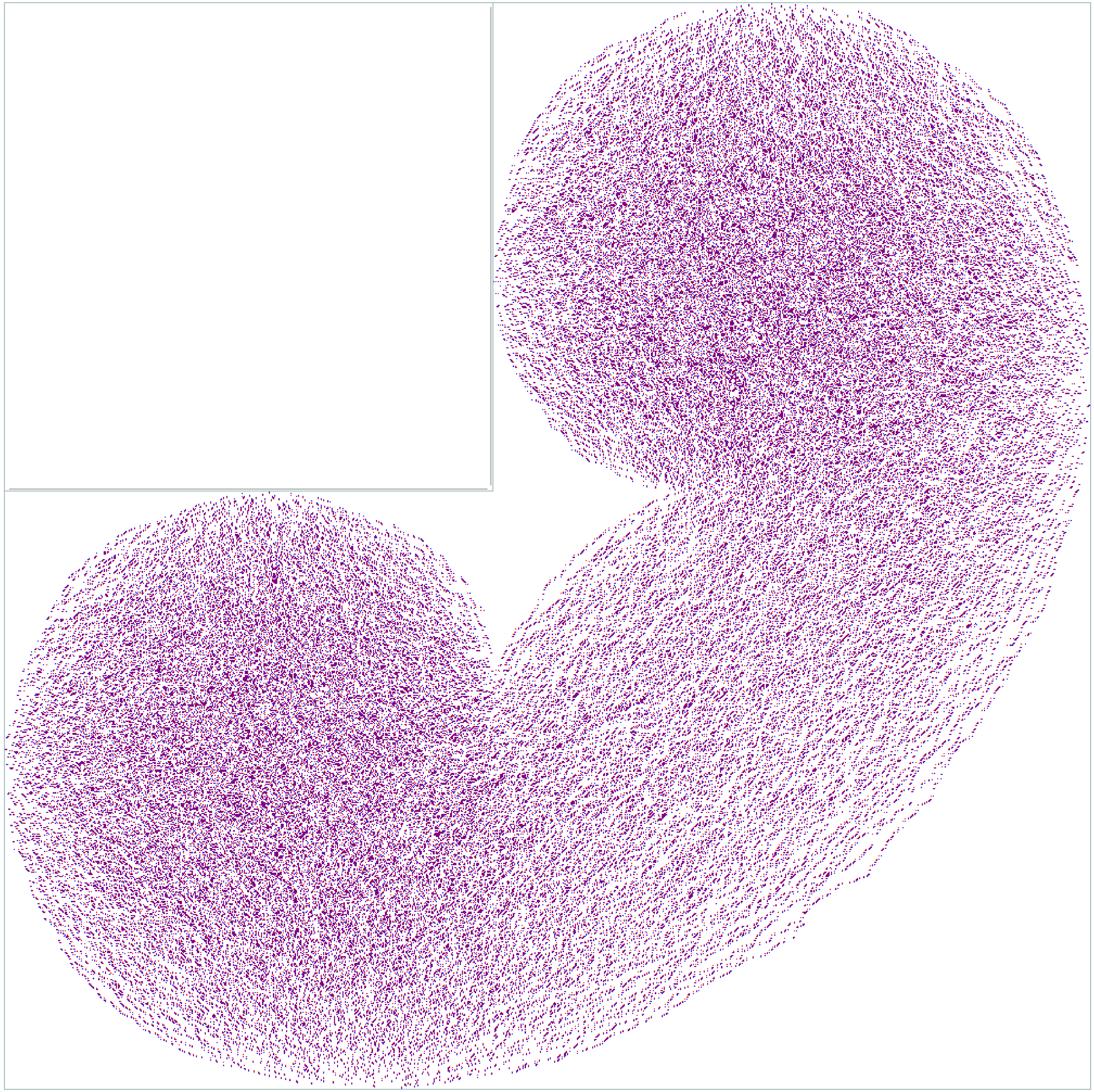
$$(2x-1)^2 + (2y-1)^2 - 4xy = 1, \qquad x, y \in \left[0, \tfrac{1}{2}\right].$$

[FC-AP,'08]



Simulation made by Ben Wieland (2008)

$\Delta = \frac{1}{2}$



$N = 1000$

$n = 550$

Remark

The condition determining the Arctic curve can be rephrased as follows:

