The estimate for the remainder term is

$$|\widetilde{R}_N(z)| = \frac{1}{2\pi}\left|\int_\gamma \frac{R_N(\zeta)}{(\zeta-z)^2}\,d\zeta\right| \leqslant c_N \varepsilon^{-1} |z|^{-1} \max_{\zeta\in\gamma} |\zeta|^{-N-1} \leqslant c'_N |z|^{-N-2},$$

since $|\zeta| \geqslant (1-\varepsilon)|z|$ for $\zeta \in \gamma$. Here c'_N is a positive constant.

43 Laplace's Method

43.1 Heuristic considerations In this section we discuss integrals of the type

$$F(\lambda) = \int_a^b f(x)\, e^{\lambda S(x)}\, dx, \tag{43.1}$$

which are known as *Laplace integrals*. Here $I = [a, b]$ is a finite segment, and λ is a large parameter. We will not consider the trivial cases $f(x) \equiv 0$ and $S(x) \equiv$ const. Everywhere in this section we assume that the function $S(x)$ admits only real values. The function $f(x)$, on the other hand, may be complex valued. We assume also that $f(x)$ and $S(x)$ are continuous for $x \in I$.

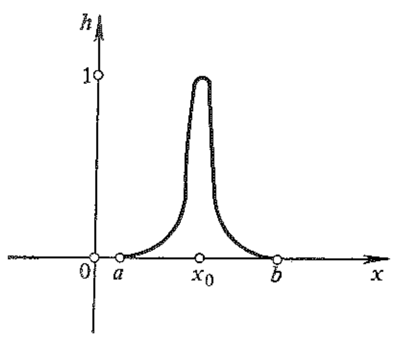

Fig. 157

We are interested in the asymptotic behavior of the integral $F(\lambda)$ as $\lambda \to +\infty$. Laplace integrals can be evaluated explicitly only in a limited number of cases, but their asymptotic behavior can be evaluated practically always. For the sake of simplicity we will assume that the peak of $S(x)$ in I is located only at one point $x_0 \in I$. Here are the two most important cases:

(1) $\max_{x\in I} S(x)$ is attained at an interior point x_0 of I, and $S''(x_0) \neq 0$.

It is clear that for large positive λ's the magnitude of the integral is determined primarily by the exponential $e^{\lambda S(x)}$. Let us consider the function

$$h(x, \lambda) = e^{\lambda(S(x)-S(x_0))}.$$

By hypothesis, $h(x_0, \lambda) = 1$, while $h(x, \lambda) < 1$ at $x \neq x_0$, $\lambda > 0$. As λ grows, the maximum at x_0 becomes more and more pronounced (Fig. 157). For this reason the value of (43.1) is approximately equal

to the value of the integral taken over a small neighborhood $(x_0 - \delta,\ x_0 + \delta)$ of point x_0. In such a neighborhood we can approximately substitute a linear function for $f(x)$ and a quadratic function for $S(x)$:

$$f(x) \approx f(x_0), \quad S(x) - S(x_0) \approx \frac{1}{2} S''(x_0)(x - x_0)^2.$$

Whence, for $f(x_0) \neq 0$ we have

$$F(\lambda) \approx e^{\lambda S(x_0)} f(x_0) \int_{x_0-\delta}^{x_0+\delta} e^{(\lambda/2) S''(x_0)(x-x_0)^2}\, dx. \tag{43.2}$$

A rigorous substantiation of these approximations will be given in Sec. 43.4.

Introducing a new variable t by the formula $x - x_0 = t/\sqrt{-\lambda S''(x_0)}$, with $S''(x_0) < 0$ since x_0 is the point of maximum, we find that the integral in (43.2) is equal to

$$\frac{1}{\sqrt{-\lambda S''(x_0)}} \int_{-\delta\sqrt{-\lambda S''(x_0)}}^{\delta\sqrt{-\lambda S''(x_0)}} e^{-t^2/2}\, dt.$$

For $\lambda \to +\infty$ the limits of integration tend to $\pm\infty$, and the integral above tends to

$$\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}.$$

Hence, the asymptotic behavior of the integral $F(\lambda)$ as $\lambda \to +\infty$ is the following:

$$F(\lambda) \approx f(x_0) \sqrt{-\frac{2\pi}{\lambda S''(x_0)}}\, e^{\lambda S(x_0)}. \tag{43.3}$$

(2) $\max_{x\in I} S(x)$ is attained only at the end point $x = a$ of segment I, and $S'(a) \neq 0$.

The same considerations as in case (1) show that for large λ's the integral $F(\lambda)$ is approximately equal to the value of the integral taken over a small segment $[a, a + \delta]$. On this segment we can replace $f(x)$ and $S(x)$ by linear functions:

$$f(x) \approx f(a), \quad S(x) \approx S(a) + (x - a) S'(a).$$

Then

$$F(\lambda) \approx e^{\lambda S(a)} f(a) \int_a^{a+\delta} e^{\lambda(x-a) S'(a)}\, dx.$$

The integral here is equal to

$$-\frac{1}{\lambda S'(a)}+\frac{e^{\lambda\delta S'(a)}}{\lambda S'(a)}\sim-\frac{1}{\lambda S'(a)},$$

since $S'(a)<0$. Hence, as $\lambda\to+\infty$,

$$F(\lambda)\approx\frac{f(a)}{-\lambda S'(a)}e^{\lambda S(a)}. \tag{43.4}$$

Formulas (43.3) and (43.4) are the basic asymptotic formulas for Laplace integrals. Let us now derive them rigorously.

43.2 The maximum of $S(x)$ at an end point of the interval Let us first obtain a rough estimate for Laplace integrals.

Lemma 1 *Let $I=(a, b)$ be a finite or infinite interval,*

$$S(x)\leqslant C,\quad x\in I, \tag{43.5}$$

and the integral (43.1) be absolutely convergent for a $\lambda_0>0$. Then for $\operatorname{Re}\lambda\geqslant\lambda_0$ *we have*

$$|F(\lambda)|\leqslant C_1e^{C\operatorname{Re}\lambda}, \tag{43.6}$$

where C_1 is a constant.

Proof. In view of (43.5) we have, for $\operatorname{Re}\lambda\geqslant\lambda_0$,

$$|e^{(\lambda-\lambda_0)S(x)}|\leqslant C_0e^{C\operatorname{Re}\lambda},\quad x\in I,$$

where $C_0=e^{-\lambda_0C}$. Hence,

$$|F(\lambda)|=\left|\int_a^b f(x)\,e^{\lambda_0S(x)}e^{(\lambda-\lambda_0)S(x)}\,dx\right|\leqslant C_0e^{C\operatorname{Re}\lambda}\int_a^b|f(x)|\,e^{\lambda_0S(x)}\,dx.$$

By hypothesis, the last integral has a finite value, which proves the validity of estimate (43.6).

From now on the interval $I=[a, b]$ is a finite segment, and the functions $f(x)$ and $S(x)$ are continuous for $x\in I$.

The asymptotic formulas for Laplace integrals, as we will show below, can be used not only when $\lambda\to+\infty$ but when $\lambda\to\infty$, $\lambda\in S_\varepsilon$, where S_ε is the sector $|\arg\lambda|\leqslant\pi/2-\varepsilon$ in the complex λ plane. Here $0<\varepsilon<\pi/2$. Note that if $\lambda\in S_\varepsilon$, then

$$|\lambda|\geqslant\operatorname{Re}\lambda\geqslant|\lambda|\sin\varepsilon.$$

For this reason, as $\lambda\to\infty$, $\lambda\in S_\varepsilon$, we have

$$(\operatorname{Re}\lambda)^{-n}=O(|\lambda|^{-n}),\quad n\geqslant0,\quad|e^{-C\lambda}|=O(|\lambda|^{-N}),$$

where N is any positive number.

Theorem 1 *Suppose*

$$S(x)<S(a),\quad x\neq a;\quad S'(a)\neq0, \tag{43.7}$$

and let us assume that the functions $f(x)$ and $S(x)$ can be differentiated any number of times in a neighborhood of point $x=a$. Then, as $\lambda\to\infty$, $\lambda\in S_\varepsilon$, the following asymptotic behavior is valid:

$$F(\lambda)\sim e^{\lambda S(a)}\sum_{n=0}^{\infty}c_n\lambda^{-n-1}. \tag{43.8}$$

This expansion can be differentiated termwise any number of times. The coefficients c_n can be calculated by the formula

$$c_n=(-1)^{n+1}\left(\frac{1}{S'(x)}\frac{d}{dx}\right)^n\left(\frac{f(x)}{S'(x)}\right)\Bigg|_{x=a}. \tag{43.9}$$

The principal term in the asymptotic expansion has the form (43.4) or, more exactly,

$$F(\lambda)=\frac{e^{\lambda S(a)}}{-\lambda S'(a)}\left[f(a)+O\left(\frac{1}{\lambda}\right)\right]. \tag{43.10}$$

Proof. Since $S'(a)\neq0$, we can select a positive δ such that $S'(x)\neq0$ for $a\leqslant x\leqslant a+\delta$. We split the integral (43.1) into two:

$$F(\lambda)=F_1(\lambda)+F_2(\lambda),$$

where $F_1(\lambda)$ is the integral taken along the segment $[a, a+\delta]$. Since $S(x)$ attains its maximum on I only at point a, we conclude that $S(x)\leqslant S(a)-c$ for $a+\delta\leqslant x\leqslant b$, where c is a positive constant. By Lemma 1, for $\lambda\in S_\varepsilon$ we have

$$|F_2(\lambda)|\leqslant c_0\,|e^{\lambda(S(a)-c)}|. \tag{43.11}$$

For this reason, the integral $F_2(\lambda)$ is exponentially small compared with $e^{\lambda S(a)}$ and, in particular, compared with any term $c_n\lambda^{-n-1}e^{\lambda S(a)}$ in the asymptotic series (43.8).

