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# Subatomic vortices

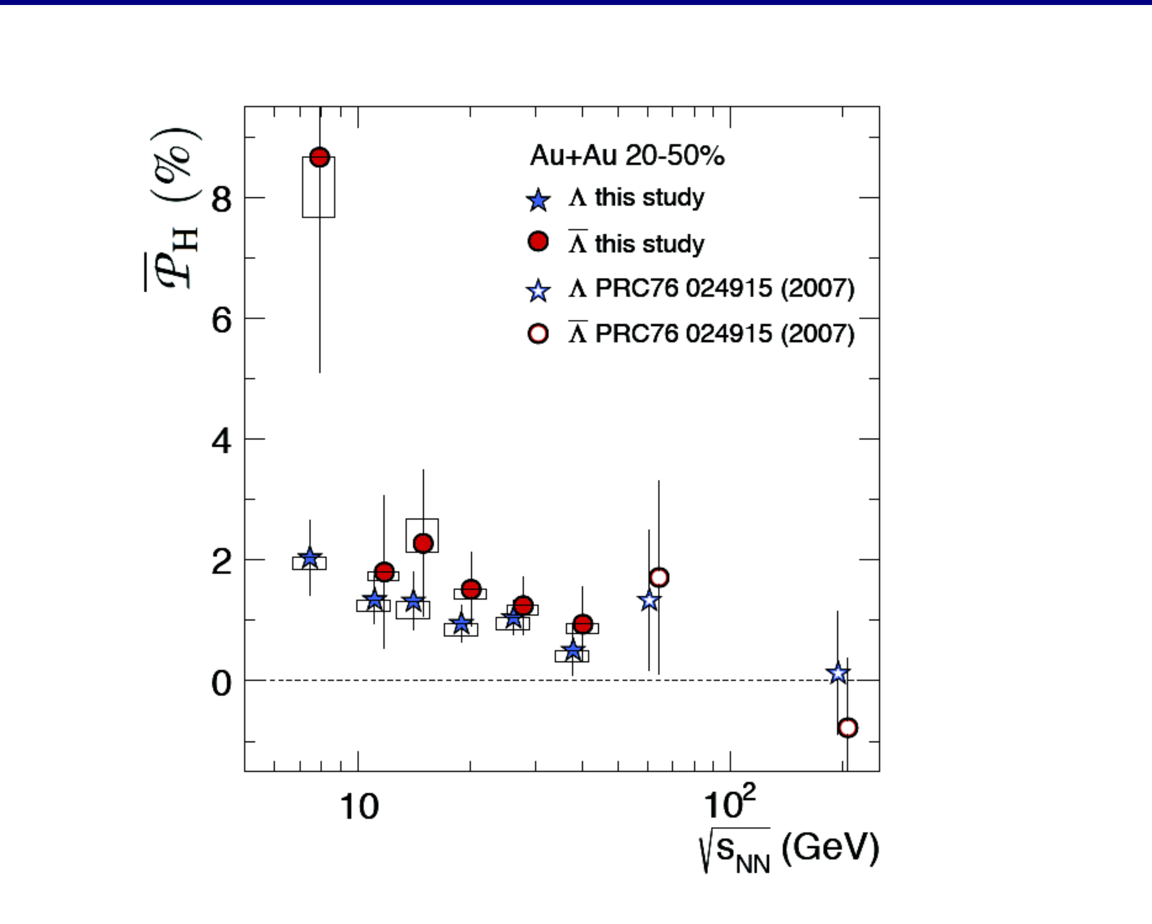
## OUTLINE

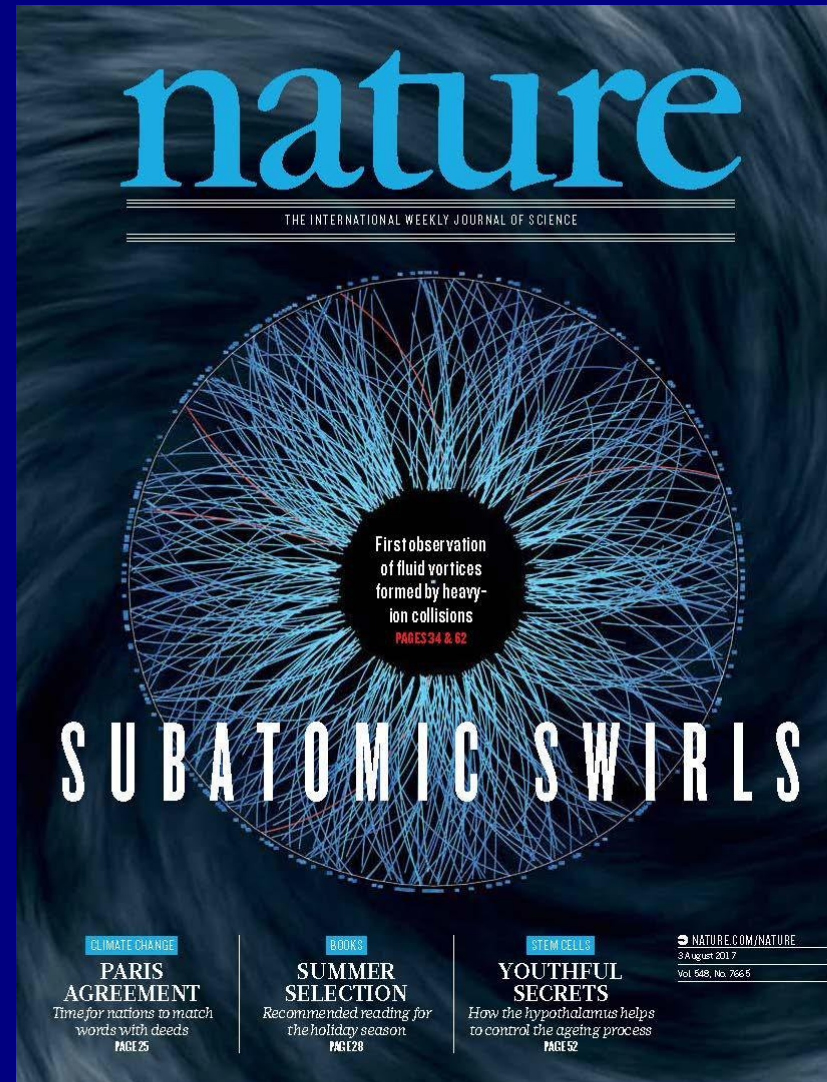
- Prologue
- QCD and relativistic nuclear collisions
- Quark Gluon Plasma as a fluid
- Polarization by rotation and acceleration: theory
- $\Lambda$ polarization
- Conclusions

Colloquium 27 -9 - 2017

# Prologue

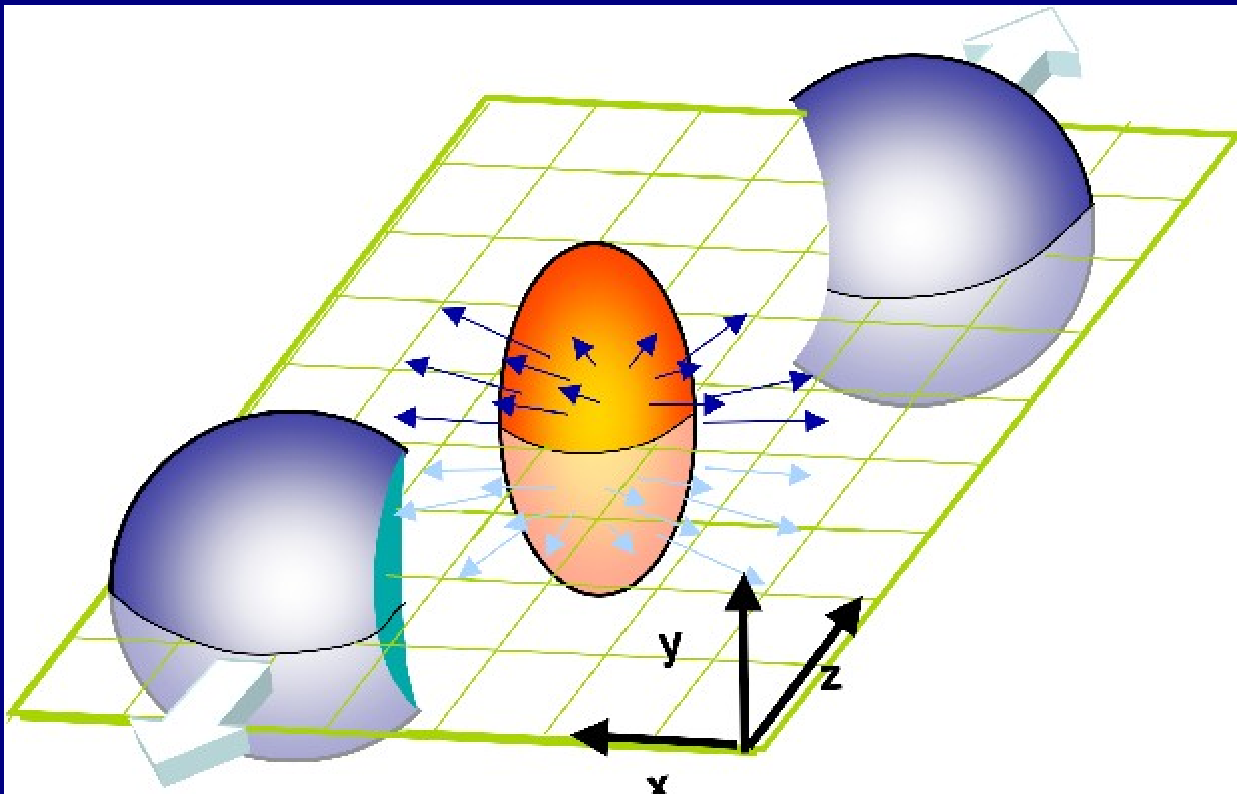
STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions,* Nature 548 62-65, 2017




First evidence of a quantum effect in (relativistic) hydrodynamics

# Introduction: relativistic nuclear collisions




$$7 < \sqrt{s_{NN}} < 200 \text{ GeV} \quad \text{RHIC}$$

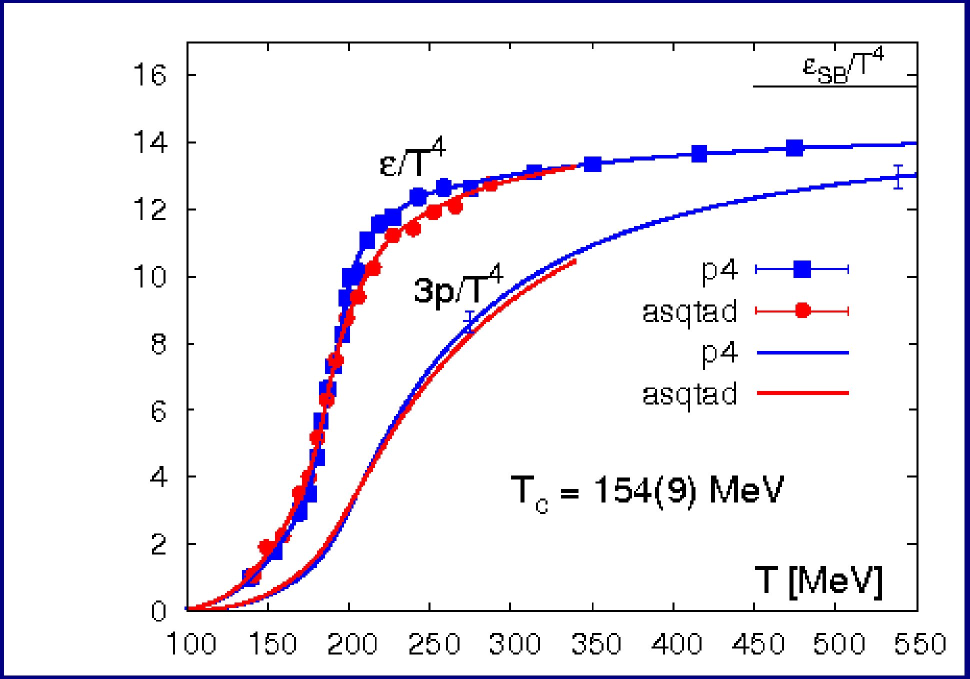
$$\sqrt{s_{NN}} \sim \ 2.76 - 5.5 \text{ TeV} \quad \text{LHC}$$