The Arctic curve is the geometric caustic (envelope) of the family of straight lines:

$$\mathcal{F}(x,y;z) := \frac{y}{z-1} - \frac{x}{z} - \frac{yt^2}{t^2 z - 2\Delta t + 1} + \lim_{N\to\infty} \frac{1}{N}\partial_z \ln h_N(z) \qquad z \in [1,+\infty)$$

It can be shown that this same condition determines the Arctic curves of the 6VM defined on generic regions (modulo some caveat) of the square lattice, with fixed, domain-wall type, boundary condition. [FC-AS, in prep.]

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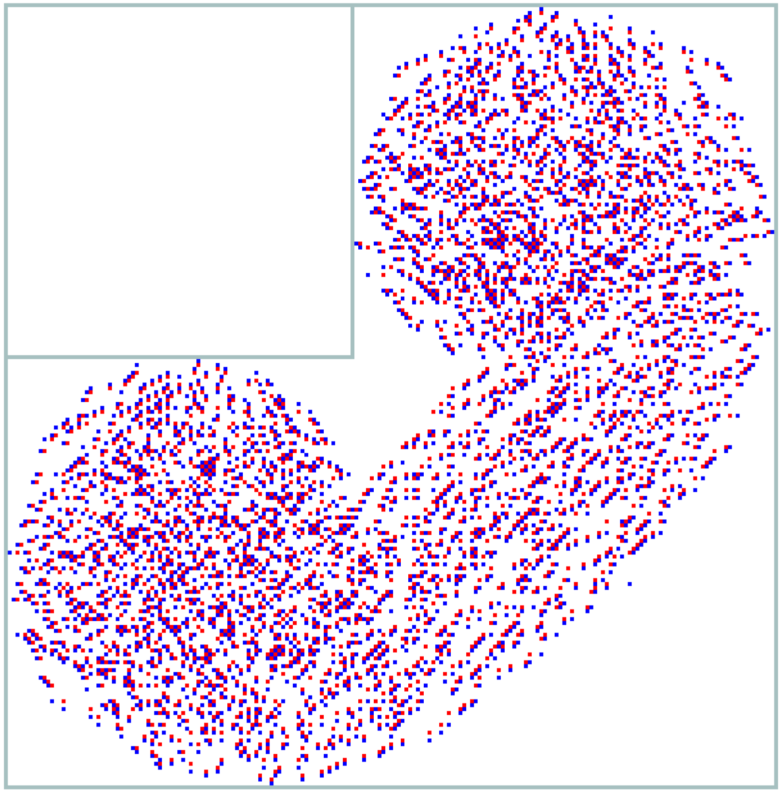
$\Delta = \frac{1}{2}$