The integral $F_1(\lambda)$, which is taken along the segment $[a, a+\delta]$, will be integrated by parts:

$$F_1(\lambda)=\int_a^{a+\delta}\frac{f(x)}{\lambda S'(x)}\,d\left(e^{\lambda S(x)}\right)=\frac{f(x)\,e^{\lambda S(x)}}{\lambda S'(x)}\Bigg|_a^{a+\delta}+\frac{1}{\lambda}F_{11}(\lambda), \tag{43.12}$$

$$F_{11}(\lambda)=\int_a^{a+\delta}e^{\lambda S(x)}f_1(x)\,dx,\quad f_1(x)=-\frac{d}{dx}\left(\frac{f(x)}{S'(x)}\right).$$

The upper limit of the first term on the right-hand side of (43.12) is exponentially small compared with $e^{\lambda S(a)}$ (as $\lambda\to\infty$, $\lambda\in S_\varepsilon$) since $S(a+\delta)-S(a)<0$.

Let us estimate the value of $F_{11}(\lambda)$. We have $S'(x) < 0$ on the segment $I_1 = [a, a+\delta]$, whereby there is a positive constant S_1 such that $S'(x) \leqslant -S_1$ for $x \in I_1$. By Lagrange's formula,

$$S(x) - S(a) = (x-a)\, S'(\xi),$$

where $\xi \in (a, a+\delta)$, and

$$S(x) - S(a) \leqslant -S_1(x-a), \quad S_1 > 0$$

on I_1. Since $f_1(x)$ is continuous, we have $|f_1(x)| \leqslant M$ for $x \in I_1$. Hence,

$$|F_{11}(\lambda)\, e^{-\lambda S(a)}| \leqslant \int_a^{a+\delta} |f_1(x)|\, |e^{\lambda(S(x)-S(a))}|\, dx$$

$$\leqslant M \int_a^{a+\delta} e^{-S_1(x-a)\operatorname{Re}\lambda}\, dx < \frac{M}{S_1 \operatorname{Re}\lambda} \leqslant \frac{c}{|\lambda|} \quad (\lambda \in S_2).$$

With this estimate in mind, we can write (43.12) thus:

$$F_1(\lambda) = e^{\lambda S(a)}\left[\frac{f(a)}{-\lambda S'(a)} + O(\lambda^{-2})\right]. \tag{43.13}$$

The last relationship, the estimate (43.11), and the fact that $F(\lambda) = F_1(\lambda) + F_2(\lambda)$ yields formula (43.10) for the principal term in the asymptotic expansion.

The integral $F_{11}(\lambda)$ has exactly the same form as $F_1(\lambda)$, and

$$F_{11}(\lambda) = \frac{f_1(x)\, e^{\lambda S(x)}}{\lambda S'(x)}\Big|_a^{a+\delta} + \frac{1}{\lambda} F_{12}(\lambda).$$

The integral $F_{12}(\lambda)$ has the same form as $F_1(\lambda)$, the only difference being that

$$f(x) \to f_2(x) = -\frac{d}{dx}\left(\frac{f_1(x)}{S'(x)}\right).$$

If we substitute f_1 for f in (43.13), the integral $F_{11}(\lambda)$ obeys (43.13), which means that

$$F(\lambda) = e^{\lambda S(a)}\left[\frac{f(a)}{-\lambda S'(a)} + \frac{f_1(a)}{-\lambda^2 S'(a)} + O(\lambda^{-3})\right].$$

If we proceed with this process still further, we arrive at the expansion (43.8) and formula (43.9).

What remains to be proved is that the series (43.8) can be differentiated term-by-term. The function $F(\lambda)$ is an entire function of λ (Theorem 1 of Sec. 16), and the asymptotic series (43.8) can be differentiated term-by-term in view of Theorem 5 of Sec. 42.

Example 1. Consider the Laplace transform of $f(x)$, i.e.

$$F(\lambda) = \int_0^\infty f(x)\, e^{-\lambda x}\, dx. \tag{43.14}$$

Let us assume that $f(x)$ is piecewise continuous for nonnegative x's, can be differentiated any number of times in a neighborhood of point $x = 0$, and satisfies the estimate

$$|f(x)| \leqslant M e^{Cx}$$

for $x \geqslant 0$. Let us show that

$$F(\lambda) \sim \sum_{n=0}^\infty f^{(n)}(0)\, \lambda^{-n-1} \tag{43.15}$$

as $\lambda \to \infty$, $\lambda \in S_\varepsilon$.

In the case at hand, $S(x) = -x$, so that $\max\limits_{x \geqslant 0} S(x) = S(0) = 0$, and $S'(0) \neq 0$. But we cannot apply Theorem 1 directly to this integral because the domain of integration is unlimited.

We split the integral $F(\lambda)$ into two: $F(\lambda) = F_1(\lambda) + F_2(\lambda)$, where $F_1(\lambda)$ is the integral taken along the segment $[0, 1]$. Since $S(x) = -x \leqslant -1$ for $x \geqslant 1$, by Lemma 1 we have

$$|F_2(\lambda)| \leqslant C\, |e^{-\lambda}| \quad (\lambda \in S_\varepsilon),$$

which is exponentially small as $\lambda \to \infty$, $\lambda \in S_\varepsilon$. Applying Theorem 1 to $F_1(\lambda)$, we arrive at (43.15). □

Example 2. Let us take the error integral

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

and establish its asymptotic behavior as $x \to +\infty$. Since

$$\int_0^\infty e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi},$$

we conclude that

$$\Phi(x) = 1 - \frac{2}{\sqrt{\pi}} F(x), \quad F(x) = \int_x^\infty e^{-t^2}\, dt.$$

Let us now represent $F(x)$ in the form of the Laplace transform (43.14).

Changing the variable according to $t = x\tau$ and putting $\tau^2 = 1 + u$, we obtain

$$F(x) = \frac{1}{2} x e^{-x^2} \int_0^\infty e^{-x^2 u} (1+u)^{-1/2}\, du.$$

The integral on the right-hand side has the form (43.14), where $\lambda = x^2$, and $f = (1 + u)^{-1/2}$, so that $f^{(k)}(0) = (-1/2)^k (2k - 1)!!$. Applying formula (43.15), we find that, as $x \to +\infty$,

$$\Phi(x) \sim 1 - \frac{1}{x\sqrt{\pi}} e^{-x^2} - \frac{1}{x\sqrt{\pi}} e^{-x^2} \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!!}{2^k x^{2k}}. \qquad (43.16)$$

The same formula holds for complex valued x's, $|x| \to \infty$, $|\arg x| \leqslant \pi/4 - \varepsilon$ $(0 < \varepsilon < \pi/4)$. Indeed, if x lies in this sector, then $\lambda = x^2$ lies in the sector $S_{2\varepsilon}$: $|\arg \lambda| \leqslant \pi/2 - 2\varepsilon$, in which formula (43.15) is valid. □

43.3 Watson's lemma The asymptotic behavior of many Laplace integrals can be examined by establishing the asymptotic behavior of the "typical" integral

$$\Phi(\lambda) = \int_0^a e^{-\lambda t^\alpha} t^{\beta-1} f(t)\, dt. \qquad (43.17)$$

Lemma 2 (Watson's lemma) *Suppose α and β are positive and let $f(t)$ be continuous for $0 \leqslant t \leqslant a$ and infinitely differentiable in a neighborhood of point $t = 0$. Then, as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, the following asymptotic expansion holds:*

$$\Phi(\lambda) \sim \frac{1}{\alpha} \sum_{n=0}^{\infty} \lambda^{-(n+\beta)/\alpha} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{f^{(n)}(0)}{n!}. \qquad (43.18)$$

This expansion can be differentiated with respect to λ any number of times.

Before proving this lemma, we will prove the validity of the formula

$$\int_0^\infty e^{-\lambda t^\alpha} t^{\beta-1}\, dt = \frac{1}{\alpha} \lambda^{-\beta/\alpha} \Gamma\left(\frac{\beta}{\alpha}\right) \qquad (43.19)$$

for Re λ positive. Here $\lambda^{-\beta/\alpha}$ is the regular branch in half-plane Re $\lambda > 0$ that is positive for positive values of λ.