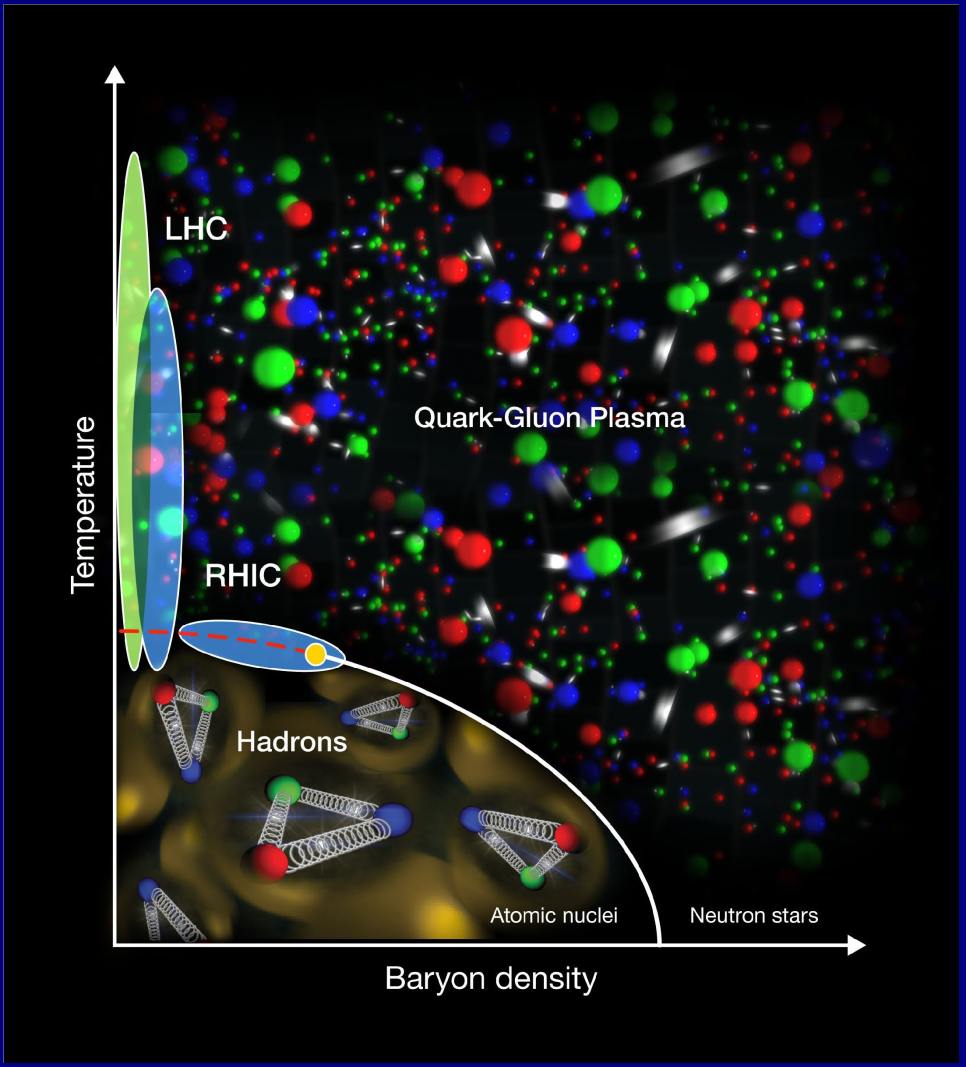
Goal: the production and study of the phases of QCD at finite T and $\mu_B$

# QCD phase diagram: lattice-QCD

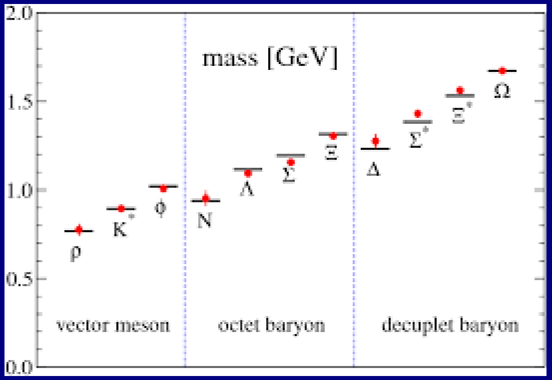
$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} &= \bar{\psi}_i \left( i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \\
&= \bar{\psi}_i (i\gamma^\mu \partial_\mu - m)\psi_i - g G^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a ,
\end{aligned}
$$

Equation of state

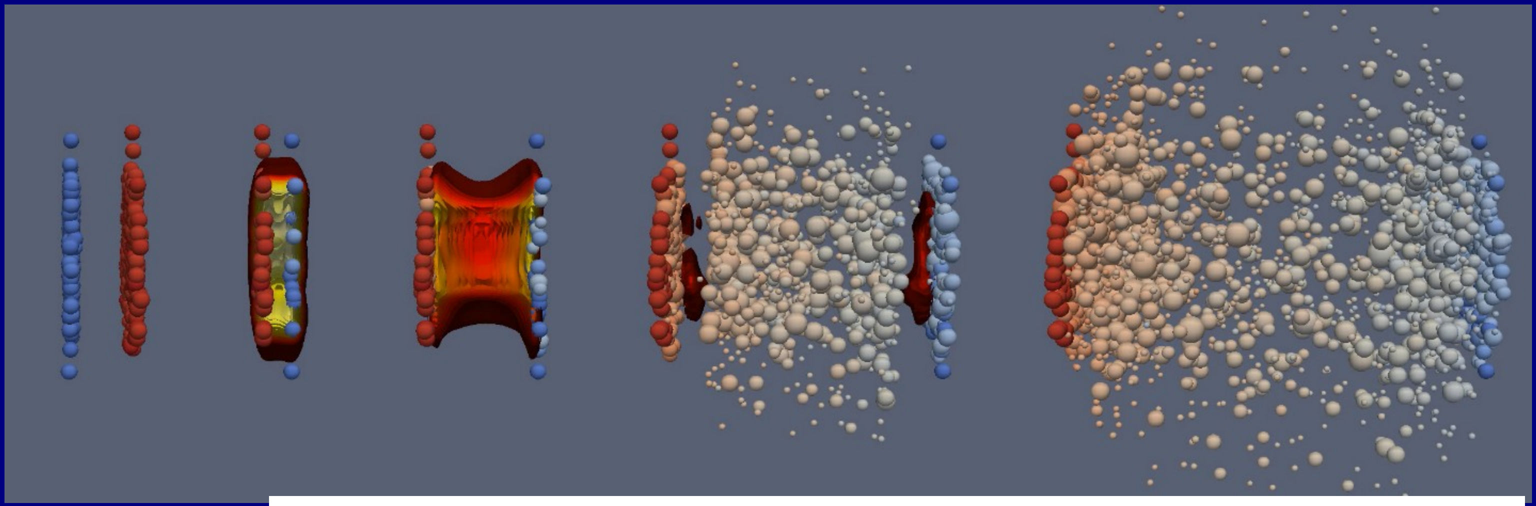


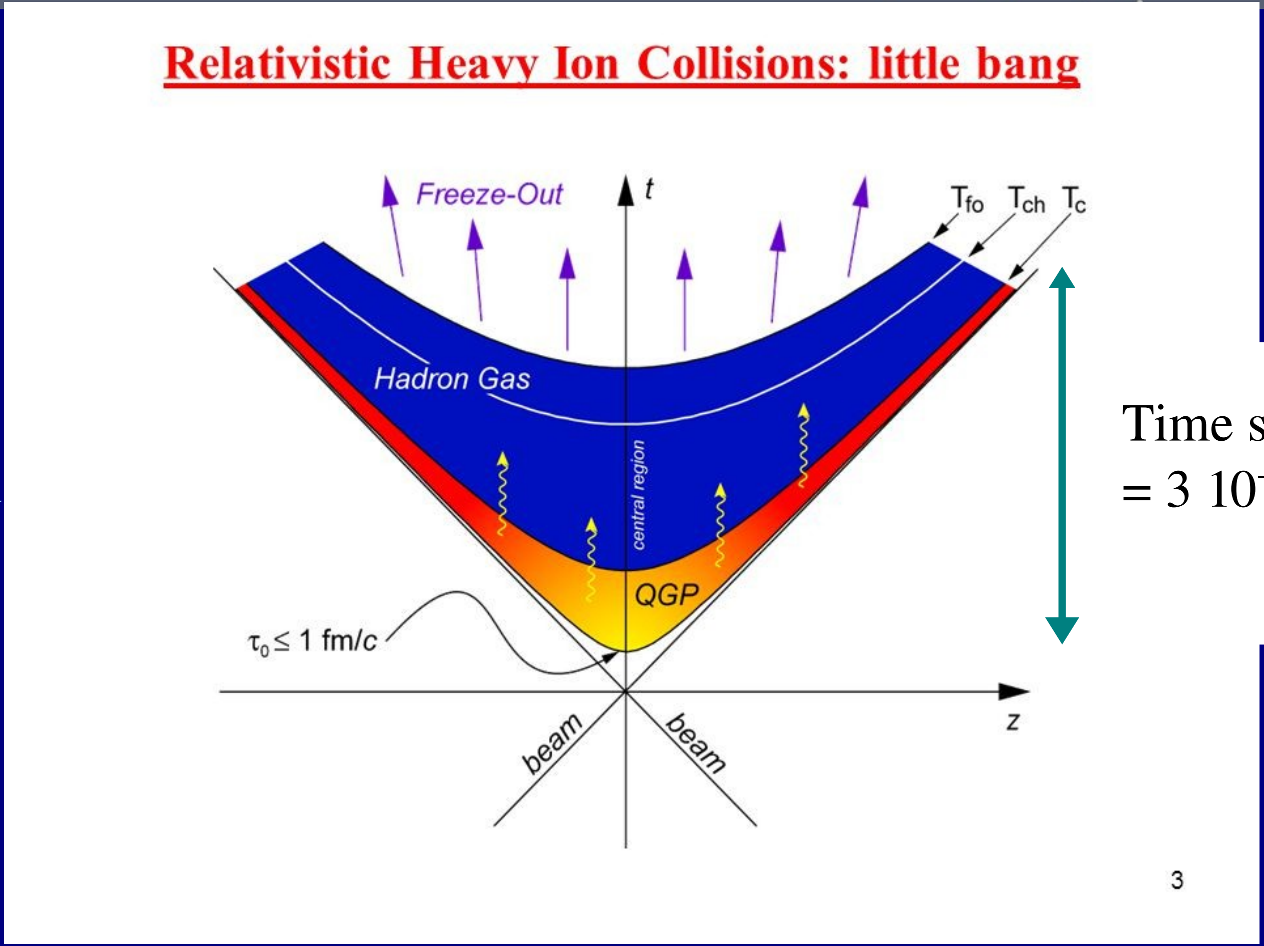




QCD reproduces baryon masses



# Relativistic nuclear collision: time evolution



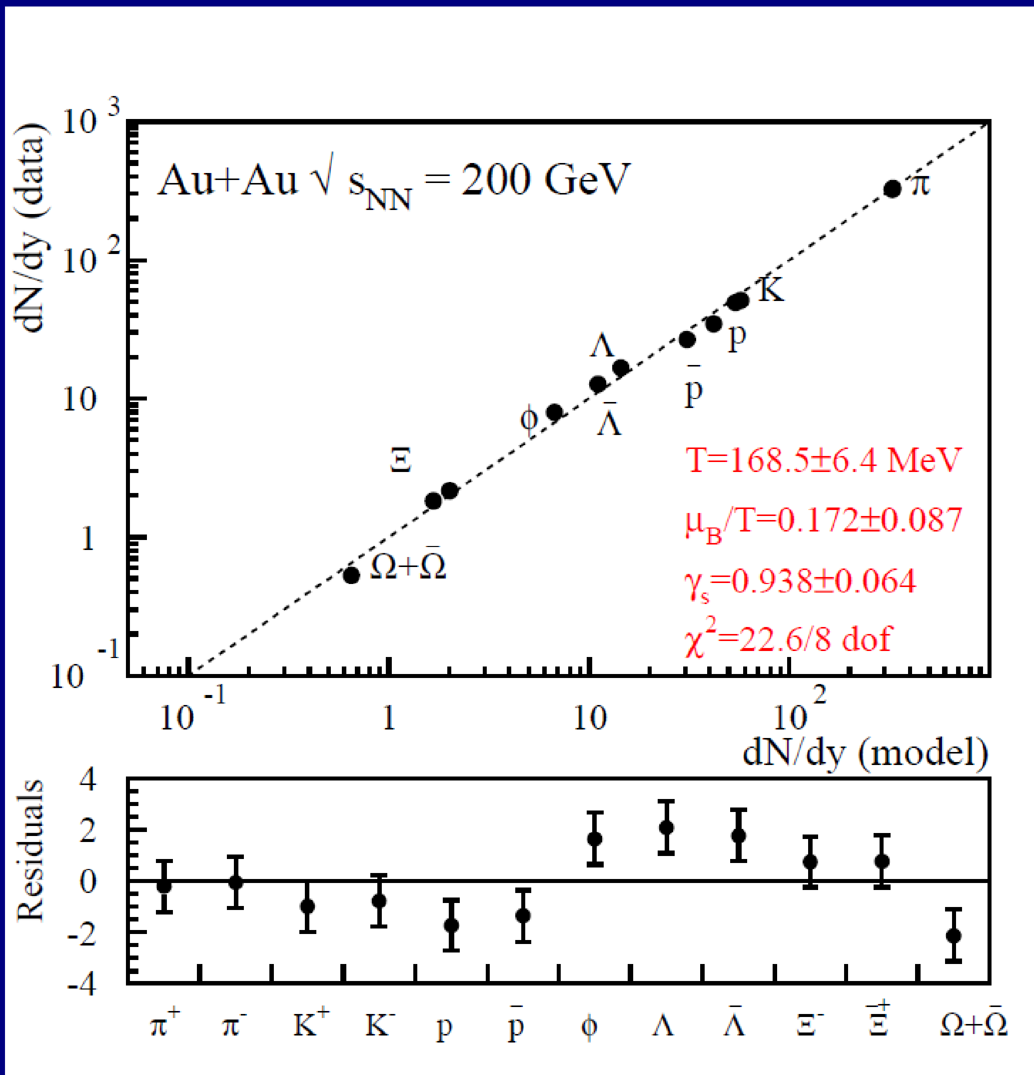



QGP = Quark Gluon Plasma

The properties of the QGP must be inferred from the spectra of final particles (final hadrons)