$N = 200$

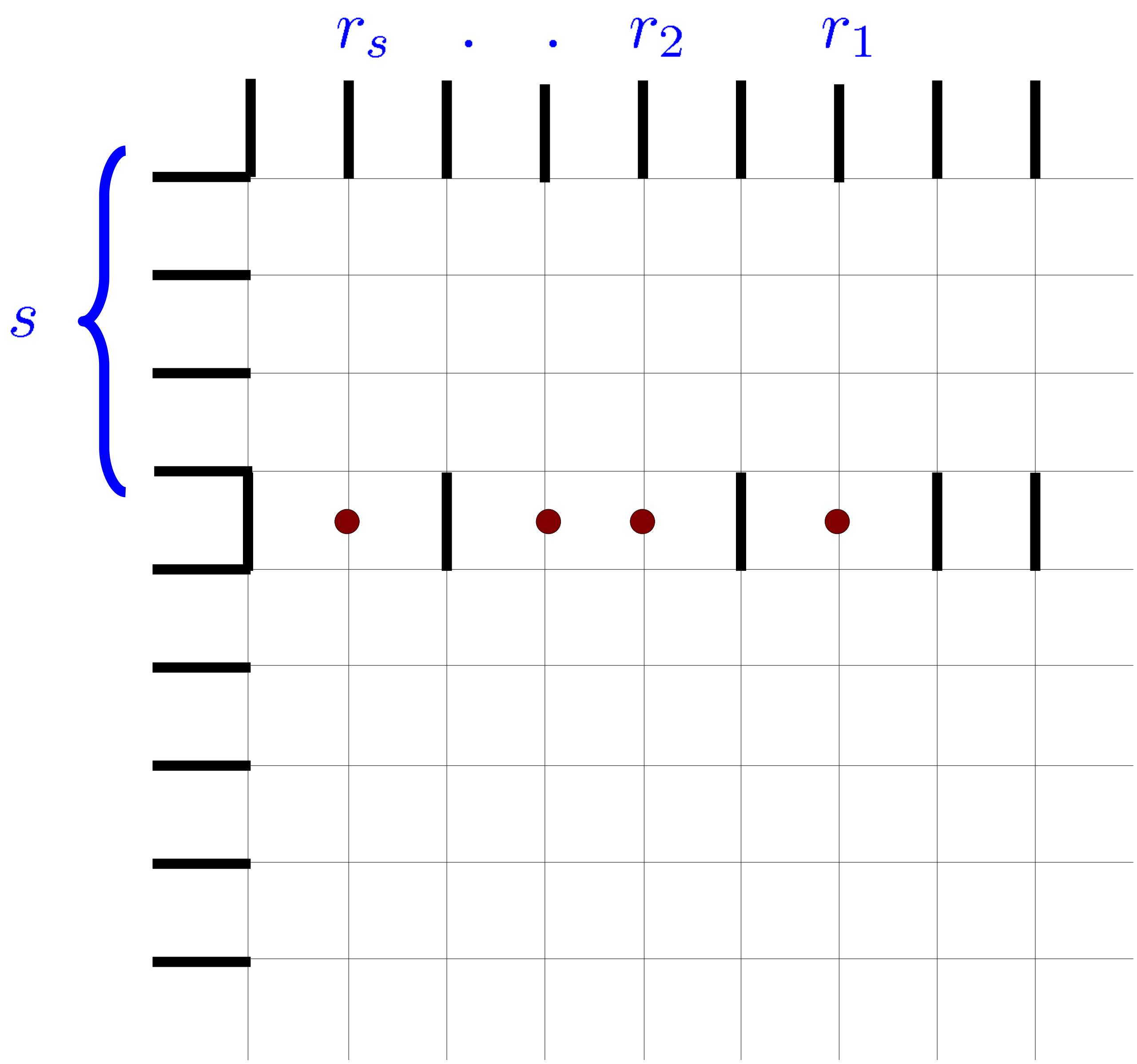
$n = 110$



$H_{N,s}^{(r_1,\ldots,r_s)}$

Row Configuration Probability

[FC-AP'11]