Suppose λ is positive. Substituting $\lambda t^\alpha = y$, we find that the integral on the left-hand side of (43.19) is equal to

$$\lambda^{-\beta/\alpha} \frac{1}{\alpha} \int_0^\infty e^{-y} y^{(\beta/\alpha)-1}\, dy = \frac{1}{\alpha} \lambda^{-\beta/\alpha} \Gamma\left(\frac{\beta}{\alpha}\right).$$

This integral is a regular function in the half-plane Re $\lambda > 0$. The right-hand side of (43.19) is analytically continuable from the semiaxis $(0, +\infty)$ into the half-plane Re $\lambda > 0$. Since both functions coincide on the semiaxis $(0, +\infty)$, by the principle of analytic continuation, they coincide in the half-plane Re $\lambda > 0$.

Proof of Lemma 2. Let us split the integral $\Phi(\lambda)$ into two:

$$\Phi(\lambda) = \Phi_1(\lambda) + \Phi_2(\lambda),$$

where the integral Φ_1 is taken along the segment $[0, \delta]$, with δ positive and small. Since $-t^\alpha \leqslant -\delta^\alpha < 0$ for $\delta \leqslant t \leqslant a$, we have, according to Lemma 1, the following estimate for $\Phi_2(\lambda)$:

$$|\Phi_2(\lambda)| \leqslant C\, |e^{-\delta^\alpha \lambda}|$$

for $\lambda \in S_\varepsilon$, $|\lambda| \geqslant 1$, and the right-hand side is exponentially small. The following expansion is valid on the segment $[0, \delta]$:

$$f(t) = \sum_{n=0}^{N} f_n t^n + \psi_N(t),$$

where $f_n = \frac{f^{(n)}(0)}{n!}$ and $|\psi_N(t)| \leqslant C_N t^{N+1}$, $0 \leqslant t \leqslant \delta$. Whence $\Phi_1(\lambda)$ is equal to the following sum:

$$\Phi_1(\lambda) = \sum_{n=0}^{N} f_n \Phi_{1n}(\lambda) + R_N(\lambda),$$

where we have introduced the notations

$$\Phi_{1n}(\lambda) = \int_0^a t^{n+\beta-1} e^{-\lambda t^\alpha}\, dt, \quad R_N(\lambda) = \int_0^a \psi_N(t)\, e^{-\lambda t^\alpha}\, dt.$$

Let us write $\Phi_{1n}(\lambda)$ in the form of the difference between respective integrals along the semiaxes $(0, +\infty)$ and $(a, +\infty)$. The first integral in this difference can be evaluated via (43.19), while the second does not exceed, by absolute value, the quantity $C\, |e^{-a^\alpha \lambda}|$, since $t^\alpha \leqslant -a^\alpha$ for $t \geqslant a$. The final result is

$$\Phi_{1n}(\lambda) = \frac{1}{\alpha} \lambda^{-(n+\beta)/\alpha} \Gamma\left(\frac{n+\beta}{\alpha}\right) + O(e^{-a^\alpha \lambda}).$$

Finally, the absolute value of the integrand in $R_N(\lambda)$ does not exceed $C_N t^{N+1} \mid e^{-\lambda t^\alpha} \mid$, whence

$$\mid R_N(\lambda) \mid \leqslant C_N \int_0^\infty t^{N+\beta} e^{-t^\alpha \operatorname{Re}\lambda}\, dt = C'_N (\operatorname{Re}\lambda)^{-(N+1+\beta)/\alpha} = O(\mid \lambda \mid^{-(N+\beta+1)/\alpha})$$

for $\lambda \in S_\varepsilon$. The final result is, for $\lambda \in S_\varepsilon$,

$$\Phi(\lambda) = \frac{1}{\alpha} \sum_{n=0}^{N} f_n \Gamma\left(\frac{n+\beta}{\alpha}\right) \lambda^{-(n+\beta)/\alpha} + O(\mid \lambda \mid^{-\frac{N+\beta+1}{\alpha}})$$

(the sum of all exponentially small terms is included in the remain-
der term). Hence we have proved the validity of the asymptotic

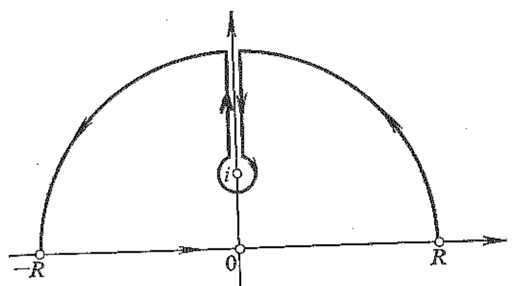

Fig. 158

expansion (43.18). The possibility of term-by-term differentiation of expansion (43.18) can be proved in the same way as in Theorem 1.

Example 3. Let us establish the asymptotic behavior, as $x \to +\infty$, of the integral

$$K_0(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{itx}}{\sqrt{1+t^2}}\, dt.$$

The integral $K_0(x)$ is known as a *modified Hankel function*, or the *MacDonald function*.

We cannot apply Watson's lemma to this integral directly. We then deform the integration contour. We cut the complex t plane along the ray $l = [i, +i\infty)$. Then the function $f(t) = (1+t^2)^{-1/2}$, $f(0) > 0$, becomes regular in the half-plane $\operatorname{Im} t > 0$ with a cut along l. For $x > 0$ the integration contour can be deformed into the cut along l. To prove this, let us consider the contour $\Gamma_{\rho, R}$ (Fig. 158). It consists of the segment $[-R, R]$, the arcs $C_R^\pm$ of the circle $\mid t \mid = R$, the circle C_ρ: $\mid t - i \mid = \rho$, and the intervals along the banks of the cut. Since $\mid f(t) \mid \leqslant c \mid t \mid^{-1}$ as $\mid t \mid \to \infty$, the integrals taken along the arcs of circle $\mid t \mid = R$ tend to zero as $R \to \infty$, according to Jordan's lemma. Further, for $t \in C_\rho$ we have

$$\mid t^2 + 1 \mid = \mid (t-i)(t+i) \mid = \rho \mid 2i + O(\rho) \mid \geqslant \rho \qquad (\rho \to 0);$$

thus, the integrand has the order of $O(1/\sqrt{\rho})$ as $\rho \to 0$, $t \in C_\rho$. Whence, the integral along C_ρ tends to zero as $\rho \to 0$, and the integral $K_0(x)$ is equal to the integral along the banks of the cut. Let us show that for $\tau > 1$ we have

$$f(i\tau + 0) = \frac{-i}{\sqrt{\tau^2 - 1}}$$

(this is the value of $f(t)$ on the right bank of the cut). We have

$$f(i\tau + 0) = \left| \frac{1}{\sqrt{\tau^2-1}} \right| e^{-i(1/2)(\varphi_1 + \varphi_2)},$$

$$\varphi_1 = \Delta_\gamma \arg(t+i), \quad \varphi_2 = \Delta_\gamma \arg(t-i).$$

Here curve γ lies in the upper half-plane and connects points 0 and $i\tau + 0$, so that $\varphi_1 = 0$ and $\varphi_2 = +\pi$. Similarly,

$$f(i\tau - 0) = \frac{i}{\sqrt{\tau^2-1}}.$$

Hence,

$$K_0(x) = -i \int_1^\infty \frac{e^{-x\tau}}{\sqrt{\tau^2-1}}\, d(i\tau) = \int_1^\infty \frac{e^{-x\tau}}{\sqrt{\tau^2-1}}\, d\tau = e^{-x} \int_0^\infty \frac{e^{-xt}\, dt}{\sqrt{t(t+2)}}$$

(we have substituted $t+1$ for τ). We can now apply Watson's lemma to the integral on the right-hand side (with $\alpha = 1$ and $\beta = 1/2$). Hence,

$$K_0(x) = e^{-x} \sqrt{\frac{\pi}{2x}} \left(1 + O\left(\frac{1}{x}\right)\right) \quad (x \to +\infty). \ \square$$

43.4 The maximum of $S(x)$ inside the interval

Theorem 2 *Suppose*

$$S(x) < S(x_0), \quad x \neq x_0, \quad a < x_0 < b, \quad S''(x_0) \neq 0, \tag{43.20}$$

and let the functions $f(x)$ and $S(x)$ be infinitely differentiable in a neighborhood of point x_0. Then, as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, the following asymptotic expansion is valid:

$$F(\lambda) \sim e^{\lambda S(x_0)} \sum_{n=0}^{\infty} c_n \lambda^{-n-1/2}. \tag{43.21}$$

This expansion can be differentiated term-by-term any number of times, and the principal term in the expansion is expressed by (43.3) *or, precisely,*

$$F(\lambda)=\sqrt{-\frac{2\pi}{\lambda S''(x_0)}}\,e^{\lambda S(x_0)}\,[f(x_0)+O(\lambda^{-1})]. \tag{43.22}$$

We will need following

Lemma 3 *Suppose $S(x)$ can be differentiated any number of times in a neighborhood of point x_0, and*

$$S'(x_0)=0,\quad S''(x_0)<0. \tag{43.23}$$

Then there are neighborhoods U and V of points $x=x_0$ and $y=0$, respectively, and a function $\varphi(y)$, such that

$$S(\varphi(y))-S(x_0)=-y^2,\quad y\in V, \tag{43.24}$$

the function $\varphi(y)$ can be differentiated any number of times for $y\in V$,

$$\varphi'(0)=\sqrt{-\frac{2}{S''(x_0)}}, \tag{43.25}$$

and the function $x=\varphi(y)$ maps V onto U in a one-to-one manner.