Time scale ~ 10 fm/c = 3 $10^{-23}$ sec

# Evidence of local thermodynamic equilibrium

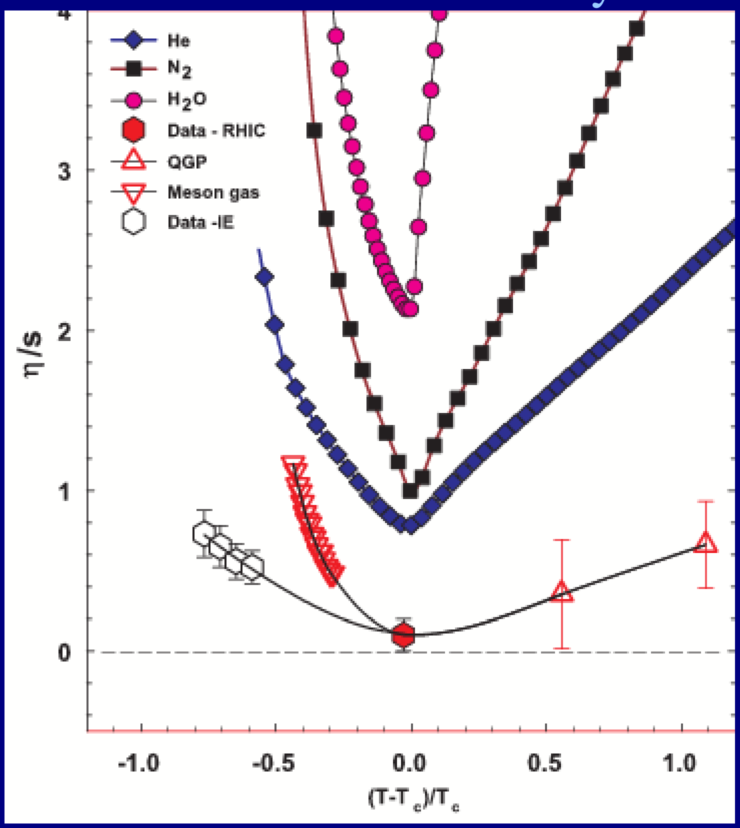



The plasma falls apart and hadronizes at the (continous) phase transition apparently in a “maximally disordered” fashion

$$\varepsilon \frac{\mathrm{d}N_j}{\mathrm{d}^3 p} = \int \mathrm{d}\Sigma \cdot p \frac{1}{\mathrm{e}^{\beta \cdot p_j - \boldsymbol{\mu} \cdot \mathbf{q}_j} \pm 1}$$

# QGP is an extraordinary fluid

- It is the hottest ever made: $T \sim 5\ 10^{12}$ K
- It is the tiniest ever made: ~10 fm across
- It has the largest initial pressure, energy density and largest initial acceleration ($a \sim 10^{30}\ g$) Surface gravity of a black hole ~ $3\ 10^{12}\ g/(M/Ms)$
- It has the lowest viscosity/entropy density ratio ever observed




$$\eta/s \sim \lambda/\lambda_T$$

If the mean free path ~ mean wavelength one cannot even think of a "particle". No kinetic description of the fluid is allowed.

*QGP around $T_c$ cannot be described in terms of colliding particles or quasiparticles and yet local thermodynamic equilibrium can be defined*

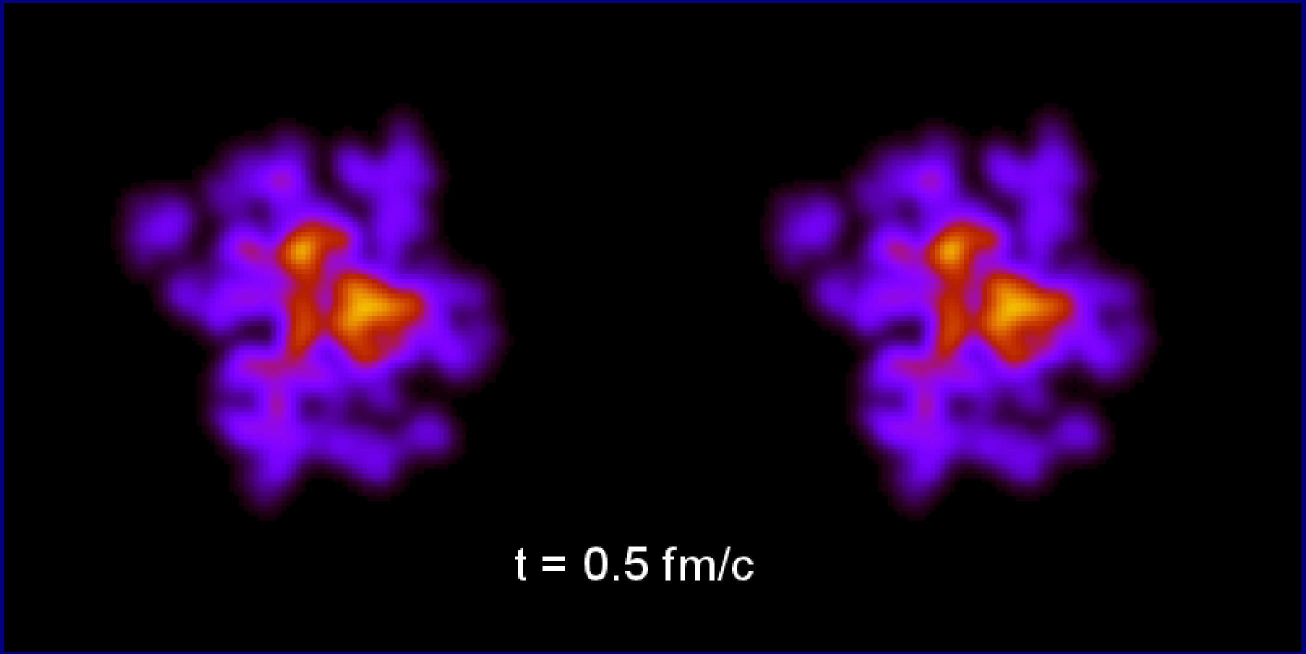
# Quark Gluon Plasma and relativistic hydrodynamics

To study the dynamics of the QGP, we rely on relativistic hydrodynamics

Effective theory working under the key assumption of *local thermodynamic equilibrium*
Separation of scales: microscopic interaction length << lenght of variation of quantities describing thermodynamic equilibrium ($T$, $u$, $\mu_B$,...)

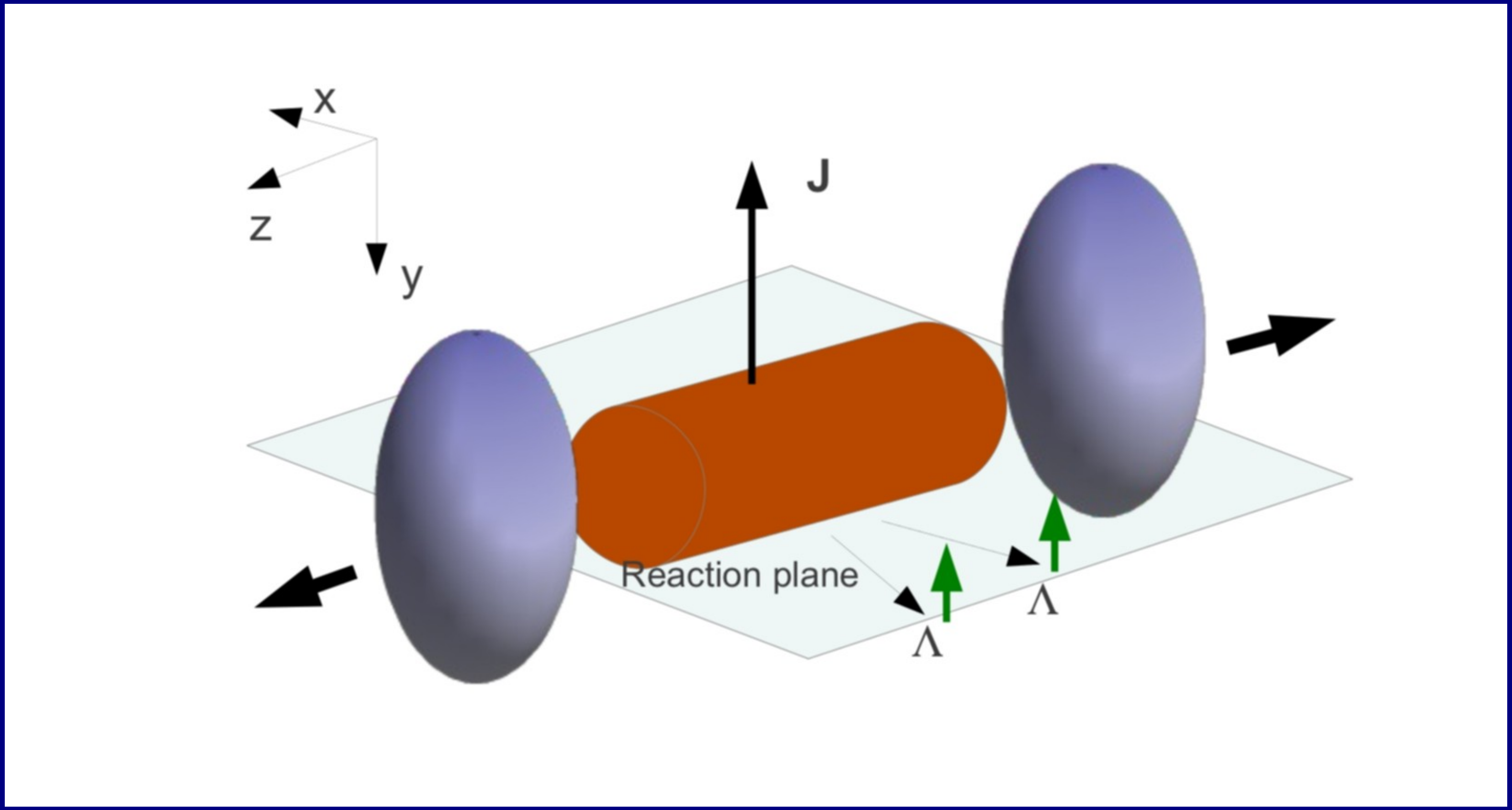
Formidable reduction of dynamical degrees of freedom:
interacting quantum fields ---> few classical fields $T(x)$, $u(x)$, etc.

$$\partial_\mu \langle \widehat{T}^{\mu\nu} \rangle = 0$$
$$\partial_\mu \langle \widehat{j}^{\mu} \rangle = 0$$



# Peripheral collisions: large angular momentum

Peripheral collisions ⇨ Angular momentum ⇨ Global polarization w.r.t reaction plane