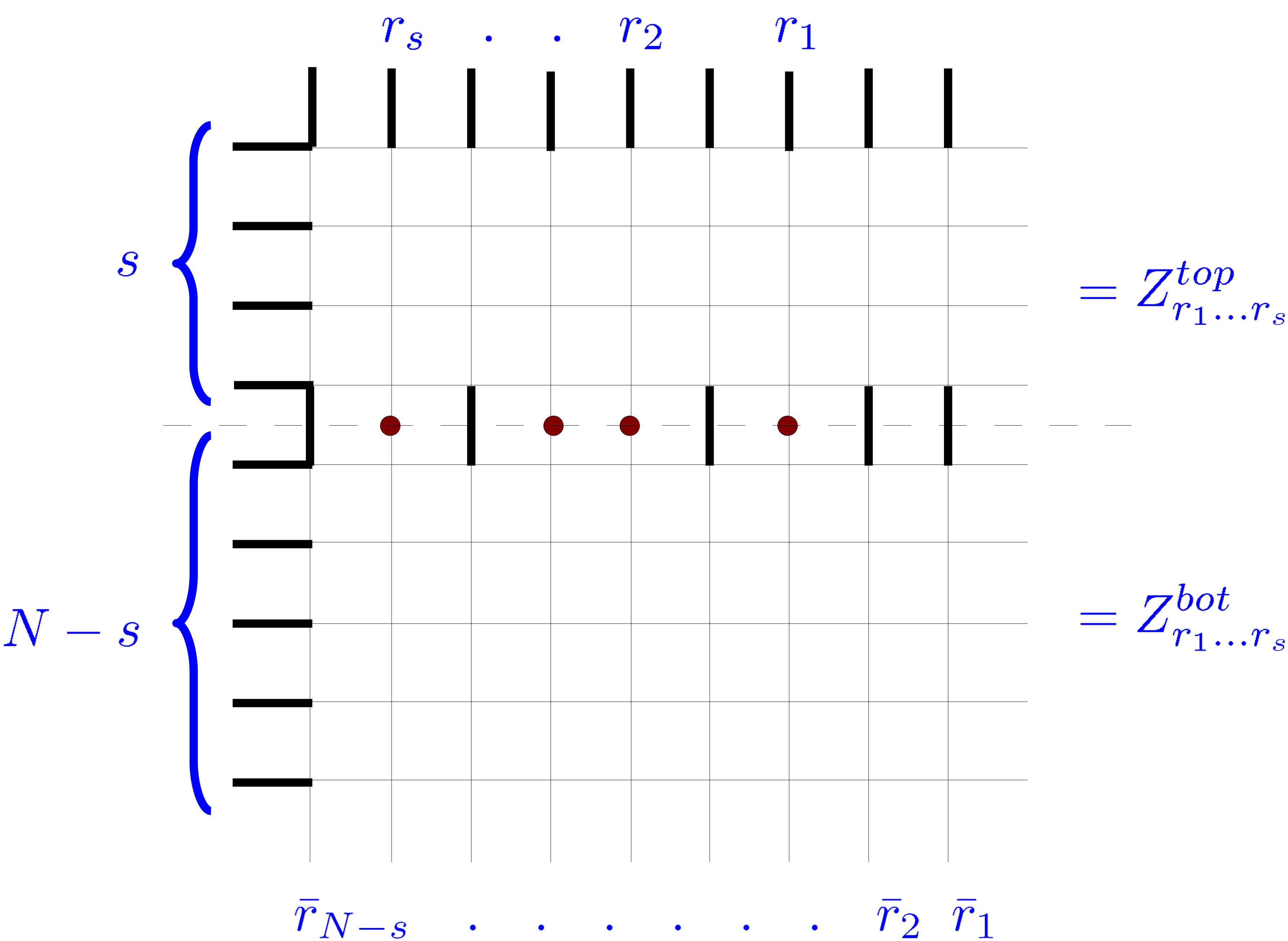


RCP as a building block for more fundamental correlation functions (e.g., EFP, polarization)

$H_{N,s}^{(r_1,\ldots,r_s)}$

Row Configuration Probability

[FC-AP'11]



Of course: $$H_{N,s}^{(r_1,\dots,r_s)} = \frac{1}{Z_N} Z^{top}_{r_1,\dots,r_s}\, Z^{bot}_{r_1,\dots,r_s}$$

Crossing symmetry implies:

$$Z^{top}_{r_1,\dots,r_s}(\lambda_1,\dots,\lambda_N;\nu_1,\dots,\nu_s) = Z^{bot}_{\bar r_1,\dots,\bar r_{N-s}}(\pi-\lambda_1,\dots,\pi-\lambda_N;-\nu_1,\dots,-\nu_s)$$

On the other hand, we can write for EFP:

$$F_N^{(r,s)} = \sum_{r_s=s}^{r} \cdots \sum_{r_2=2}^{r_3-1} \sum_{r_1=1}^{r_2-1} H_{N,s}^{(r_1,\dots,r_s)}$$