Proof of Lemma 3 Without loss of generality, we can assume that $x_0=0$ and $S(x_0)=0$. By Taylor's formula,

$$S(x)=\int_0^x (x-t)\,S''(t)\,dt=x^2\int_0^1 (1-t)\,S''(xt)\,dt\equiv -x^2h(x).$$

If U_0 is a small neighborhood of point $x=0$, then $S''(xt)$ is negative for $x\in U_0$, $0\leqslant t\leqslant 1$, since $S''(0)$ is negative. For this reason, the function

$$h(x)=\int_0^1 (t-1)\,S''(xt)\,dt$$

is positive for $x\in U_0$ and can be differentiated any number of times. We put

$$x\sqrt{h(x)}=y, \tag{43.26}$$

i.e. $x^2h(x)=y^2$ or, which is the same, $S(x)=-y^2$. Here $\sqrt{h(x)}$ is positive. Since

$$\frac{d}{dx}\left(x\sqrt{h(x)}\right)\Big|_{x=0}=\sqrt{h(0)}=\sqrt{-\frac{S''(0)}{2}}\neq 0,$$

by the inverse function theorem we conclude that Eq. (43.26) has a solution $x=\varphi(y)$, $\varphi(0)=0$ with the required properties. The proof of Lemma 3 is complete.

Proof of Theorem 2. Let $x_0=0$ and $S(x_0)=0$. We select a small neighborhood $[-\delta_1,\delta_2]$ of point $x=0$ and split the integral $F(\lambda)$ into three integrals:

$$F(\lambda)=F_1(\lambda)+F_2(\lambda)+F_3(\lambda).$$

Here $F_1(\lambda)$ is the integral taken along the segment $[a,-\delta_1]$, $F_2(\lambda)$ the integral taken along the segment $[-\delta_1,\delta_2]$, and $F_3(\lambda)$ the integral taken along the segment $[\delta_2,b]$. Since $S(x)<S(0)$ for $x\in I$, $x\neq 0$, we conclude that the integrals $F_1(\lambda)$ and $F_3(\lambda)$ are exponentially small as $\lambda\to\infty$, $\lambda\in S_\varepsilon$, i.e.

$$F_j(\lambda)=O(e^{-\lambda c})\quad (c>0),\quad j=1,3.$$

The proof is the same as in Theorem 1.

Now let us select δ_1 and δ_2 in a way such that $S(-\delta_2)=S(\delta_1)$. We have $S(\delta_1)=-\varepsilon^2$, with ε positive, since $x=0$ is the point at which $S(x)$ attains its maximum. We substitute $\varphi(y)$ for x in the integral $F_2(\lambda)$, i.e.

$$S(\varphi(y))=-y^2,$$

which is justified in view of Lemma 2. Then

$$F_2(\lambda)=\int_{-\varepsilon}^{\varepsilon} e^{-\lambda y^2}h(y)\,dy,\quad h(y)=f(\varphi(y))\,\varphi'(y).$$

Further,

$$F_2(\lambda)=\int_0^{\varepsilon} e^{-\lambda y^2}g(y)\,dy,$$

where $g(y)=h(y)+h(-y)$.

Now we need only to apply Watson's lemma to the integral $F_2(\lambda)$. Here $\alpha=2$ and $\beta=1$, besides, $g(y)$ is an even function, so that $g^{(k)}(0)=0$ for all odd values of k. The final result is the expansion (43.21) for $F_2(\lambda)$, where the coefficients c_n are given by the formula

$$c_n=\Gamma\left(n+\frac{1}{2}\right)\frac{h^{(2n)}(0)}{(2n)!}. \tag{43.27}$$

Here we took into account that $g^{(2n)}(0)=2h^{(2n)}(0)$. The coefficient c_0 is given by the formula

$$c_0=\Gamma\left(\frac{1}{2}\right)h(0)=\Gamma\left(\frac{1}{2}\right)f(x_0)\,\varphi'(0)=\sqrt{\frac{2\pi}{-S''(x_0)}}\,f(x_0),$$

since $\Gamma(1/2)=\sqrt{\pi}$, and $\varphi'(0)$ is given by (43.25).

The proof of Theorem 2 leads to

Corollary 1 *Suppose* $\max_{x\in I} S(x)$ *is attained at the end point* $x=a$ *of the segment* I, *and* $S'(a)=0$ *and* $S''(a)\neq 0$. *Then, as* $\lambda\to\infty$, $\lambda\in S_\varepsilon$, *the following asymptotic expansion is valid:*

$$F(\lambda)\sim e^{\lambda S(a)}\sum_{n=0}^{\infty} d_n\lambda^{-(n+1)/2}. \tag{43.28}$$

The principal term in the asymptotic expansion is

$$F(\lambda)=\frac{1}{2}e^{\lambda S(a)}\sqrt{-\frac{2\pi}{\lambda S''(a)}}\left[f(a)+O\left(\frac{1}{\sqrt{\lambda}}\right)\right]. \tag{43.29}$$

Example 4. Consider the gamma function

$$\Gamma(x+1)=\int_0^\infty t^x e^{-t}\,dt.$$

Let us prove Stirling's formula

$$\Gamma(x+1)=x^x e^{-x}\sqrt{2\pi x}\left(1+O\left(\frac{1}{x}\right)\right)\quad (x\to+\infty). \tag{43.30}$$

This integral is not of the (43.1) type, but we will transform it into such a type. The integrand $t^x e^{-t}$ attains the maximal value on the semiaxis $t>0$ at point $t_0(x)=x$, which tends to infinity as $x\to+\infty$. To restrict its variation, we substitute xt' for t. Then

$$\Gamma(x)=x^{x+1}\int_0^\infty e^{x(\ln t-t)}\,dt.$$

This integral has the form (43.1): $\lambda\to x$, $S=\ln t-t$, and $f(t)\equiv 1$. The point of maximum is $t_0=1$, with $S(t_0)=-1$ and $S''(t_0)=-1$.

To apply Theorem 2, we partition the region of integration into three parts: $(0, 1/2)$, $(1/2, 3/2)$, and $(3/2, \infty)$. The integrals taken over the first and third parts are exponentially small compared with $e^{-x}=e^{xS(t_0)}$, according to Lemma 1. The asymptotic behavior of the integral taken along the segment $[1/2, 3/2]$ is determined from (43.22), and we arrive at (43.30).

Formula (43.30) yields Stirling's formula for factorial $n!$:

$$n!\sim n^n e^{-n}\sqrt{2\pi n}\quad (n\to+\infty).$$

The asymptotic formula (43.30) is also valid for complex values of z, with $z\to\infty$, $z\in S_\varepsilon$, where S_ε is the sector $|\arg z|\leqslant\pi-\varepsilon$ (see Evgrafov [2]). Here ε is fixed and $0<\varepsilon<\pi$. There is also a more exact asymptotic expansion for the logarithm of the gamma function (see Fedoryuk [1]):

$$\ln\Gamma(z)\sim\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{2n(2n-1)z^{2n-1}}$$

$$(z\in S_\varepsilon,\ z\to\infty),$$

where the B_n are the Bernoulli number (see Example 4 in Sec. 12). For the remainder term in (43.30) the following estimate has been found:

$$|O(1/x)|\leqslant 1/12x.\ \square$$

Example 5. Let us establish the asymptotic behavior of the sum

$$F(n)=\sum_{k=0}^{n}\binom{n}{k}k!n^{-k}$$

as $n\to\infty$. Let us transform this sum into an integral. Using the identity $k!n^{-k-1}=\int_0^\infty e^{-nx}x^k\,dx$, we obtain

$$F(n)=n\int_0^\infty e^{-nx}(1+x)^n\,dx,$$

whence

$$F(n)=n\int_0^\infty e^{nS(x)}\,dx,$$

where $S(x)=-x+\ln(1+x)$. The function $S(x)$ attains its maximum on the semiaxis $x\geqslant 0$ only at point $x=0$, with $S(0)=0$ and $S''(0)=-1$. Applying Corollary 1 of Theorem 2, we find that

$$F(n)=\sqrt{\frac{\pi n}{2}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\quad (n\to\infty).\ \square$$

44 The Method of Stationary Phase

44.1 Statement of the problem Let us take the integral

$$F(\lambda)=\int_a^b f(x)\,e^{i\lambda S(x)}\,dx. \tag{44.1}$$

Here $I=[a, b]$ is a finite segment, the function $S(x)$ admits only real values, and λ is a large positive parameter. Integrals of the type (44.1) are known as *Fourier integrals*, and $S(x)$ is known as the *phase*, or *phase function*. We are interested in the asymptotic behavior of $F(\lambda)$ as $\lambda \to +\infty$. We will not consider the trivial cases with $f(x) \equiv 0$ or $S(x) \equiv \text{const}$.