Idea: equipartition of angular momentum among momentum and spin degrees of freedom

# Theoretical approaches to global polarization

- Polarization estimated at quark level by spin-orbit coupling

Z. T. Liang, X. N. Wang, Phys. Rev. Lett. 94 (2005) 102301

- By local thermodynamic equilibrium of the spin degrees of freedom

F. B., F. Piccinini, Ann. Phys. 323 (2008) 2452; F. B., F. Piccinini, J. Rizzo, Phys. Rev. C 77 (2008) 024906

Spin µ (thermal) vorticity

# Polarization by rotation

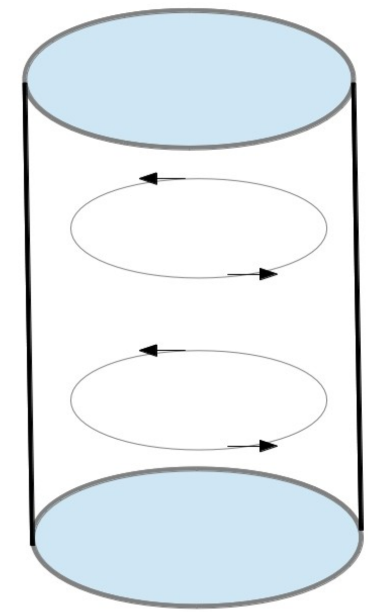
Take an ideal gas in a rigidly rotating vessel. At thermodynamical equilibrium (Landau) the gas will also have a velocity field

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

WARNING The potential term has a + sign as it stems from both centrifugal and Coriolis potentials

For the *comoving* observer the equilibrium particle distribution function will be given by:

$$f(\mathbf{x}', \mathbf{p}') \propto \exp[-\mathbf{p}'^2/2mT + m(\boldsymbol{\omega} \times \mathbf{x}')^2/2T]$$



If we calculate the distribution function seen by the external inertial observer

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}) \quad &\propto \exp[-\mathbf{p}^2/2mT + \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{x})/T] \\ &= \exp[-\mathbf{p}^2/2mT + \boldsymbol{\omega} \cdot \mathbf{L}/T] \end{aligned}$$

It seems quite *natural* to extend this to particle with spin

$$f(\mathbf{x},\mathbf{p},\mathbf{S}) \propto \exp[-\mathbf{p}^2/2mT + \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S})/T]$$

which implies that particles (and antiparticles) are *POLARIZED*, in a rotating ideal gas, along the direction of the angular velocity vector by an amount

$$P \simeq \frac{S+1}{3}\frac{\hbar\omega}{KT}$$

For a gas at STP with $\omega = 1000$ Hz, P ~ $10^{-11}$

For relativistic nuclear collisions:

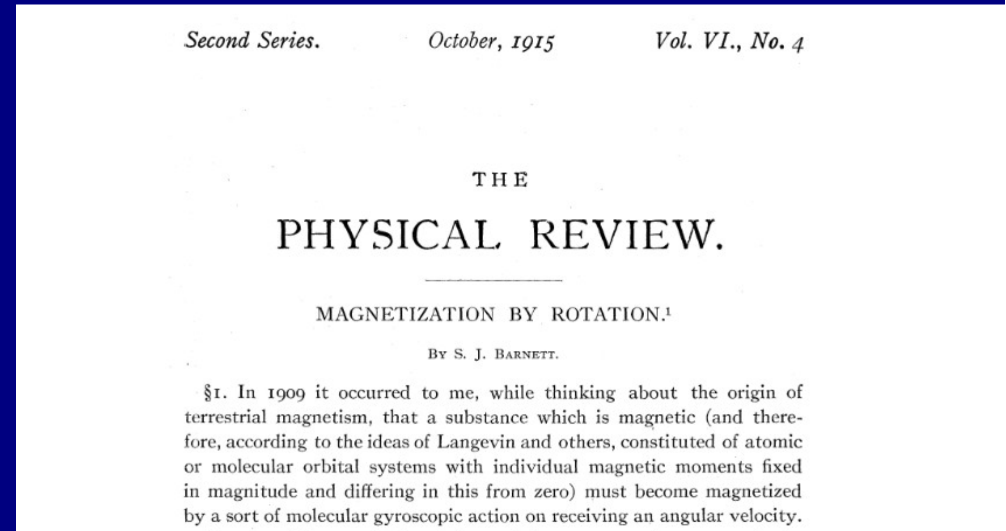
$$\frac{\hbar\omega}{KT} \approx \frac{c}{12\mathrm{fm}200\mathrm{MeV}} \approx 0.08$$

$$a \approx 10^{30} g \implies \frac{\hbar a}{cKT} \approx 0.06$$

# Barnett effect

S. J. Barnett, *Magnetization by Rotation,*

Phys. Rev.. 6, 239–270 (1915).



Second Series. October, 1915 Vol. VI., No. 4

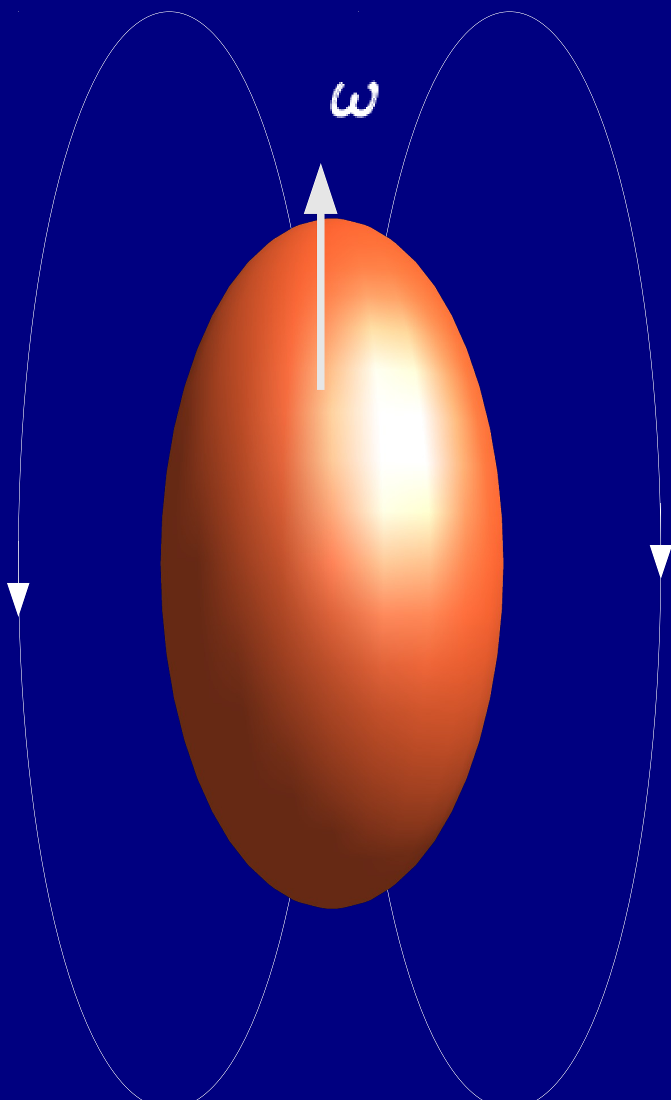
THE

PHYSICAL REVIEW.

MAGNETIZATION BY ROTATION.[1]

By S. J. Barnett.

§1. In 1909 it occurred to me, while thinking about the origin of terrestrial magnetism, that a substance which is magnetic (and therefore, according to the ideas of Langevin and others, constituted of atomic or molecular orbital systems with individual magnetic moments fixed in magnitude and differing in this from zero) must become magnetized by a sort of molecular gyroscopic action on receiving an angular velocity.




Spontaneous magnetization of an uncharged body when spun around its axis, in quantitative agreement with the previous polarization formula

$$M = \frac{\chi}{g}\omega$$

It can be seen as a dissipative transformation of the orbital angular momentum into spin of the constituents. The angular velocity decreases and a small magnetic field appears; this phenomenon is accompanied by a heating of the sample. Requires a spin-orbit coupling.

# Converse: Einstein-De Haas effect

*the only Einstein's non-gedanken experiment*

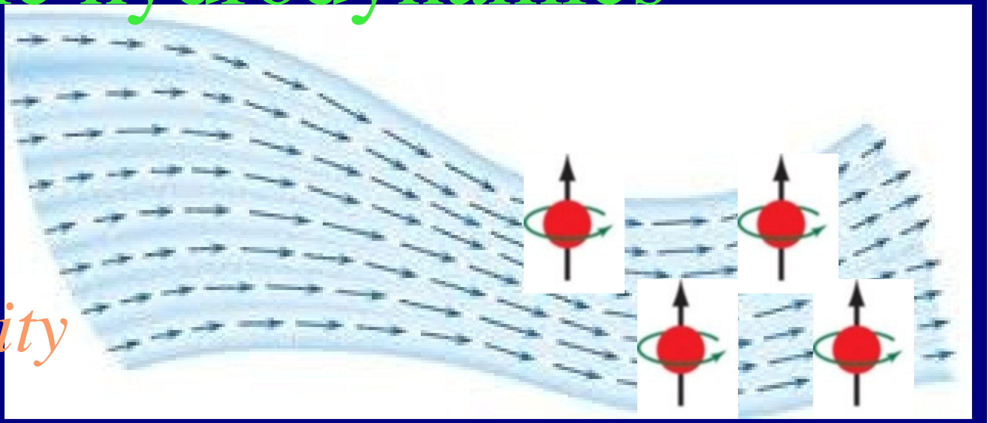
A. Einstein, W. J. de Haas, Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings, 18 I, 696-711 (1915)


Rotation of a ferromagnet originally at rest when put into an external H field

An effect of angular momentum conservation:
spins get aligned with H (irreversibly) and this must be compensated by a on overall orbital angular momentum

# Polarization and relativistic hydrodynamics

F. B., V. Chandra, L. Del Zanna, E. Grossi,
Ann. Phys. 338 (2013) 32



*Spin, local equilibrium and relativity*

It is crucial to use a *quantum-relativistic* formalism from the onset
Definition of a *relativistic spin* four-vector

For a single particle

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\lambda\rho}\langle \widehat{J}_{\nu\lambda}\widehat{P}_\rho \rangle$$

$$\langle \widehat{X} \rangle = \mathrm{tr}(\widehat{\rho}\widehat{X})$$

Relativistic Spin vs Pauli-Lubanski vs Polarization

$$S^\mu = \frac{1}{m} W^\mu = S P^\mu$$

# The density operator

Covariant form of the local thermodynamical equilibrium quantum density operator.

$$\widehat{\rho} = \frac{1}{Z} \exp\left[ - \int_\Sigma \mathrm{d}\Sigma_\mu \left( \widehat{T}^{\mu\nu}\beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

Extension of the known:

$$\widehat{\rho} = \frac{1}{Z} \exp[-\widehat{H}/T + \mu \widehat{Q}/T]$$

Hydrodynamic limit: Taylor expansion of the β and ζ fields around the point *x* where *Local operators* are to be calculated.

$$\mathrm{tr}(\widehat{O}(x)\widehat{\rho}) = O(x)$$

Local values of *T*,*u*,μ and their local derivatives (antisymmetric part: local thermal vorticity)

F. B., L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75 (2015) 191 (β frame)

$$\widehat{\rho} = \frac{1}{Z} \exp\left[ -\beta(x)\cdot\widehat{P} + \zeta(x)\widehat{Q} + \frac{1}{2}\varpi_{\mu\nu}(x)\widehat{J}^{\mu\nu}_x + \text{terms vanishing at global equilibrium} \right]$$

$$\beta^\mu = \frac{1}{T}u^\mu \qquad \zeta = \mu/T$$

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu\beta_\mu - \partial_\mu\beta_\nu)$$

*Thermal vorticity*

Adimensional in natural units

# Local polarization and spin tensor

For a particle with momentum $p$

$$S^\mu(p)N(p) = -\frac{1}{2m}\epsilon^{\mu\nu\lambda\rho}\int_\Sigma \mathrm{d}\Sigma_\tau \, \langle \widehat{\mathcal{S}}^\tau_{\nu\lambda}\rangle_p p_\rho$$

The rank 3 operator is the SPIN TENSOR and we need *its momentum-resolved mean value*

For the Dirac field

$$\widehat{\mathcal{S}}^{\lambda\mu\nu} = \frac{i}{8}\bar\Psi\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi$$