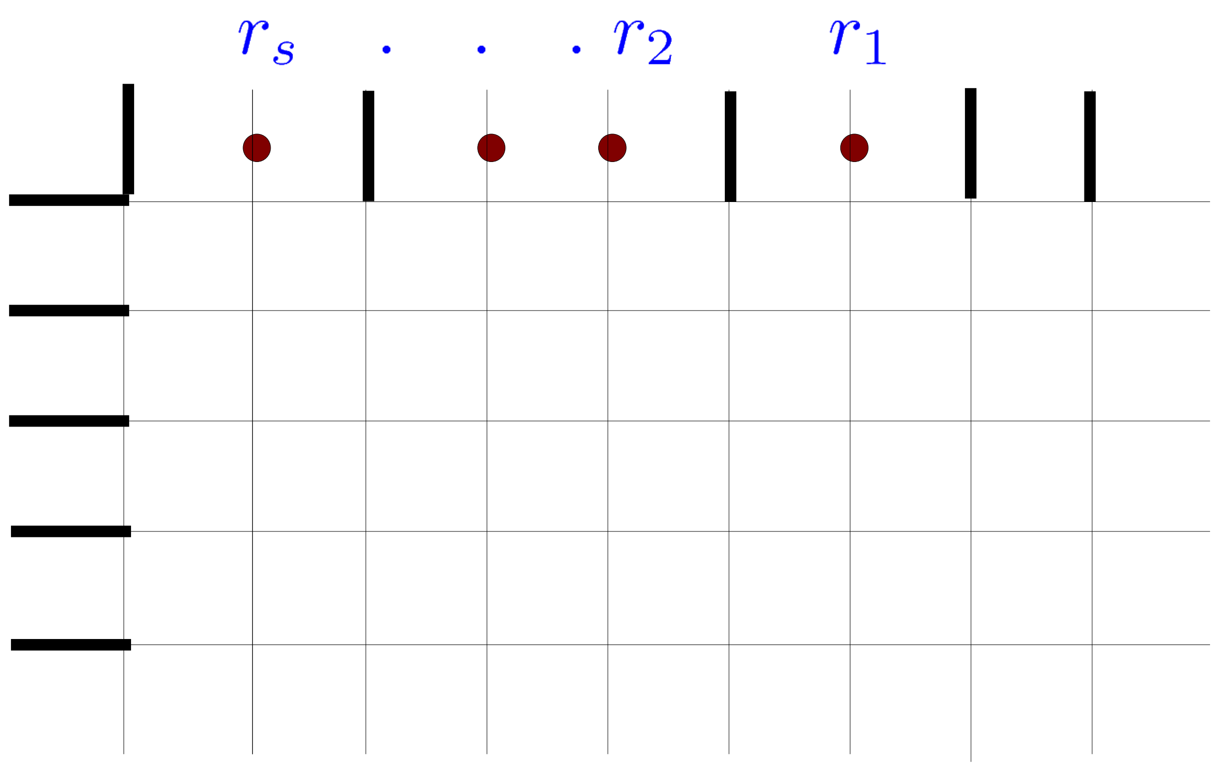
or polarization (probability that arrow on vertical edge at position (r,s) points down):

$$H_N^{(r,s)} = \sum_{j=0}^{s} \sum_{r_s=r+s-j}^{N} \cdots \sum_{r_{j+1}=r+1}^{r_{j+2}-1} \sum_{r_j=j}^{r-1} \cdots \sum_{r_1=1}^{r_2-1} H_{N,s}^{(r_1,\dots,r_s)}$$

- To be able to perform these sums, a necessary condition is that $H_{N,s}^{(r_1,\dots,r_s)}$ should depend only on $r_1,\dots,r_s$ and not on $\bar r_1,\dots \bar r_{N-s}$.

- We thus need two different, complementary representations, for $Z^{top}_{r_1,\dots,r_s}$ and $Z^{bot}_{r_1,\dots,r_s}$ These two representations are not related by crossing symmetry.