A particular case of Fourier integrals is the Fourier transform

$$F(\lambda) = \int_a^b f(x)\, e^{i\lambda x}\, dx. \tag{44.2}$$

Let the function $f(x)$ be continuous for $a \leqslant x \leqslant b$. Then $F(\lambda)$ tends to zero as $\lambda \to +\infty$. Indeed, for large λ's the function $\mathrm{Re}\,(f(x)\, e^{i\lambda x})$ rapidly oscillates, and two neighboring half-waves have areas that are approximately equal in absolute value but are opposite in sign. For this reason the sum of these areas constitute a small quantity, in view of which the entire integral

$$\int_a^b \mathrm{Re}\,(f(x)\, e^{i\lambda x})\, dx$$

is small, too.

The following proposition provides the most general result concerning the asymptotic behavior of integrals of type (43.2):

The Riemann-Lebesgue lemma (see Nikol'skii [1]) *Suppose the integral* $\int_a^b |f(x)|\, dx$ *has a finite value. Then*

$$\int_a^b f(x)\, e^{i\lambda x}\, dx \to 0 \quad (\lambda \to +\infty).$$

The Riemann-Lebesgue lemma says nothing of the rate with which the integral $F(\lambda)$ tends to zero. The fact is that this rate depends essentially on the properties of the derivatives of $f(x)$ and can be very small. It was found that asymptotic expansions for Fourier integrals can be found only when $f(x)$ and $S(x)$ are smooth functions. We will consider only the case where both functions can be differentiated any number of times on I.

Theorem 1 *Suppose the functions $f(x)$ and $S(x)$ are infinitely differentiable and $S'(x) \neq 0$ for $x \in I$. Then, as $\lambda \to +\infty$, the integral (44.1) possesses the following asymptotic expansion:*

$$F(\lambda) \sim \frac{1}{i\lambda} e^{i\lambda S(b)} \sum_{n=0}^{\infty} b_n (i\lambda)^{-n} - \frac{1}{i\lambda} e^{i\lambda S(a)} \sum_{n=0}^{\infty} a_n (i\lambda)^{-n}. \tag{44.3}$$

This expansion can be differentiated with respect to λ any number of times. The principal term in the asymptotic expansion has the form

$$F(\lambda) = \frac{f(b)}{i\lambda S'(b)} e^{i\lambda S(b)} - \frac{f(a)}{i\lambda S'(a)} e^{i\lambda S(a)} + O(\lambda^{-2}). \tag{44.4}$$

The coefficients a_n and b_n are calculated by the following formulas:

$$a_n = (-1)^n M^n \left(\frac{f(x)}{S'(x)}\right)\Big|_{x=a}, \quad b_n = (-1)^n M^n \left(\frac{f(x)}{S'(x)}\right)\Big|_{x=b}, \tag{44.5}$$
$$M = \frac{1}{S'(x)} \frac{d}{dx}.$$

Note that the formulas for the a_n coincide with those for the c_n, (44.9).

Proof of Theorem 1 We integrate (44.1) by parts in the same way as we did in the proof of Theorem 1 of Sec. 43:

$$F(\lambda) = \frac{1}{i\lambda} \int_a^b \frac{f(x)}{S'(x)}\, d\,(e^{i\lambda S(x)}) = \frac{1}{i\lambda} e^{i\lambda S(x)} \frac{f(x)}{S'(x)}\Big|_a^b + \frac{1}{i\lambda} F_1(\lambda),$$

$$F_1(\lambda) = -\int_a^b e^{i\lambda S(x)} \frac{d}{dx}\left(\frac{f(x)}{S'(x)}\right) dx.$$

By the Riemann-Lebesgue lemma, $F_1(\lambda) = o(1)$ $(\lambda \to +\infty)$, and we have thus proved the validity of (44.4) with the remainder term of the order of $o(1/\lambda)$. The integral $F_1(\lambda)$ has exactly the same form as $F(\lambda)$; integration by parts once more yields

$$F_1(\lambda) = \frac{f_1(x)}{i\lambda S'(x)} e^{i\lambda S(x)}\Big|_a^b + \frac{1}{i\lambda} F_2(\lambda).$$

Here $f_1(x) = -(f(x)/S'(x))'$, and F_2 is obtained from F_1 by substituting f_1 for f. Since $F_2(\lambda) = o(1)$ $(\lambda \to +\infty)$ in view of the Riemann-Lebesgue lemma, we conclude that $F_1(\lambda) = O(\lambda^{-1})$, which proves the validity of (44.4) completely. We have also proved that

$$F(\lambda) = \frac{1}{i\lambda}\left[\left(b_0 + \frac{b_1}{i\lambda}\right) e^{i\lambda S(b)} - \left(a_0 + \frac{a_1}{i\lambda}\right) e^{i\lambda S(a)}\right] + \frac{1}{(i\lambda)^2} F_2(\lambda),$$

where the coefficients a_j and b_j have the form (44.5). Continuing the process of integration by parts, we arrive at the expansion (44.3).

Since the integral $F(\lambda)$ has a finite value for all complex valued λ's, we conclude that $F(\lambda)$ is an entire function of λ (see Theorem 1 of Sec. 16). The possibility of term-by-term differentiation of (44.3) follows from Theorem 5 of Sec. 42.

The proof of Theorem 1 leads to

Corollary 1 *Suppose $f(x)$ and $S(x)$ can be differentiated k and $k+1$ times, respectively (with k a positive integer) on the segment $[a, b]$. Then, as $\lambda \to +\infty$,*

$$F(\lambda)\frac{1}{i\lambda}e^{i\lambda S(b)}\sum_{n=0}^{k-1} b_n (i\lambda)^{-n} - \frac{1}{i\lambda}e^{i\lambda S(a)}\sum_{n=0}^{k-1} a_n (i\lambda)^{-n} + o(\lambda^{-k}). \tag{44.6}$$

A particular case of this corollary is the asymptotic estimate for Fourier coefficients, which is known from mathematical analysis.

Corollary 2 *Suppose the function $f(x)$ is k times continuously differentiable on the segment $[0, 2\pi]$ and*

$$f^{(j)}(0) = f^{(j)}(2\pi), \quad 0 \leqslant j \leqslant k. \tag{44.7}$$

Then

$$c_m = \int_0^{2\pi} e^{imx} f(x)\,dx = o(m^{-k}) \tag{44.8}$$

as $m \to +\infty$.

Indeed, since $e^{i2m\pi} = 1$ when m is an integer and condition (44.7) is met, all terms in (44.6) except the remainder term cancel out.

Integration by parts enables us to find the asymptotic behavior of some other classes of integrals of rapidly oscillating functions.

Example 1. Let us consider the integral

$$\Phi(x) = \int_x^\infty e^{it^2}\,dt$$

and establish its asymptotic behavior as $x \to +\infty$. Integration by parts yields

$$\Phi(x) = \int_x^\infty \frac{1}{2it}\,d(e^{it^2}) = -\frac{e^{ix^2}}{2xi} + \frac{1}{2i}\int_x^\infty e^{it^2}\frac{dt}{t^2}.$$

Let us estimate the value of the integral on the right-hand side. We have

$$\left|\int_x^\infty e^{it^2}\frac{dt}{t^2}\right| \leqslant \int_x^\infty \frac{dt}{t^2} = \frac{1}{x}.$$

We have therefore found that

$$\Phi(x) = -\frac{e^{ix^2}}{2ix} + O\left(\frac{1}{x}\right)$$

as $x \to +\infty$. Both terms on the right-hand side are of the same order of magnitude. Hence,

$$\Phi(x) = O\left(\frac{1}{x}\right) \quad (x \to +\infty).$$

To obtain a more precise estimate, we integrate by parts once more:

$$\int_x^\infty t^{-2}e^{it^2}\,dt = \frac{1}{2i}\int_x^\infty t^{-3}d(e^{it^2}) = -\frac{1}{2ix^3}e^{ix^2} + \frac{3}{2i}\int_x^\infty t^{-4}e^{it^2}\,dt.$$

The absolute value of the integral on the right-hand side does not exceed

$$\int_x^\infty t^{-4}\,dt = O(x^{-3}) \quad (x \to +\infty).$$

This yields

$$\Phi(x) = \frac{ie^{ix^2}}{2x} + O\left(\frac{1}{x^3}\right) \quad (x \to +\infty).$$

Continuing the process of integration by parts, we arrive at the asymptotic expansion for $\Phi(x)$ as $x \to +\infty$. We give this expansion with the first two terms:

$$\Phi(x) = e^{ix^2}\left(\frac{i}{2x} + \frac{1}{4x^3}\right) + O\left(\frac{1}{x^5}\right). \ \square$$

44.2 The contribution from a nondegenerate stationary point The hypothesis of Theorem 1 contains one important restriction, namely, $S'(x) \neq 0$ for $x \in I$, i.e. $S(x)$ (the phase) does not have a stationary point within the interval. But if such stationary points exist, then the asymptotic behavior of $F(\lambda)$ has a different nature than that stated in Theorem 1. The phase $S(x) = x^2$ has a stationary point $x = 0$. In the neighborhood of this point (over an interval of the order of $1/\sqrt{\lambda}$) the function $\cos \lambda x^2$ does not oscillate, while the sum of the areas of the other waves in the cosine are of the order of $O(\lambda^{-1})$, i.e. is considerably less than the area about the stationary point. For this reason the value of the integral $F(\lambda)$ will be of the order of $1/\sqrt{\lambda}$. Let us give a rigorous foundation to these heuristic considerations.