An useful tool: the covariant Wigner function

$$\begin{aligned} W(x,k)_{AB} &= -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y \; \mathrm{e}^{-ik\cdot y}\langle :\Psi_A(x-y/2)\overline{\Psi}_B(x+y/2):\rangle \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4y \; \mathrm{e}^{-ik\cdot y}\langle :\overline{\Psi}_B(x+y/2)\Psi_A(x-y/2):\rangle \end{aligned}$$

$$\langle\widehat{\mathcal{S}}^{\lambda\mu\nu}\rangle = \frac{i}{8}\langle\bar\Psi\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi\rangle = \frac{i}{8}\int \mathrm{d}^4k \; \mathrm{tr}_4(\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}W(x,k))$$

# Spin four-vector for spin ½ particles

Approximation at first order in the gradients

$$n_F = (\mathrm{e}^{\beta\cdot p - \xi} + 1)^{-1}$$

$$S^\mu(x,p) = -\frac{1}{8m}(1-n_F)\epsilon^{\mu\rho\sigma\tau} p_\tau \varpi_{\rho\sigma}$$

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

$$S^\mu(p) = \frac{1}{8m}\epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau n_F (1-n_F)\partial_\nu \beta_\rho}{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau \; n_F}$$

Same formula obtained with a perturbative expansion of the solution of the Wigner function e.o.m. in

R. h. Fang, L. g. Pang, Q Wang and X.n. Wang, Phys. Rev. C 94, 024904 (2016) arXiv:1604.04036

# Contributions of vorticity, acceleration and Grad T

$$\partial_\mu \beta_\nu = \partial_\mu \left(\frac{1}{T}\right) + \frac{1}{T}\partial_\mu u_\nu$$

$$\begin{aligned} A^\mu &= u \cdot \partial u^\mu \\ \omega^\mu &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu u_\rho u_\sigma \end{aligned}$$

$$\begin{aligned} S^\mu(p)\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau \, n_F &= \frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \int_\Sigma \mathrm{d}\Sigma \cdot p \, n_F(1-n_F)\nabla_\nu(1/T)u_\rho && \text{Grad T} \\ &+ \frac{1}{8m}\int_\Sigma \mathrm{d}\Sigma \cdot p \, n_F(1-n_F) 2\frac{\omega^\mu u \cdot p - u^\mu \omega \cdot p}{T} && \text{Vorticity} \\ &- \frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \int_\Sigma \mathrm{d}\Sigma \cdot p \, n_F(1-n_F)\frac{1}{T}A_\nu u_\rho && \text{Acceleration} \end{aligned}$$

In the rest frame of the particle:

$$\mathbf{S}^* \propto \frac{\hbar}{KT^2}\mathbf{u} \times \nabla T + \frac{\hbar}{KT}(\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\mathbf{v}\mathbf{u}/c^2) + \frac{\hbar}{KT}\mathbf{A}\times\mathbf{u}/c^2$$

Thermal term (new)

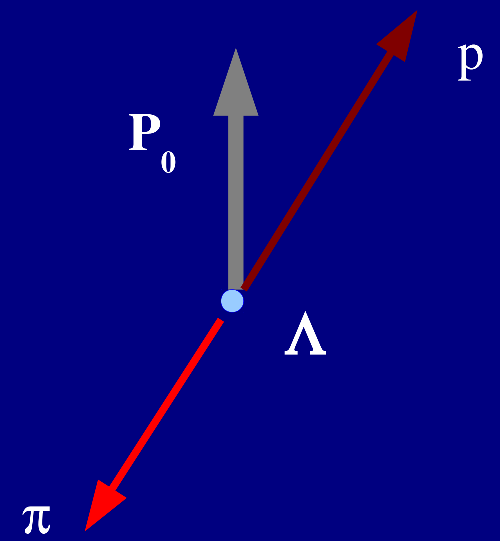
Vorticous term (known)

Acceleration term (purely relativistic)

$$\frac{\hbar\omega}{KT} \approx \frac{c}{12\mathrm{fm}200\mathrm{MeV}} \approx 0.08 \qquad a \approx 10^{30} g \implies \frac{\hbar a}{cKT} \approx 0.06$$

# How to observe it: global Λ polarization

Because of parity violation, the polarization vector of Λ can be measured in its decay
Into a proton and a pion




Distribution of protons in the Λ rest frame

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{1}{4\pi}(1 + \alpha \mathbf{P}_0 \cdot \hat{\mathbf{p}}^*) \qquad \mathbf{P}_0(p) = \mathbf{P}(p) - \frac{\mathbf{p}}{\varepsilon(\varepsilon + m)}\mathbf{P}(p) \cdot \mathbf{p}$$

$$\alpha = 0.642$$

# Global Λ polarization prediction at $\sqrt{s_{NN}}$ = 200 GeV

"Minimal" initial
Vorticity scenario

40-80 % centrality

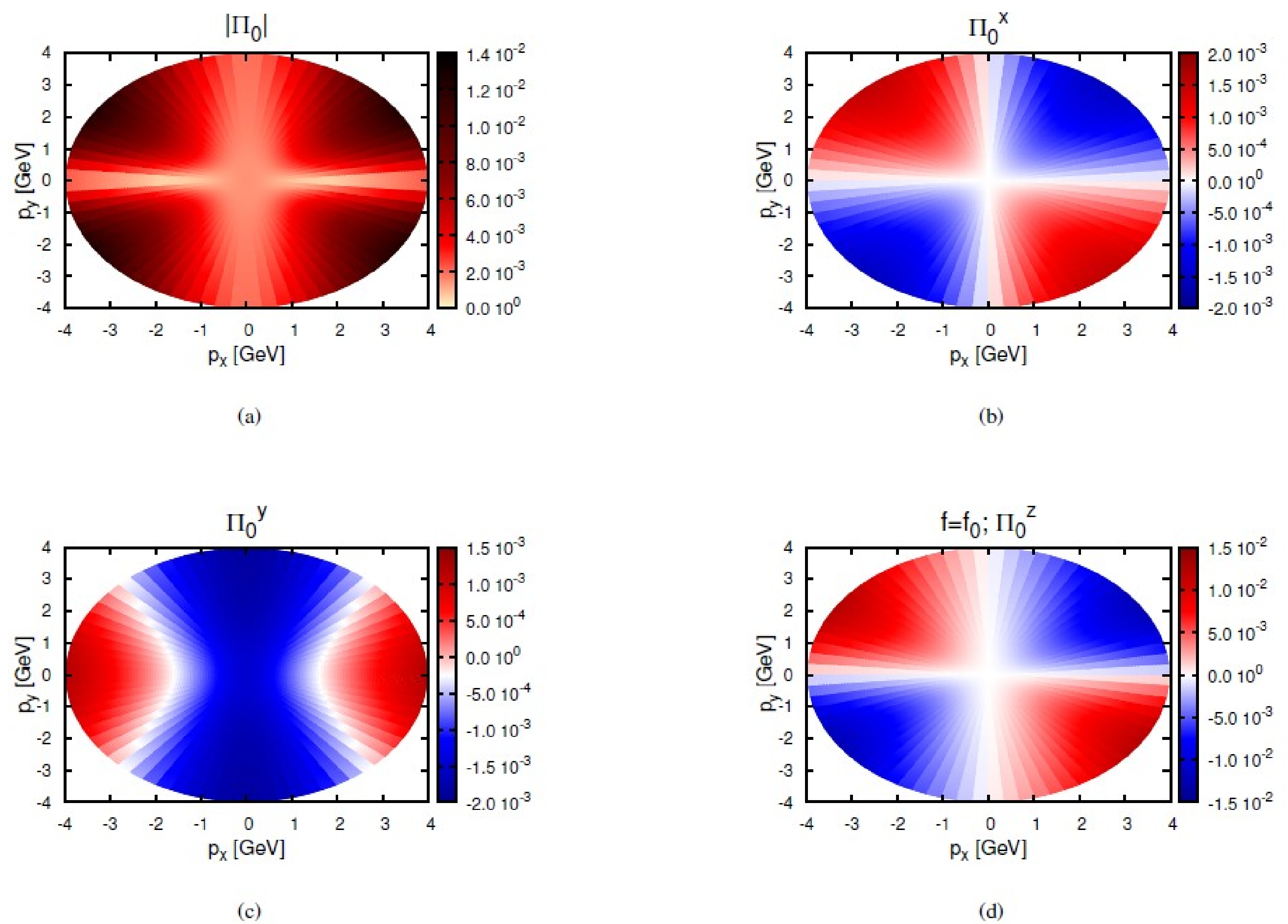
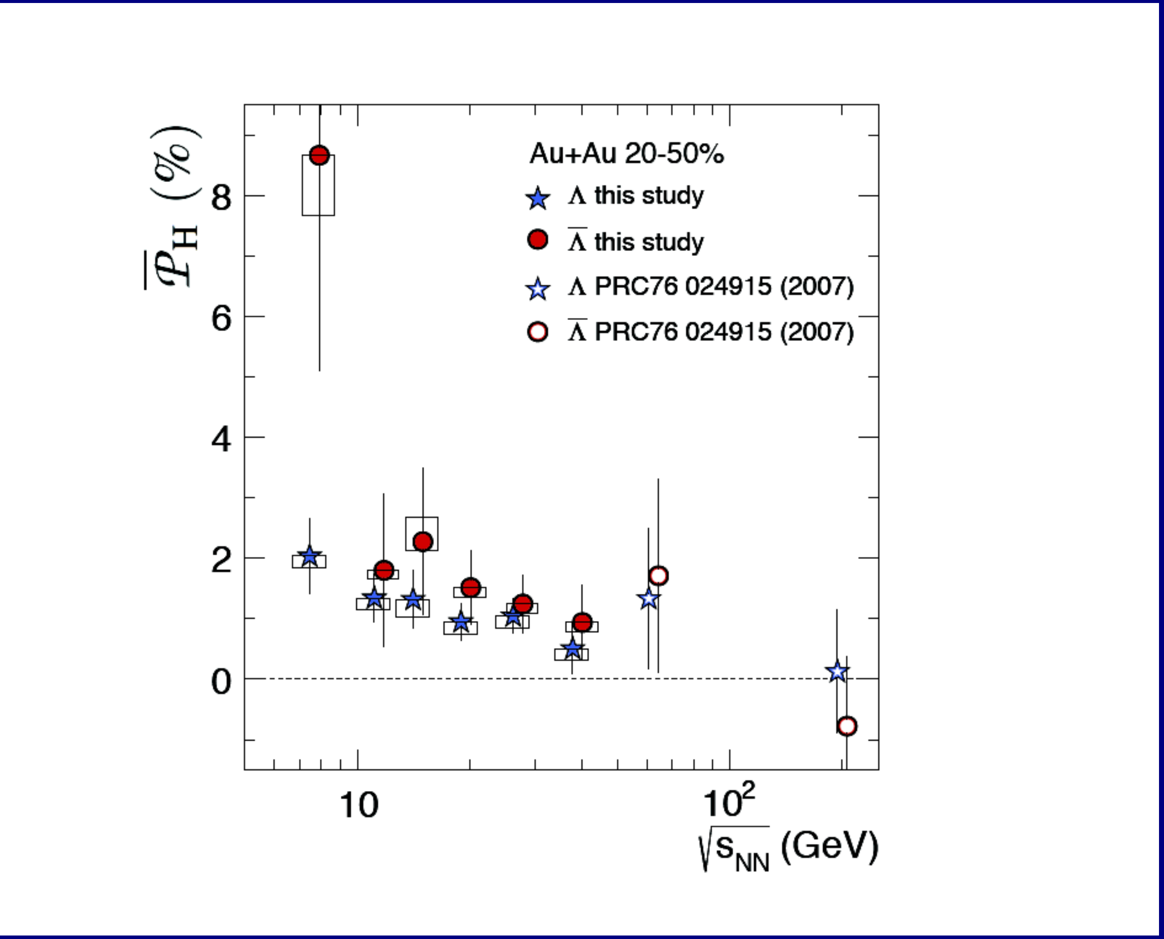