$$Z^{bot}_{r_1,\dots,r_s} =$$

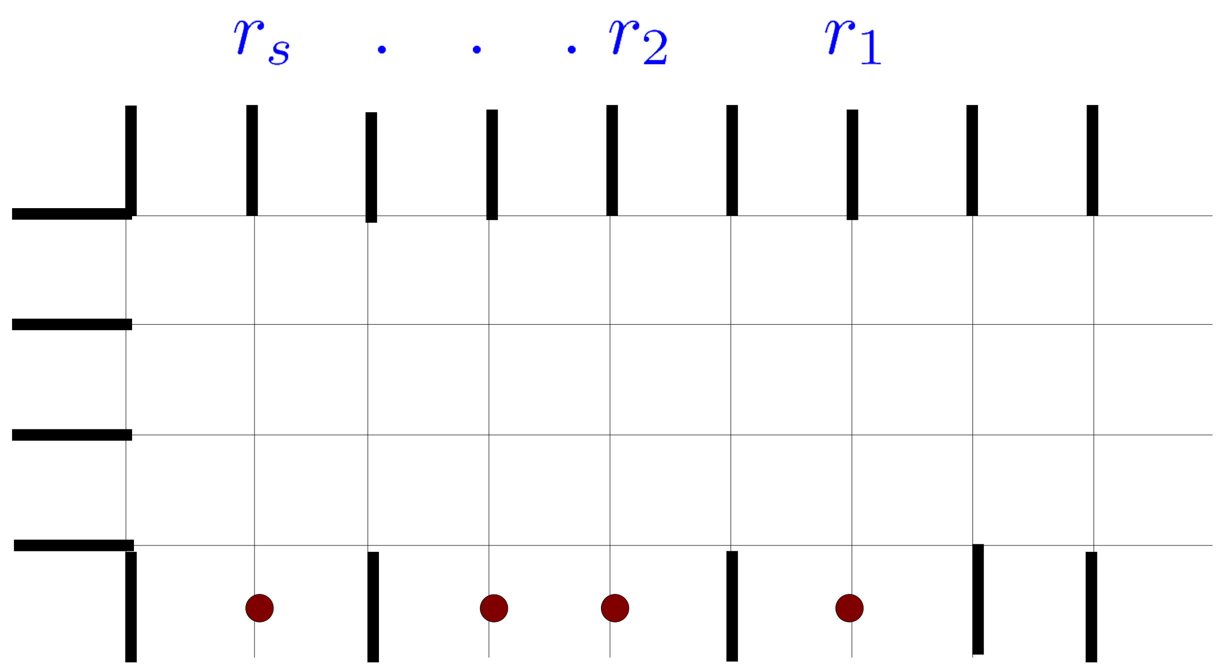


$$= \langle \Downarrow_{bot} | B_{bot}(\lambda_N) \cdots B_{bot}(\lambda_{r_s+1})\, A_{bot}(\lambda_{r_s})\, B_{bot}(\lambda_{r_s-1}) \cdots$$
$$\cdots B_{bot}(\lambda_{r_1+1})\, A_{bot}(\lambda_{r_1})\, B_{bot}(\lambda_{r_1-1}) \cdots B_{bot}(\lambda_1) | \Uparrow_{bot} \rangle$$

Use:

$$A(\lambda)B(\lambda') = f(\lambda,\lambda')B(\lambda')A(\lambda) + g(\lambda,\lambda')B(\lambda)A(\lambda')$$

We get an s-fold sum of partition functions on the $(N-s)\times(N-s)$ lattice (which can then be reinterpreted as the Laplace expansion of $N\times N$ determinant and treated as previous cases).



$$Z^{top}_{r_1,\ldots,r_s} =$$

$$= \langle \Downarrow_{top} | D_\top(\lambda_N) \cdots D_{top}(\lambda_{r_s+1}) B_{top}(\lambda_{r_s}) D_{top}(\lambda_{r_s-1}) \cdots$$
$$\times \cdots D_{top}(\lambda_{r_1+1}) B_{top}(\lambda_{r_1}) D_{top}(\lambda_{r_1-1}) \cdots D_\top(\lambda_1) | \Uparrow_{top}\rangle$$