Lemma 1 *Suppose a function $f(x)$ is infinitely differentiable on the segment $[0, a]$ and $\alpha \neq 0$. Then, as $\lambda \to +\infty$,*

$$\Phi(\lambda) = \int_0^a f(x)\,e^{(i/2)\alpha\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{2\pi}{|\alpha|\lambda}}\,e^{i(\pi/4)\delta(\alpha)}f(0) + O\left(\frac{1}{\lambda}\right), \tag{44.9}$$

$$\delta(\alpha) = \operatorname{sign}\alpha.$$

Proof. Let $\alpha > 0$ and $f(x) \equiv 1$. Introducing the variable $t = x\sqrt{\alpha\lambda}$, we have

$$\int_0^a e^{(i/2)\alpha\lambda x^2}\,dx = \frac{1}{\sqrt{\alpha\lambda}} \int_0^{a\sqrt{\alpha\lambda}} e^{it^2/2}\,dt = \frac{1}{\sqrt{\alpha\lambda}} \left[\int_0^\infty e^{it^2/2}\,dt - \int_{a\sqrt{\alpha\lambda}}^\infty e^{it^2/2}\,dt \right].$$

The first integral in the brackets is a Fresnel integral and is equal to $(1/2)\, e^{i\pi/4}\sqrt{2\pi}$ (see Sec. 29). The other is an integral whose value is of the order of $O(1/\sqrt{\lambda})$ as $\lambda \to +\infty$, in view of Example 1, so that

$$\int_0^a e^{(i/2)\alpha\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{2\pi}{\alpha\lambda}}\, e^{i\pi/4} O\left(\frac{1}{\lambda}\right) \quad (\lambda \to +\infty). \tag{44.10}$$

Now suppose α is negative. Then

$$\overline{\int_0^a e^{i\alpha x^2}\,dx} = \int_0^a e^{i\beta x^2}\,dx,$$

with $\beta = -\alpha > 0$. Hence, formula (44.10) remains valid for $\alpha < 0$ if we substitute $-\alpha = |\alpha|$ for α and $e^{-i\pi/4}$ for $e^{i\pi/4}$.

Let us represent $f(x)$ in the form

$$f(x) = f(0) + [f(x) - f(0)] = f(0) + xg(x),$$

where $g(x) = \frac{f(x) - f(0)}{x}$ is a function that is infinitely differentiable for $0 \leqslant x \leqslant a$. Then

$$\Phi(\lambda) = \frac{1}{2} f(0) \sqrt{\frac{2\pi}{|\alpha|\lambda}}\, e^{i\frac{\pi}{4}\delta(\alpha)} + O\left(\frac{1}{\lambda}\right) + \Phi_1(\lambda), \tag{44.11}$$

$$\Phi_1(\lambda) = \int_0^a e^{\frac{i}{2}\alpha\lambda x^2} x g(x)\,dx.$$

Let us estimate $\Phi_1(\lambda)$. We have

$$|\Phi_1(\lambda)| = \left| \frac{1}{i\alpha\lambda} \int_0^a g(x)\, d\left(e^{(i/2)\alpha\lambda x^2}\right) \right|$$

$$= \frac{1}{|\alpha|\lambda} \left| g(a)\, e^{(i/2)\alpha\lambda a^2} - g(0) - \int_0^a e^{(i/2)\alpha\lambda x^2} g'(x)\,dx \right|$$

$$\leqslant \frac{1}{|\alpha|\lambda} \left[|g(a)| + |g(0)| + \int_0^a |g'(x)|\,|dx| \right] = O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to +\infty$. Substituting this estimate into (44.11), we arrive at (44.9). The proof of the lemma is complete.

Remark 1. The proof of Lemma 1 implies that (44.9) is valid if $f(x)$ is doubly differentiable on $[0, a]$.

Theorem 2 *Suppose $f(x)$ and $S(x)$ are infinitely differentiable on the segment $[a, b]$ and the function $S(x)$ has only one stationary point $x_0 \in [a, b]$, with $a < x_0 < b$. If $S''(x_0) \neq 0$, then the integral (44.1) possesses the following formula:*

$$F(\lambda) = e^{i\lambda S(x_0)} e^{i(\pi/4)\delta_0} \sqrt{\frac{2\pi}{\lambda\,|S''(x_0)|}}\, f(x_0) + O\left(\frac{1}{\lambda}\right) \quad (\lambda \to +\infty). \tag{44.12}$$

Here $\delta_0 = \operatorname{sign} S''(x_0)$.

Proof. Let us split the integration domain into two segments, $[a, x_0]$ and $[x_0, b]$, and the integral $F(\lambda)$ into two integrals, $F_1(\lambda)$ and $F_2(\lambda)$, respectively. Suppose $S''(x_0)$ is positive, for the sake of definiteness. Then $S'(x_0)$ is positive for $x_0 < x \leqslant b$, and the function $S(x)$ monotonically increases for $x_0 < x \leqslant b$, i.e. $S(x) > S(x_0)$ in this interval. In the integral $F_2(\lambda)$ (taken along the segment $[x_0, b]$) we introduce a new variable t by the relationship $x = \varphi(t)$ in a way such that $S(x) - S(x_0) = t^2$ (see Sec. 43). Then

$$F_2(\lambda) = e^{i\lambda S(x_0)} \int_0^{b'} e^{i\lambda t^2} g(t)\,dt.$$

Here

$$g(t) = f(\varphi(t))\,\varphi'(t), \quad b' = \sqrt{S(b) - S(x_0)} > 0.$$

By Lemma 1, we have

$$F_2(\lambda) = \frac{1}{2}\, e^{i\lambda S(x_0)} e^{i\pi/4} \sqrt{\frac{\pi}{\lambda}}\, g(0) + O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to +\infty$, with $g(0) = f(x_0)\sqrt{\frac{2}{S''(x)_0}}$. A similar formula exists for $F_1(\lambda)$. Combining the two, we arrive at (44.12). The case with $S''(x_0) < 0$ can be reduced to the case with $S''(x_0) > 0$:

$$\overline{F(\lambda)} = \int_a^b e^{i\lambda \tilde{S}(x)} \overline{f(x)}\,dx, \quad \tilde{S}(x) = -S(x),$$

and $\tilde{S}''(x_0) > 0$. The proof of the theorem is complete.

Example 2. Let us calculate the asymptotic behavior of the Bessel function

$$J_n(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\varphi-n\varphi)}\,d\varphi$$

as $x\to+\infty$, with n a nonnegative integer. In the case at hand the phase $S(\varphi)=\sin\varphi$, and there are two stationary points, $\varphi_1=\pi/2$ and $\varphi_2=3\pi/2$, with

$$S(\varphi_1)=1,\quad S''(\varphi_1)=-1,\quad S(\varphi_2)=-1,\quad S''(\varphi_2)=1.$$

The asymptotic behavior of $J_n(x)$ is given by the sum of the contributions from the points φ_1 and φ_2 (i.e. expressions of the type (44.12)) and a term of the order of $O(1/x)$, i.e.

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{x}\right)\quad(x\to+\infty).\ \square$$

44.3 Poisson's summation formula This formula enables us to replace a series of the form $\sum_{n=-\infty}^{\infty} f(n)$ with another series, namely

$$\sum_{n=-\infty}^{\infty} f(n)=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi ikx}f(x)\,dx. \tag{44.13}$$

This formula is valid if

(a) $f(x)$ is continuously differentiable for $-\infty<x<\infty$,

(b) the series $\sum_{n=-\infty}^{\infty} f(n)$ is convergent,

(c) the series $\sum_{n=-\infty}^{\infty} f'(n+x)$ is uniformly convergent for $0\leqslant x\leqslant 1$.

The proof of the validity of (44.13) under these or other conditions is given in Evgrafov [2]. We will restrict our discussion to a formal derivation of (44.13). Consider the function $\varphi(x)=\sum_{n=-\infty}^{\infty} f(n+x)$. This function is periodic with a period equal to unity. Let us expand $\varphi(x)$ in a Fourier series:

$$\varphi(x)=\sum_{k=-\infty}^{\infty}\varphi_k e^{2\pi ikx},$$

whence

$$\sum_{k=-\infty}^{\infty}\varphi_k=\sum_{n=-\infty}^{\infty} f(n). \tag{44.14}$$

Let us show that formula (44.14) leads to Poisson's summation formula. We have

$$\varphi_k=\int_0^1 e^{-2\pi ikx}\varphi(x)\,dx=\int_0^1\sum_{n=-\infty}^{\infty} f(n+x)\,e^{-2\pi ikx}\,dx$$
$$=\sum_{n=-\infty}^{\infty}\int_n^{n+1} f(x)\,e^{-2\pi ikx}\,dx=\int_{-\infty}^{\infty} e^{-2\pi ikx}f(x)\,dx,$$

and, substituting the φ_k into (44.14), we arrive at (44.13).