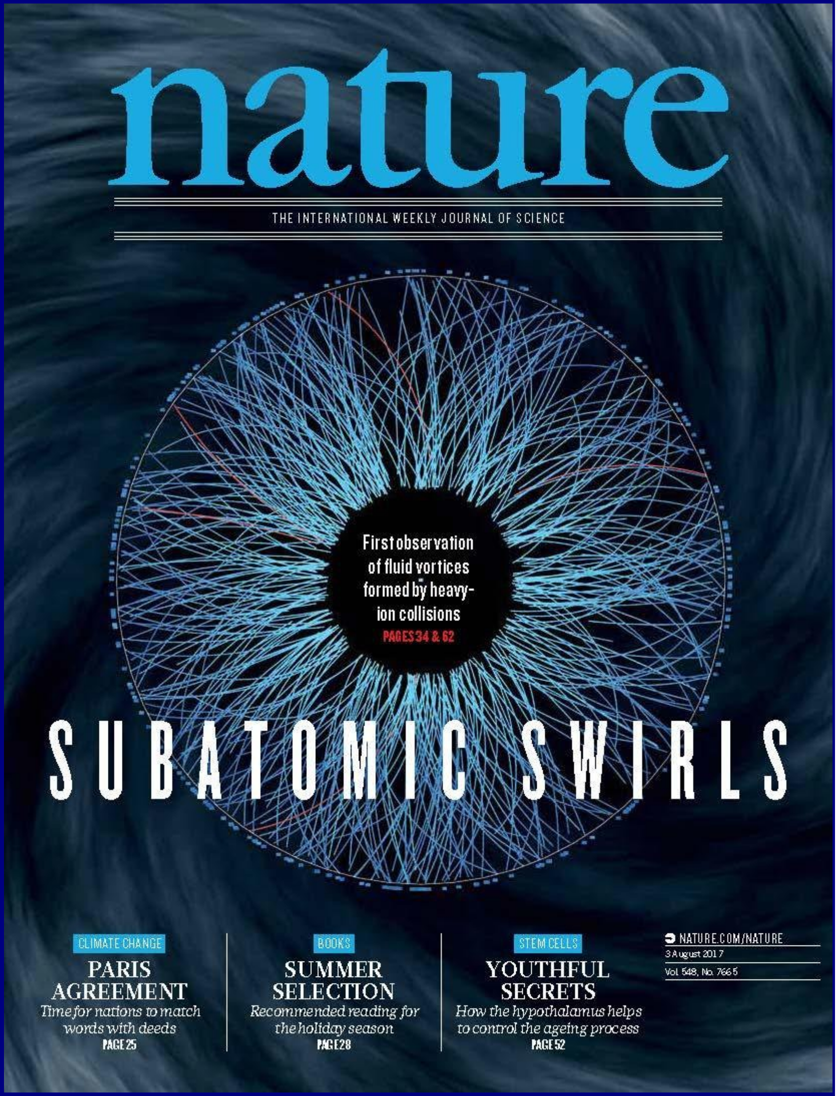

Figure 14: (color online) Magnitude (panel a) and components (panels b,c,d) of the polarization vector of the Λ hyperon in its rest frame.

F. B., G. Inghirami, V. Rolando, A. Beraudo, L. Del Zanna, A. De Pace, M. Nardi, G. Pagliara, V. Chandra
Eur. Phys. J C 75 (2015) 46

# First positive signal of this phenomenon

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions,* Nature 548 62-65, 2017

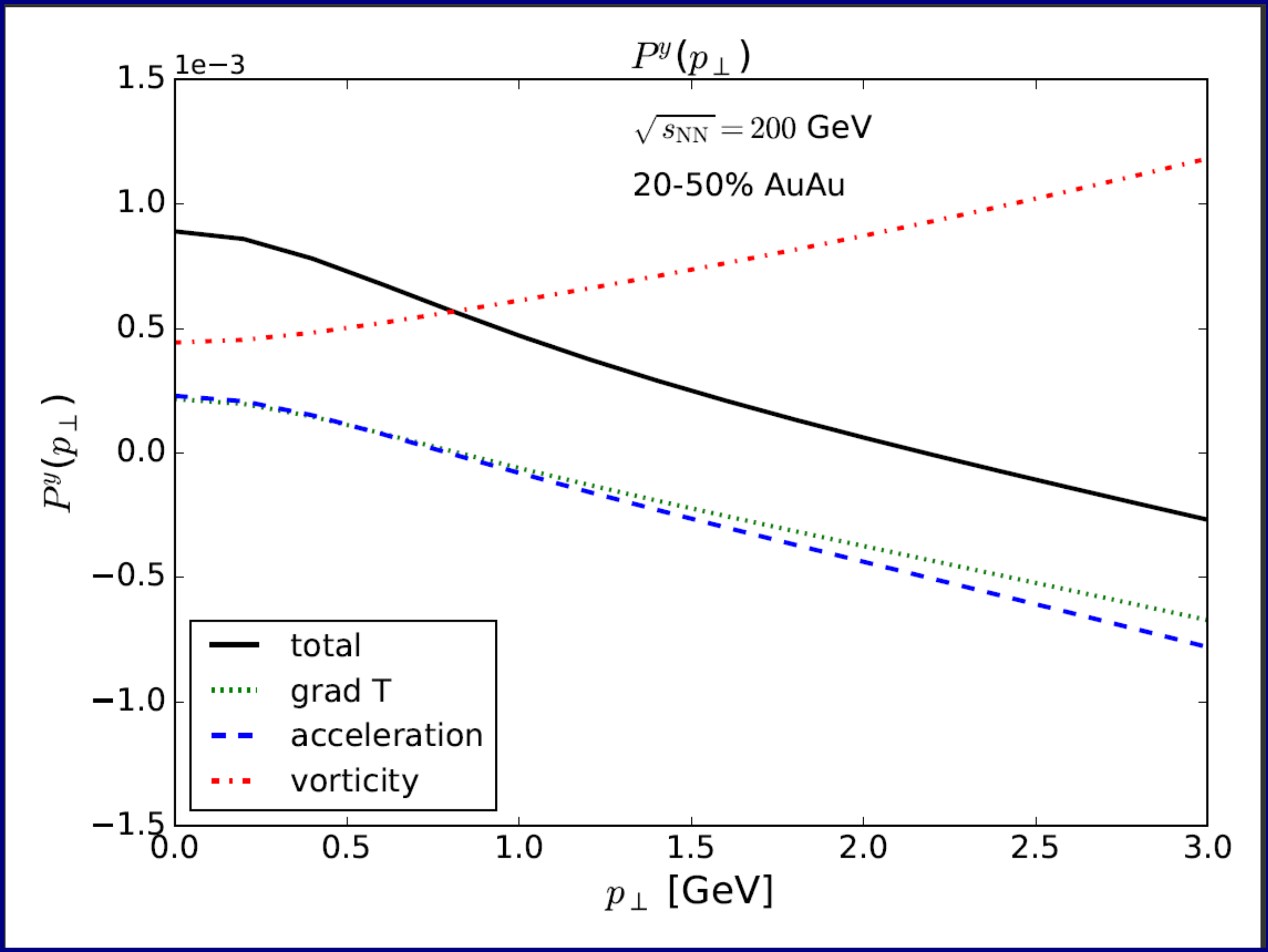



Particle and antiparticle have the same polarization sign.
This shows that the phenomenon cannot be driven
by a mean field (such as EM) whose coupling is *C-odd.*
Definitely favours the thermodynamic (equipartition) interpretation

# Contributions to polarization

$$\mathbf{S}^* \propto \frac{\hbar}{KT^2}\mathbf{u}\times\nabla T + \frac{\hbar}{KT}(\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\mathbf{v}\mathbf{u}/c^2) + \frac{\hbar}{KT}\mathbf{A}\times\mathbf{u}/c^2$$




I. Karpenko, INFN Florence

Rule of thumb

$$\omega/T \simeq P_\Lambda$$

At T=160 MeV this corresponds to

$\omega \sim 2.4\ 10^{-21}\ \mathrm{sec}^{-1}$

Relativistic effects are crucial

# Comparison of theoretical calculations with STAR result

I. Karpenko and F. B., Eur. Phys. J. C 77 (2017) 213

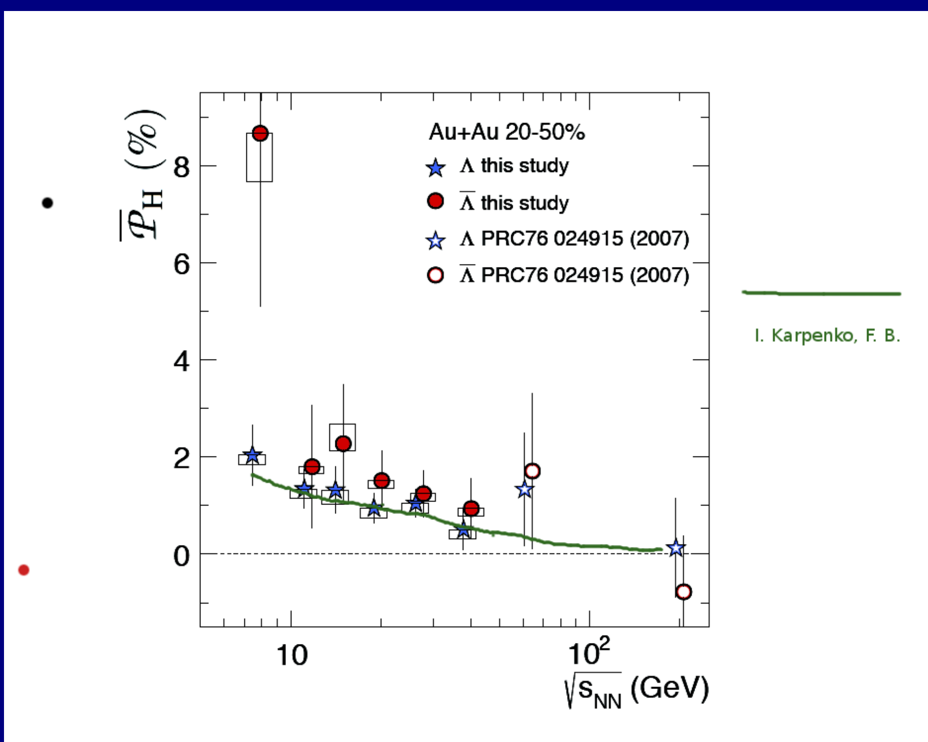
Y. Xie, D. Wang and L. P. Csernai, Phys. Rev. C 95 031901 (2017) [arXiv:1703.03770]

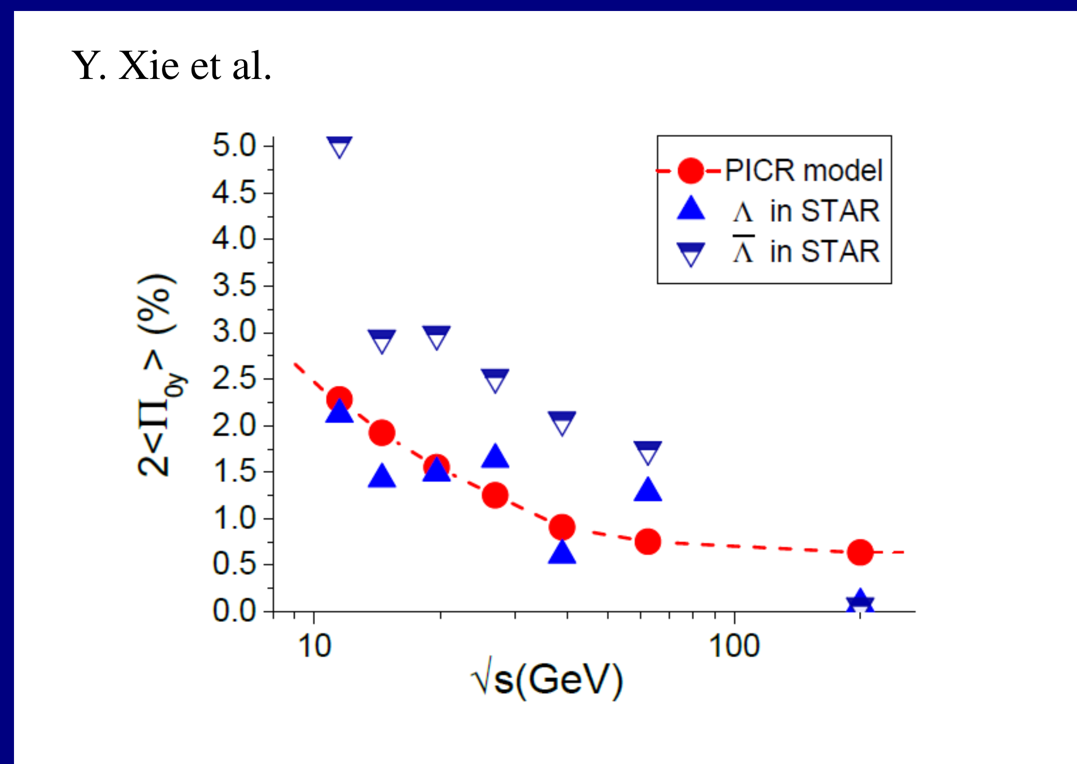
H. Li, H. Petersen, L. G. Pang, Q. Wang, X. L. Xia and X. N. Wang, arXiv:1704.03569

H. Li, L. G. Pang, Q. Wang and X. L. Xia, arXiv:1704.01507

Y. Sun and C. M. Ko, arXiv:1706.09467

Same thermal vorticity-related formula, but different initial conditions, evolution models as well as hadronization pictures.