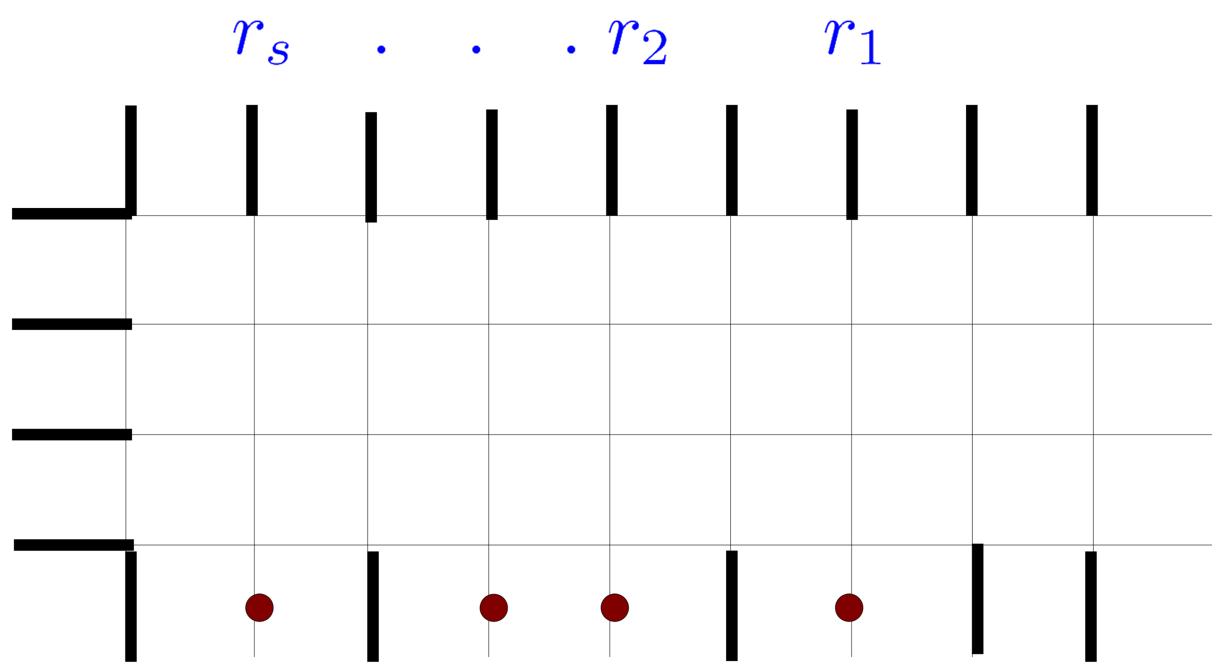
Use

$$D(\lambda)B(\lambda') = f(\lambda',\lambda)B(\lambda')D(\lambda) + g(\lambda,\lambda')B(\lambda)D(\lambda')$$

We get an $(N-s)$-fold sum of partition function on $s \times s$ lattice, which can again be reinterpreted as Laplace expansion of $N \times N$ determinant, etc....

BUT now dependence is on $\bar{r}_j$'s (rather than r_j's). It is just the crossing symmetry transformed of previous example. Useless!

$$Z^{top}_{r_1,\ldots,r_s} =$$



$$= \langle \Downarrow_{top} | D_\top(\lambda_N) \cdots D_{top}(\lambda_{r_s+1}) B_{top}(\lambda_{r_s}) D_{top}(\lambda_{r_s-1}) \cdots$$
$$\times \cdots D_{top}(\lambda_{r_1+1}) B_{top}(\lambda_{r_1}) D_{top}(\lambda_{r_1-1}) \cdots D_\top(\lambda_1) | \Uparrow_{top} \rangle$$

Let us use instead:

$$B(\lambda)D(\lambda') = f(\lambda',\lambda)D(\lambda')B(\lambda) + g(\lambda,\lambda')D(\lambda)B(\lambda')$$

We obtain an $s-$fold sum of partition functions on the $s \times s$ lattice. This is a different representation, complementary to the previous one, and depending on the correct set of r_j's (not on $\bar{r}_j$'s).

We correspondingly obtain two different s-fold multiple integral representations:

$$Z_{r_1,\dots,r_s}^{bot} = Z_N \frac{\prod_{j=1}^{s} t^{j-r_j}}{a^{s(N-1)} c^s (2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{z_j^{r_j}} \prod_{1\le j<k\le s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} \times h_{N,s}(z_1,\dots,z_s)\, dz_1 \cdots dz_s.$$

and

$$Z_{r_1,\dots,r_s}^{top} = \frac{Z_s a^{s(N-s)}}{(2\pi i)^s} \prod_{j=1}^{s} t^{j-r_j} \oint_{C_1} \cdots \oint_{C_1} \prod_{j=1}^{s} \frac{\left(t^2 w_j - 2\Delta t + 1\right)^{r_j-1}}{(w_j-1)^{r_j}} \times \prod_{1\le j<k\le s} \frac{w_k - w_j}{t^2 w_j w_k - 2\Delta t w_j + 1} h_{s,s}(w_1,\dots,w_s)\, dw_1 \cdots dw_s.$$

both depending only on r_j's.