Formula (44.13) proves to be convenient when the integrals

$$\varphi_n=\int_{-\infty}^{\infty} e^{-2\pi inx}f(x)\,dx$$

decrease as $n\to\infty$ faster than $f(n)$ (i.e. when the Fourier transform of $f(x)$ decreases as $|x|\to\infty$ faster than $f(x)$). For one, this is the case for a rapidly oscillating $f(x)$.

Example 3. Let us consider the series

$$F(t)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{n^2+t^2}}$$

and establish the asymptotic behavior of $F(t)$ as $t\to\pm\infty$. Here

$$f(x,t)=\frac{e^{i\pi x}}{\sqrt{x^2+t^2}}.$$

We apply Poisson's summation formula. Conditions (a) and (b) are met. Let us see whether condition (c) is (for a fixed $t>0$). We have

$$f'_x(x,t)=\pi ie^{i\pi x}(x^2+t^2)^{-1/2}-xe^{i\pi x}(x^2+t^2)^{-3/2}.$$

Consider the series

$$S_1=\sum_{k=1}^{\infty}(-1)^k a_k,\quad a_k=[(x+k)^2+t^2]^{-1/2}.$$

Since the functions $a_k(x)$ decrease monotonically in k for each fixed $x\in[0,t]$, and the partial sums of the series $\sum_{k=1}^{\infty}(-1)^k$ are limited, we conclude, from Dirichlet's test (see Kudryavtsev [1]), that the series S_1 is uniformly convergent on $[0, 1]$. The uniform convergence of the series

$$S_2=\sum_{k=1}^{\infty}(-1)^k(x+k)[(x+k)^2+t^2]^{-3/2}$$

for $x \in [0, 1]$ can be proved similarly. The same is true for the series S_1 and S_2 when the summation is carried out from $-\infty$ to -1. Hence, conditions (a), (b), and (c) are met. Applying (44.13), we find that

$$F(t) = \sum_{k=-\infty}^{\infty} \varphi_k(t), \quad \varphi_k(t) = \int_{-\infty}^{\infty} e^{-2\pi ikx+\pi ix}(x^2+t^2)^{-1/2}\,dx.$$

Substituting ty for x, we obtain

$$\varphi_k(t) = \int_{-\infty}^{\infty} e^{-it\pi(2k-1)y}(y^2+1)^{-1/2}\,dy,$$

so that $\varphi_k(t) = 2K_0((2k-1)\pi t)$ (see Example 3 in Sec. 43). In Example 3 in Sec. 43 it was shown that $K_0(b)$ is an even function and that

$$K_0(b) = \sqrt{\frac{\pi}{2b}}\, e^{-b}[1+O(b^{-1})] \quad (b \to +\infty).$$

Hence, for $|b| \geqslant 1$ we have

$$|K_0(b)| \leqslant Ce^{-|b|},$$

where C does not depend on b, and

$$|F(t)-\varphi_0(t)-\varphi_1(t)| \leqslant 4C\sum_{k=2}^{\infty}|K_0((2k-1)\pi t)|$$
$$\leqslant 4C\sum_{k=2}^{\infty} e^{-(2k-1)\pi t} \leqslant 8Ce^{-3\pi t}.$$

We finally obtain

$$F(t) = 2\varphi_0(t) + O(e^{-3\pi t}) = 2\sqrt{\frac{1}{2t}}\, e^{-\pi t}[1+O(t^{-1})] \quad (t \to +\infty). \quad \square$$

45 The Saddle-Point Method

45.1 Preliminary considerations Let us take the integral

$$F(\lambda) = \int_{\gamma} f(z)\, e^{\lambda S(z)}\, dz, \tag{45.1}$$

where γ is a piecewise smooth curve in the complex z plane, and the functions $f(z)$ and $S(z)$ are regular in a domain D that contains γ. We are interested in the asymptotic behavior of $F(\lambda)$ as $\lambda \to +\infty$. The trivial cases $f(z) \equiv 0$ or $S(z) \equiv$ const are of no interest to us.

In Sec. 43 it was found that if γ is a segment and $S(z)$ admits real values on γ, the asymptotic behavior of (45.1) can be established

by applying Laplace's method. We will try to transform the integral in (45.1) in such a way as to make it possible to apply Laplace's method. Since $f(z)$ and $S(z)$ are regular in D, we can deform γ in D (with the end points remaining fixed) without changing the **value** of $F(\lambda)$. Suppose we can deform γ into a contour $\tilde{\gamma}$ such that

(1) $\max\limits_{z\in\tilde{\gamma}} |e^{\lambda S(\lambda)}|$ is attained only at one point $z_0 \in \tilde{\gamma}$ (z_0 an interior point of the contour),

(2) $\operatorname{Im} S(z) \equiv$ const for $z \in \tilde{\gamma}$ in a neighborhood of point z_0.

Suppose $\tilde{\gamma}_0$ is a small arc of $\tilde{\gamma}$ that contains point z_0. Then $\operatorname{Re} S(z) \leqslant \operatorname{Re} S(z_0) - \delta$, where $\delta > 0$, for $z \bar{\in} \tilde{\gamma}_0$, $z \in \tilde{\gamma}$. This follows from the fact that $\max\limits_{z\in\tilde{\gamma}} \operatorname{Re} S(z)$ is attained only at point z_0, according to condition (1). For this reason the integral taken along the arc $\tilde{\gamma} - \tilde{\gamma}_0$ is of the order of $O(|e^{\lambda(S(z_0)-\delta)}|)$ as $\lambda \to +\infty$ (see Lemma 1 of Sec. 43). Let us consider the integral taken along the arc $\tilde{\gamma}_0$; suppose $z = \varphi(t)$, with $-t_0 \leqslant t \leqslant t_0$ and $\varphi(0) = z_0$, is the equation of this arc. By condition (2), $\operatorname{Im} S(z) \equiv \operatorname{Im} S(z_0)$ on $\tilde{\gamma}_0$, so that the integral taken along this arc is

$$F_1(\lambda) = e^{i\lambda \operatorname{Im} S(z_0)} \int_{-t_0}^{t_0} \tilde{f}(t)\, e^{\lambda\tilde{S}(t)}\, dt,$$

where $\tilde{f}(t) = f(\varphi(t))\,\varphi'(t)$ and $\tilde{S}(t) = \operatorname{Re} S(\varphi(t))$. In $F_1(\lambda)$ the function $\tilde{S}(t)$ assumes only real values; hence, $F_1(\lambda)$ belongs to the class of integrals discussed in Sec. 43 and its asymptotic behavior can be established by Laplace's method.

Note, in addition, that $S'(z_0) = 0$. Indeed, $\frac{d}{dt}\operatorname{Im} S(z)|_{t=0} = 0$, by condition (2), and since $\max\limits_{z\in\tilde{\gamma}} \operatorname{Re} S(z)$ is attained at point z_0 (condition (1), we conclude that $\frac{d}{dt}\operatorname{Re} S(z)|_{t=0} = 0$. Hence, $\frac{d}{dt} S(z)|_{t=0} = 0$, so that $S'(z_0) = 0$.

A point z_0 at which $S'(z_0) = 0$ is said to be a *saddle point*, and a contour that obeys conditions (1) and (2) must pass through the saddle point of $S(z)$.

In a similar manner we can establish the asymptotic behavior of the integral (45.1) when $\max\limits_{z\in\gamma} \operatorname{Re} S(z)$ is attained only at one end point of γ. In this case the point, z_0, may not be a saddle point.

Thus, if the function $\operatorname{Re} S(z)$ on contour γ attains its maximum only at a finite number of points, which are either saddle points or end points of the contour (we call such a contour a *saddle contour*), then the asymptotic behavior of integral (45.1) can be established by Laplace's method. The most difficult problem in applying the saddle-point method is finding a saddle contour $\tilde{\gamma}$ equivalent to the initial contour γ (the equivalence of γ and $\tilde{\gamma}$ means that the integrals of type (45.1) are equal along these contours). Many problems have been solved by the saddle-point method (e.g. see Evgrafov [2], Fedoryuk [1], Lavrent'ev and Shabat [1], Morse and Feshbach [1], and Whittaker and Watson [1]), but there is not a single general technique that would enable us to find an equivalent saddle contour $\tilde{\gamma}$ from given functions $f(z)$ and $S(z)$ and a given contour γ.

We will now give a rigorous derivation of the asymptotic formulas for integral (45.1) taken along a saddle contour. But first we will study the local structure of the curves along which $\text{Re } S(z)$ or $\text{Im } S(z)$ remain constant (the *level curves*).

45.2 **The structure of level curves of harmonic functions** Suppose $S(z)$ is regular in a neighborhood of point z_0. Let us study the level curves of $\text{Re } S(z) = \text{Re } S(z_0) + \varepsilon$ and $\text{Im } S(z) = \text{Im } S(z_0) + \varepsilon$ for small ε's in a neighborhood of point z_0.