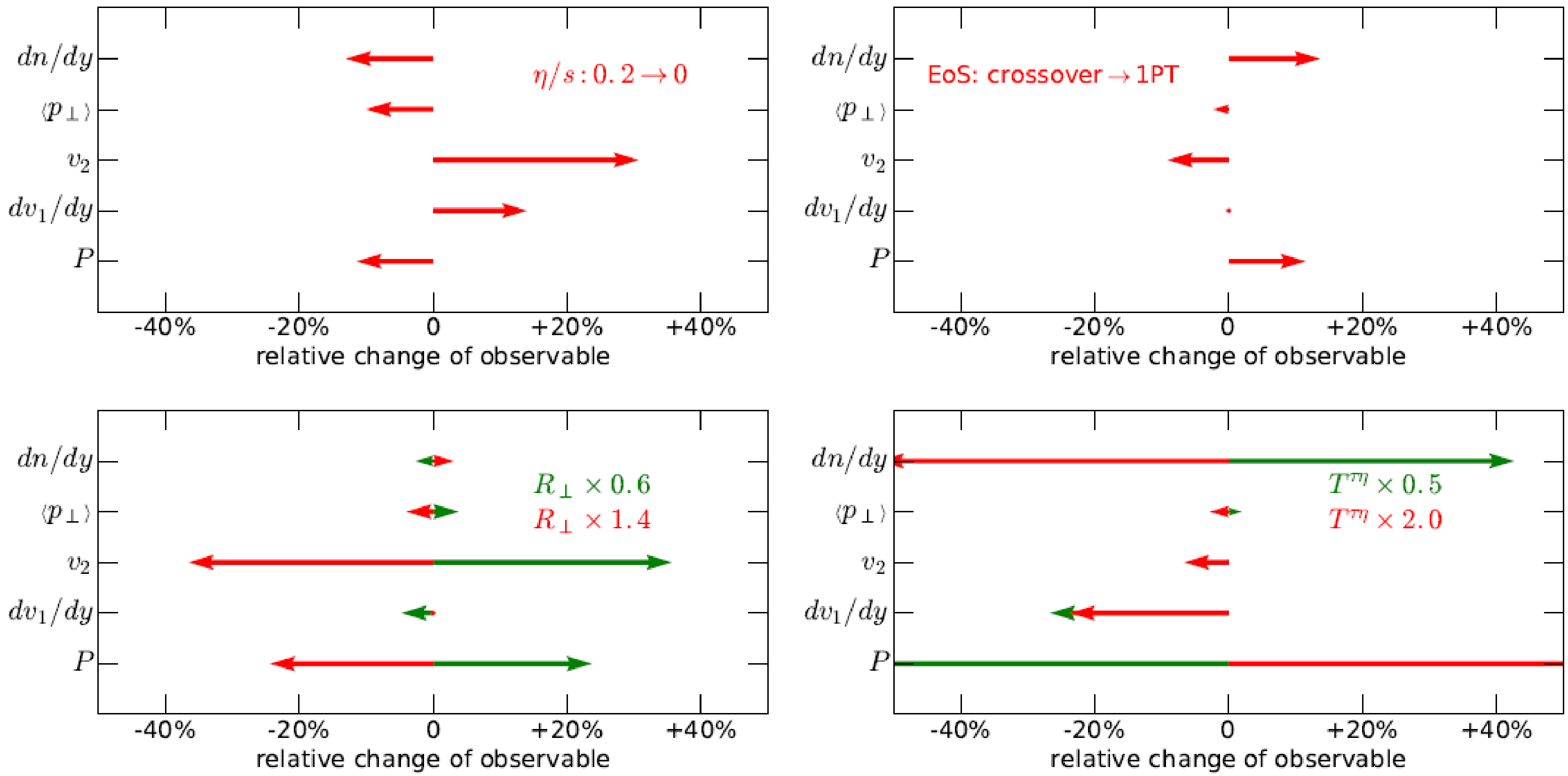
# Sensitivity to the initial hydro conditions

I. Karpenko, INFN Florence

## A closer look at the parameter dependence

NEW

$\sqrt{s_{\mathrm{NN}}} = 7.7$ GeV



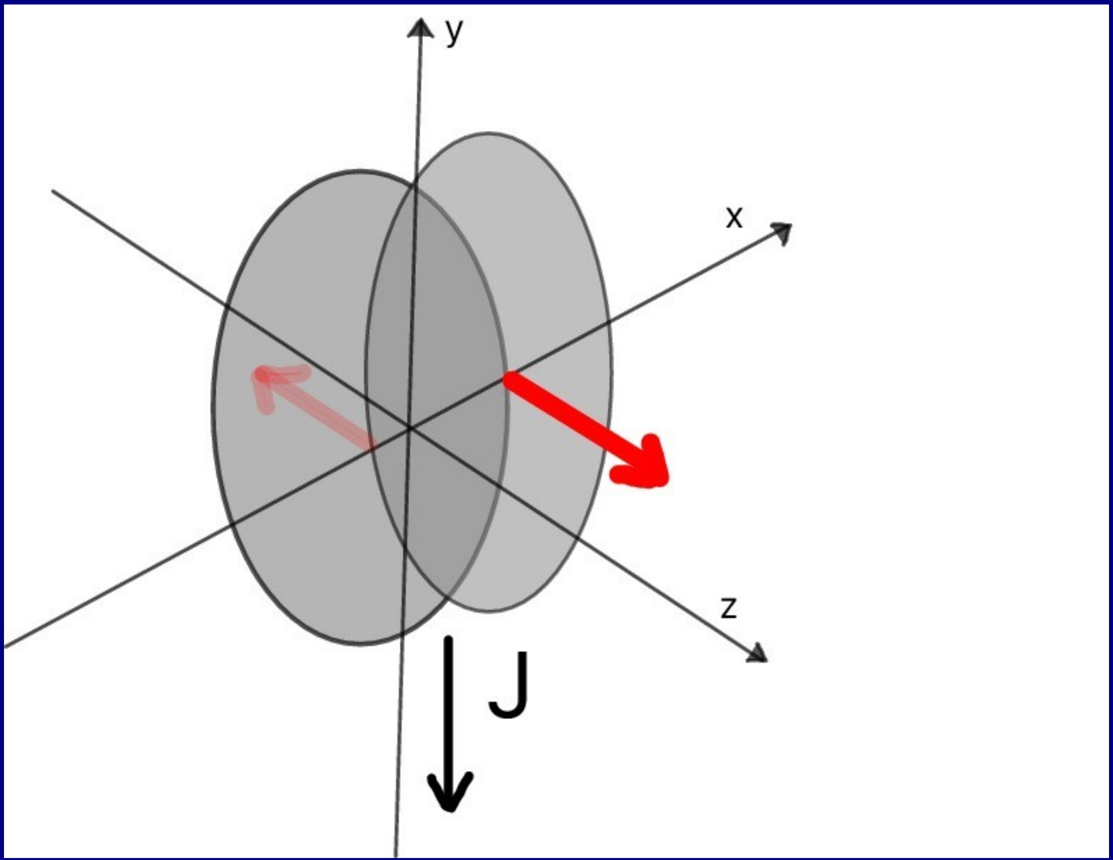

- Polarization observable is more sensitive to details of initial state rather than to details of hydro evolution.
- No sensitivity on the value of particlization energy density $\varepsilon_{\mathrm{sw}}$.

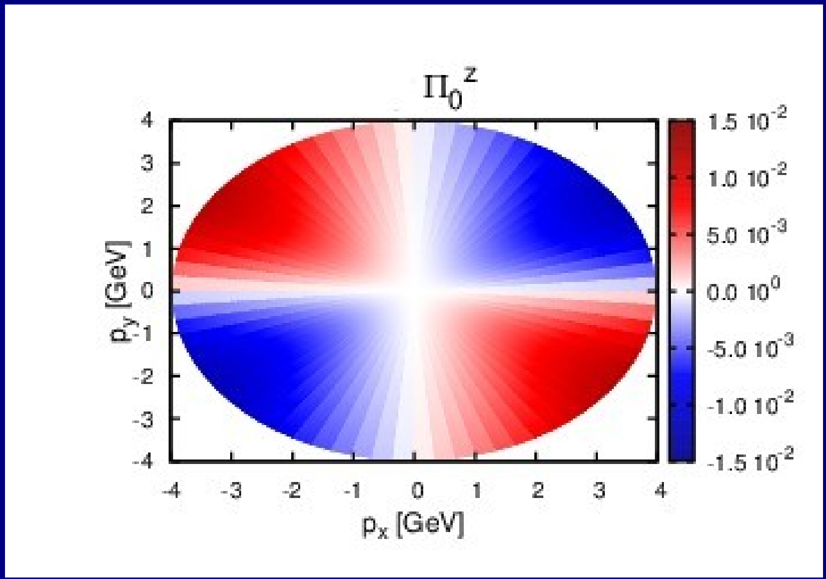
# Outlook

- New measurements by STAR experiment expected to confirm or question the origin of this phenomenon and the hydrodynamic picture
- Polarization is sensitive to the gradients of $T$ and $u$ and it is thus a sensitive probe of the QGP dynamics
- If QGP gets polarized, other possible consequences related to the long-sought chiral anomalous effects
- Fundamental theoretical issues raised about the description of a C-even polarized relativistic matter (spin tensor?)

# Global *longitudinal* polarization: quadrupole structure

F. B., I. Karpenko, arXiv:1707.07984




*200 GeV: larger magnitude than $S_J$!*

Peripheral heavy ion collisions feature two discrete symmetries: reflection w.r.t. reaction plane and rotation by 180 around its perpendicular direction. This reflects into the quadrupole pattern of the global Λ polarization at midrapidity

$$S^z(\mathbf{p}_T, Y=0) = \frac{1}{2}\sum_{k=1}^{\infty} f_{2k}(p_T)\sin 2k\varphi$$

# Longitudinal boost invariant scenario

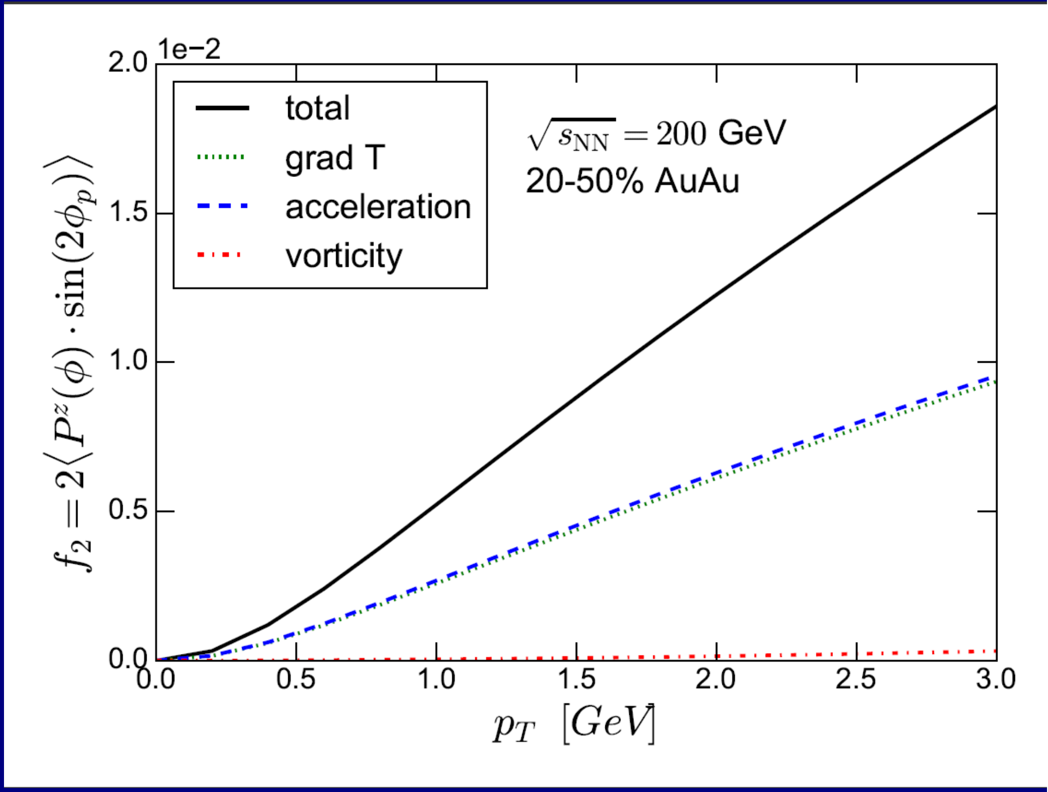
Invariance by longitudinal boost implies

$$\varpi_{\mu\nu} = \frac{1}{T}\left(A_\mu u_\nu - A_\nu u_\mu\right)$$

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\rho\sigma\tau}p_\tau \frac{\int_\Sigma \mathrm{d}\Sigma_\lambda p^\lambda A_\rho \beta_\sigma n_F(1-n_F)}{\int_\Sigma \mathrm{d}\Sigma_\lambda p^\lambda n_F} = \frac{1}{4mT}\epsilon^{\mu\rho\sigma\tau}p_\tau \frac{\frac{\partial}{\partial p^\sigma}\int_\Sigma \mathrm{d}\Sigma_\lambda p^\lambda n_F \partial_\rho T}{\int_\Sigma \mathrm{d}\Sigma_\lambda p^\lambda n_F}$$