We correspondingly obtain two different s-fold multiple integral representations:

$$Z^{bot}_{r_1,\dots,r_s} = Z_N \frac{\prod_{j=1}^{s} t^{j-r_j}}{a^{s(N-1)} c^s (2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{z_j^{r_j}} \prod_{1\le j<k\le s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} \times h_{N,s}(z_1,\dots,z_s)\, dz_1 \cdots dz_s.$$

and

$$Z^{top}_{r_1,\dots,r_s} = \frac{Z_s a^{s(N-s)}}{(2\pi i)^s} \prod_{j=1}^{s} t^{j-r_j} \oint_{C_1} \cdots \oint_{C_1} \prod_{j=1}^{s} \frac{\left(t^2 w_j - 2\Delta t + 1\right)^{r_j-1}}{(w_j-1)^{r_j}} \times \prod_{1\le j<k\le s} \frac{w_k - w_j}{t^2 w_j w_k - 2\Delta t w_j + 1}\, h_{s,s}(w_1,\dots,w_s)\, dw_1 \cdots dw_s.$$

both depending only on r_j's.

To be compared to symmetrized MIR of EFP:

$$F_N(r,s) \propto \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{j=1}^{s} \frac{[(t^2-2\Delta t) z_j + 1]^{s-1}}{z_j^r (z_j-1)^s} \prod_{\substack{j,k=1\\ j\ne k}}^{s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} \times h_{N,s}(z_1,\dots,z_s) h_{s,s}(u_1,\dots,u_s).$$

where $\quad u_j := \dfrac{1-z_j}{(t^2-2\Delta t) z_j + 1}\,.$

Indeed the sum in:

$$F_N^{(r,s)} = \sum_{r_s=s}^{r} \cdots \sum_{r_2=2}^{r_3-1} \sum_{r_1=1}^{r_2-1} H_{N,s}^{(r_1,\dots,r_s)}$$

can be performed, and the correct result for EFP is recovered.

Hopefully, the same can be done for the sums defining polarization (probability that arrow on vertical edge at position (r, s) points down):

$$H_N^{(r,s)} = \sum_{j=0}^{s} \sum_{r_s=r+s-j}^{N} \cdots \sum_{r_{j+1}=r+1}^{r_{j+2}-1} \sum_{r_j=j}^{r-1} \cdots \sum_{r_1=1}^{r_2-1} H_{N,s}^{(r_1,\dots,r_s)}$$

and the scaling limit behaviour of the resulting representation can be studied.

Final comments

1. Concerning the above representation for $Z^{top}_{r_1,\ldots,r_s}$ it should be mentioned that it is related to the so-called `coordinate wavefunction' representation [Izergin-Korepin-Reshetikhin'87] by a deformation of the integration contour.

2. Note that the existence of various equivalent representations implies a plethora of non-trivial identities (of course, somehow already encoded in the RTT relations)

In particular, if instead of our representation we use for, say, $Z^{top}_{r_1,\ldots,r_s}$, the `coordinate wavefunction' representation, we are unable to reproduce EFP after summation; this last representation seems thus less convenient.

The equivalence of the result of of procedure at point 3. with EFP implies a non-trivial identity. Indeed, define:

$$f(w_1,\ldots,w_s;z_1,\ldots,z_s) := \prod_{1\le j<k\le s} \frac{1}{w_j - w_k}$$

$$\times \operatorname*{Asym}_{w_1,\ldots,w_s} \left[\prod_{j=1}^{s} \frac{1}{\prod_{l=1}^{j} w_l - \prod_{l=1}^{j} z_l} \prod_{1\le j<k\le s} (t^2 w_j w_k - 2\Delta t w_j + 1) \right]$$

Then:

$$s! \operatorname*{Asym}_{z_1,\ldots,z_s} \left[f(1,\ldots,1;z_1,\ldots,z_s) \prod_{1\le j<k\le s} [z_j(t^2 z_j z_k - 2\Delta t z_k + 1)] \right]$$

$$= \frac{1}{\prod_{j=1}^{s}(1-z_j)} \operatorname*{Asym}_{z_1,\ldots,z_s} \left[\prod_{1\le j<k\le s} \frac{[(t^2 - 2\Delta t) z_j + 1](t^2 z_j z_k - 2\Delta t z_k + 1)}{(z_j - 1)} \right]$$

Very similar identities (but simpler!) have appeared in [Zinn-Justin Di Francesco'08] [Zeilberger'07] [Tracy-Widom'08]