In the simple case of $T=T(\tau)$ (Bjorken isocronous f.o.)

$$S^z(\mathbf{p}_T, Y=0) = -\frac{\mathrm{d}T/\mathrm{d}\tau}{4mT}\frac{\partial}{\partial\varphi}2v_2(p_T)\cos 2\varphi$$
$$= \frac{\mathrm{d}T}{\mathrm{d}\tau}\frac{1}{mT}v_2(p_T)\sin 2\varphi$$



# Summary

- Evidence for global particle-antiparticle polarization in relativistic nuclear collisions in agreement with the predictions of relativistic hydrodynamics and local thermodynamic equilibrium/equipartition of angular momentum

- Polarization driven by acceleration, vorticity and temperature gradients: 1st order quantum effect in (relativistic) hydrodynamics

- New promising probe to study the QGP dynamics and finite temperature QCD

- Interesting theoretical problems related to the quantum field theoretical foundation of relativistic hydrodynamics

If you want to know more:

*QCD chirality workshop*

March 2018 GGI

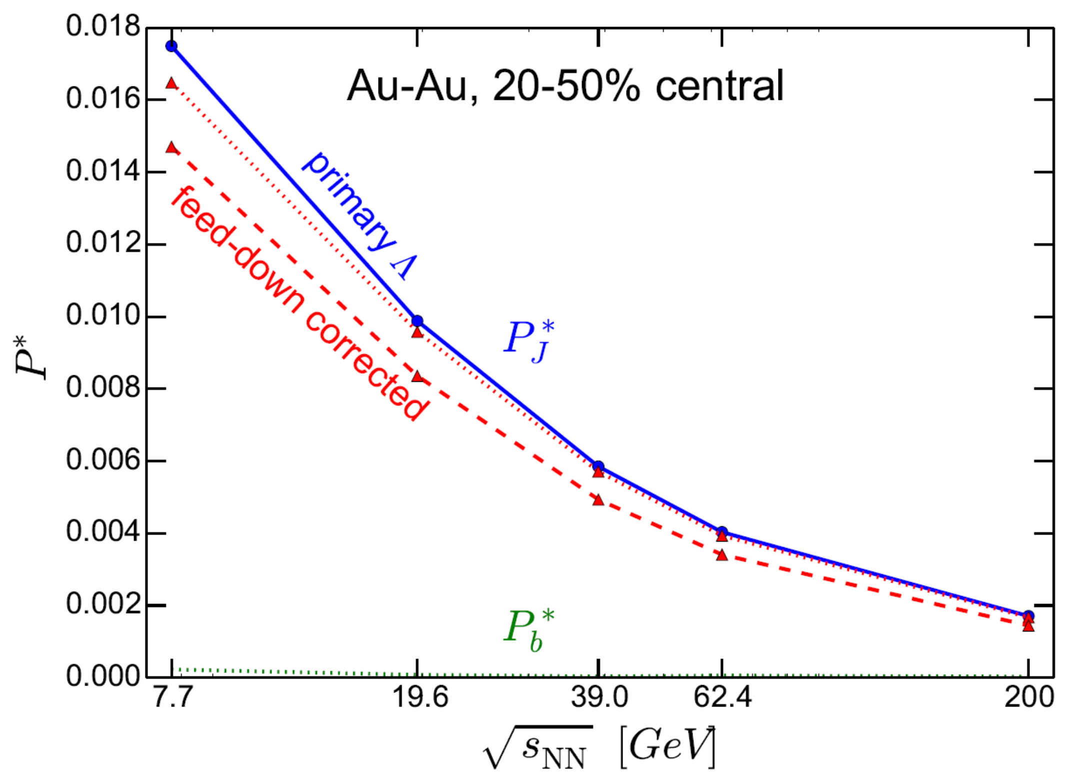
# $\Lambda$ polarization transfer by decay

It can be shown that in two-body decays there is a simple relation between the spin Vector of the decaying particle and each of the decay products

$$\mathbf{S}^*_{\text{daughter}} = C\mathbf{S}^*_{\text{parent}}$$

$$C = \sum_{\lambda_A,\lambda_B,\lambda'_A} T^J(\lambda_A,\lambda_B) T^J(\lambda'_A,\lambda_B)^* \sum_{n=-1}^{1} \langle \lambda'_A | \widehat{S}_{A,-n} | \lambda_A \rangle \times \frac{c_n}{\sqrt{J(J+1)}} \langle J\lambda | J1 | \lambda' n \rangle \left( \sum_{\lambda_A,\lambda_B} |T^J(\lambda_A,\lambda_B)|^2 \right)^{-1}$$

F. B., I. Karpenko, M. Lisa, I. Upsal,
S. Voloshin, Phys. Rev. C 95 054902 (2017)




This allows to take into account contribution of higher lying states

# Open theoretical problems

$$S^\mu(p) \simeq \frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau n_F(1-n_F)\partial_\nu\beta_\rho}{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau \; n_F}$$

- Do we really need a spin tensor to get this formula?

Subtle and profound theoretical issue: can the spin tensor be measured?

- Relation between anomaly and polarization

- What is the exact expression of spin vector at global equilibrium with rotation?

A theoretical challenge

# Covariant Wigner function for the free Dirac field

$$W(x,k)_{AB} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \Psi_A(x-y/2)\overline{\Psi}_B(x+y/2) : \rangle$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \overline{\Psi}_B(x+y/2)\Psi_A(x-y/2) : \rangle$$

$$(m - \not{k} - \frac{i}{2}\not{\partial})W(x,k) = 0$$

1 – On mass shell *(De Groot et al., Relativistic kinetic theory)*

$$W^+(x,k) \equiv \theta(k^0)W(x,k) = \frac{1}{2}\int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\,\delta^4(k-p)\sum_{r,s} u_r(p) f_{rs}(x,p)\bar{u}_s(p)$$
$$W^-(x,k) \equiv \theta(-k^0)W(x,k) = -\frac{1}{2}\int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\,\delta^4(k+p)\sum_{r,s} v_s(p) \bar{f}_{rs}(x,p)\bar{v}_r(p)$$

2 – Ansatz to the quantum statistics extension of the ideal Boltzmann relativistic gas with spin (F. B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338 (2013) 32)

$$f(x,p) = \frac{1}{2m}\bar{U}(p)\left(\exp[\beta(x)\cdot p - \xi(x)]\exp[-\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} U(p)$$
$$\bar{f}(x,p) = -\frac{1}{2m}(\bar{V}(p)\left(\exp[\beta(x)\cdot p + \xi(x)]\exp[\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} V(p))^T$$

Still, the above W(x,k) is not an exact solution of the free spinor Wigner equation

# Global quantum-relativistic equilibrium

General covariant expression of an equilibrium density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[ - \int_{\Sigma} \mathrm{d}\Sigma_\mu \; \left( \widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

Zubarev 1979
Weert 1982

Obtained by maximizing the entropy $S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$ with respect to $\widehat{\rho}$ with the constraints of fixed energy, momentum and charge density.

Global equilibrium requires:

$$\boxed{\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0}$$

Killing equation

$$\partial_\mu \zeta = 0$$

Solution of the Killing equation in Minkowski spacetime:

$$\beta^\nu = b^\nu + \varpi^{\nu\mu} x_\mu$$

constants

Density operator becomes:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[ -b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right]$$

# Ansatz for the LTE distribution function with FD statistics

The explicit calculation of *W(x,k)* and the extraction of *f* in the most general case is difficult. One can make a reasonable ansatz extending previous special cases.

The general solution must:

- reduce to the global equilibrium solution with rotation in the Boltzmann limit
- reduce to the known Fermi-Juttner or Bose-Juttner formulae at the LTE in the non-rotating case

$$f(x,p) = \frac{1}{2m}\bar{U}(p)\left(\exp[\beta(x)\cdot p - \xi(x)]\exp[-\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} U(p)$$

$$\bar{f}(x,p) = -\frac{1}{2m}(\bar{V}(p)\left(\exp[\beta(x)\cdot p + \xi(x)]\exp[\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} V(p))^{T}$$

*U, V* 4x2 Dirac spinors and $\Sigma$ the generators of the Lorentz transformation in the fundamental representation

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$$

# What is the spin tensor?

In Minkowski space-time, from translational and Lorentz invariance one obtains two conserved Noether currents:

$$\widehat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^a)} \partial^\nu \Psi^a - g^{\mu\nu} \mathcal{L}$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Psi^a)} D^A (J^{\mu\nu})^a_b \Psi^b$$

It is very important to stress that these are operators (henceforth denoted with a hat)

$$\partial_\mu \widehat{T}^{\mu\nu} = 0$$

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = \partial_\lambda \left( \widehat{\mathcal{S}}^{\lambda,\mu\nu} + x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} + \widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = 0$$

# Pseudo-gauge transformations with a *superpotential* $\widehat{\Phi}$

F.W. Hehl, Rep. Mat. Phys. 9 (1976) 55

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left( \widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$
$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu} + \partial_\alpha \widehat{\Xi}^{\alpha\lambda,\mu\nu}$$

With (we confine ourselves to $\Xi = 0$):

$$\int_{\partial\Omega} \mathrm{d}S \; \left( \widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) n_i = 0$$
$$\int_{\partial\Omega} \mathrm{d}S \; \left[ x^\mu \left( \widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) - x^\nu \left( \widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{\mu,i0} \right) \right] n_i = 0$$

They leave the conservation equations and spacial integrals (=generators, or total energy, momentum and angular momentum) invariant.

This seems to be enough for a quantum relativistic field theory. It is not in gravity but, as long as we disregard it, different couples of tensors related by a pseudo-gauge transformation cannot be distinguished

# Example: Belinfante symmetrization procedure

Just take $\widehat{\Phi} = \widehat{\mathcal{S}}$

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha\left(\widehat{\mathcal{S}}^{\alpha,\mu\nu} - \widehat{\mathcal{S}}^{\mu,\alpha\nu} - \widehat{\mathcal{S}}^{\nu,\alpha\mu}\right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

This is a way of getting rid of the spin tensor, whose physical meaning seems to be thus very limited in QFT (eliminated by a pseudo-gauge transformation).

Free Dirac field

Canonical couple

$$\widehat{T}^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{1}{2}\overline{\Psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi = \frac{i}{8}\overline{\Psi}\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi$$

$$\widehat{T}'^{\mu\nu} = \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi\right]$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

Belinfante couple

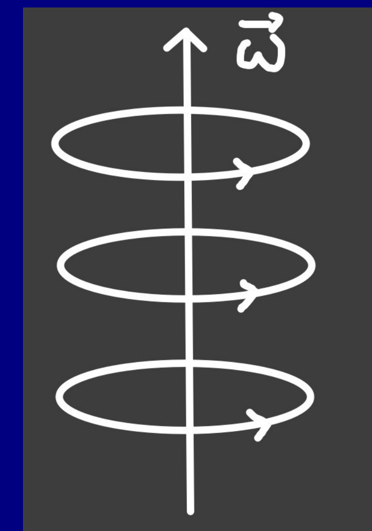
# Special cases

## Pure rotation (just seen)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (\omega/T_0)(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu})$$

$$\beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + \omega \widehat{J}_z/T_0]$$




## Pure acceleration

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (a/T_0)(g_{0\mu}g_{3\nu} - g_{3\mu}g_{0\nu})$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$