Lemma 1 *Suppose $S'(z_0)$ is not zero. Then in a small neighborhood of point z_0 the level curves* $\text{Re } S(z) = \text{const}$ *and* $\text{Im } S(z) = \text{const}$ *are smooth curves.*

Proof. The function $S(z)$ is univalent at point z_0, since $S'(z_0) \neq 0$. For this reason the function $w = S(z)$ maps a small neighborhood U of point z_0 conformally and in a one-to-one manner onto a small neighborhood V of point $w_0 = S(z_0)$. We select U in a way such that V is the square $|u - u_0| < \delta$, $|v - v_0| < \delta$, where $w = u + iv$ and $w_0 = u_0 + iv_0$. Under such a mapping the level curves of the functions $\text{Re } S(z)$ and $\text{Im } S(z)$ lying in U are mapped into segments of straight lines $u = \text{const}$ and $v = \text{const}$ lying in V. These segments are, obviously, smooth curves (lines), and their preimages are smooth curves, too, since $z = S^{-1}(w)$, which is the inverse of $S(z)$, is regular at point w_0 (see Theorem 1 of Sec. 13). The proof of the lemma is complete.

We have thus established that the local structure of the level curves of $\text{Re } S(z)$ and $\text{Im } S(z)$ in a neighborhood of a point that is not a saddle point is exactly the same as that of the function $S(z) = z$ (Fig. 159).

Now let us study the structure of the level curves of $\text{Re } S(z)$ and $\text{Im } S(z)$ in a neighborhood of a saddle point. But first let us consider a simple case.

Example 1. Let us study the level curves of the real and imaginary parts of the function $S(z) = -z^2$. Point $z = 0$ is a saddle point. Assuming that $z = x + iy$ and $S = u + iv$, we find that $u = y^2 - x^2$ and $v = -2xy$. The family of level curves has the form

$$x^2 - y^2 = C_1, \quad 2xy = C_2,$$

where C_1 and C_2 are constants. If both C_1 and C_2 are not zeros, each of the curves $\text{Re } S = C_1$ and $\text{Im } S = C_2$ is a hyperbola, while the curve $u = 0$ consists of the two straight lines $x - y = 0$ and $x + y = 0$ and the curve $v = 0$ of the two straight lines $x = 0$ and $y = 0$ (Fig. 160). The level curves $\text{Re } S(z) = \text{Re } S(0)$ (i.e. the straight lines $x \pm y = 0$) divide the complex z plane into four sectors, with the signs of $\text{Re}(S(z) - S(0))$ in two neighboring sectors being different (Fig. 160). Suppose D_0 is the sector $|\arg z| < \pi/4$ and D_1 is the sector $|\arg(-z)| < \pi/4$; in these sectors $\text{Re}(-z^2) < 0$. The level curve that passes through the saddle point $z = 0$ is $\text{Im } S(z) = \text{Im } S(0)$, which is the straight line l: $y = 0$. Along this line we have $\text{Re } S(z) = -x^2$, i.e. the function $\text{Re } S(z)$ decreases strictly monotonically as point z moves along l away from the saddle point $z = 0$. Line (in general a curve) l is known as the *path of steepest descent.* □

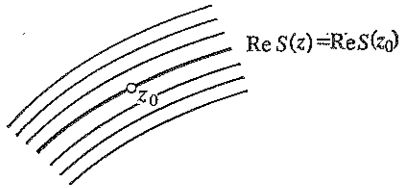

Fig. 159

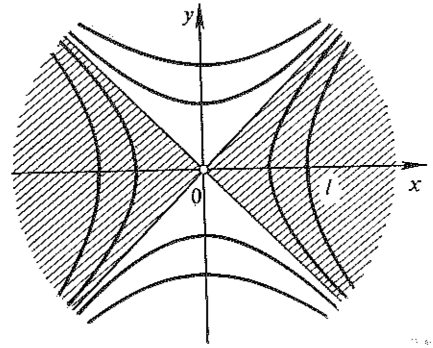

Fig. 160

Take a three-dimensional space with coordinates x, y, and $\text{Re } S$ and the surface $\text{Re } S = \text{Re}(-z^2)$, i.e. $\text{Re } S = y^2 - x^2$. This surface is a hyperbolic paraboloid (Fig. 161), and the origin of coordinates is the saddle point. A mountain pass or a saddle resemble such a surface; hence the names "saddle-point method" and "saddle point". The path of steepest descent from the saddle point is projected onto the (x, y) plane into l.

Let us now show that if z_0 is a simple saddle point for $S(z)$, i.e. if $S''(z_0) \neq 0$, then in a neighborhood of this point the level curves of $\text{Re } S(z)$ and $\text{Im } S(z)$ have the same structure as those in the case of $S(z) = -z^2$.

Lemma 2 *Suppose point z_0 is a simple saddle point of $S(z)$, i.e. $S'(z_0) = 0$ and $S''(z_0) \neq 0$. Then in a small neighborhood U the* *level curve* $\operatorname{Re} S(z) = \operatorname{Re} S(z_0)$ *consists of two smooth curves* l_1 *and* l_2 *that are orthogonal to each other at point* z_0 *and divide* U *into four sectors. In neighboring sectors the signs of* $\operatorname{Re}(S(z) - S(z_0))$ *are different. This situation is depicted in Fig. 162.*

Proof. Suppose U is a small neighborhood of point z_0. Then there is a function $\varphi(\zeta)$ that is regular in a neighborhood V of point $\zeta = 0$ and such that

$$S(\varphi(\zeta)) = S(z_0) - \zeta^2, \quad \zeta \in V \tag{45.2}$$

(see Corollary 2 in Sec. 32). Moreover, $\varphi'(0) \neq 0$, and the function $z = \varphi(\zeta)$ maps V onto U in a one-to-one manner. The level curves of $\operatorname{Re} S(z)$ and $\operatorname{Im} S(z)$ are mapped by $\zeta = \varphi^{-1}(z)$ into the curves

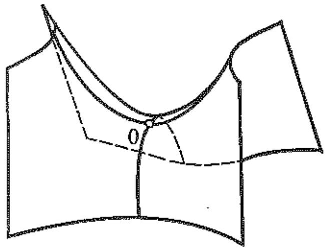

Fig. 161

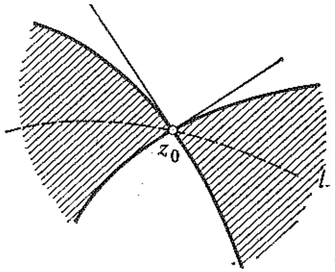

Fig. 162

$\operatorname{Re} \zeta^2 = \text{const}$ and $\operatorname{Im} \zeta^2 = \text{const}$, whose structure was studied earlier. Returning to the variable z, we complete the proof of the lemma.

Corollary 1 *Through the sectors in which* $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$ *there passes a smooth curve* l *such that* $\operatorname{Im} S(z) = \operatorname{Im} S(z_0)$ *for* $z \in l$. *The function* $\operatorname{Re} S(z)$ *decreases strictly monotonically as* z *moves along* l *away from* z_0.

The curve l is the path of steepest descent (Fig. 162; the dashed line).

45.3 The contribution from an end point of the integration path In what follows we will always assume that γ is a finite curve and that the functions $f(z)$ and $S(z)$ are regular in a domain D containing γ.

Theorem 1 *Suppose* $\max\limits_{z \in \gamma} \operatorname{Re} S(z)$ *is attained only at the initial point* a *of* γ *and* $S'(a) \neq 0$. *Then, as* $\lambda \to +\infty$, *the following asymptotic expansion is valid:*

$$F(\lambda) \equiv \int_\gamma f(z) e^{\lambda S(z)}\, dz \sim \lambda^{-1} e^{\lambda S(a)} \sum_{n=0}^{\infty} c_n \lambda^{-n}. \tag{45.3}$$

This expansion can be differentiated with respect to λ *term-by-term any number of times. The principal term of the asymptotic expansion has the form*

$$F(\lambda) = \frac{1}{-\lambda S'(a)} e^{\lambda S(a)} \left[f(a) + O\left(\frac{1}{\lambda}\right) \right]. \tag{45.4}$$

The expansion coefficients in (45.3) are given by the formula

$$c_n = (-1)^n M^n \left(\frac{f(z)}{S'(z)} \right) \Big|_{z=a}, \quad M = \frac{1}{S'(z)} \frac{d}{dz}. \tag{45.5}$$

Note that for Laplace integrals (45.3)-(45.5) coincide with formulas (43.8) and (43.9). The proof is similar to the proof of Theorem 1 of Sec. 43.

45.4 The contribution from a simple saddle point Let us establish the asymptotic behavior of (45.1) in the case where $\max\limits_{z \in \gamma} \operatorname{Re} S(z)$ is attained at an interior point of the integration path. Precisely, let the following conditions be met:

(a) $\max\limits_{z \in \gamma} \operatorname{Re} S(z)$ is attained only at a point z_0 that is an interior point of γ and a simple saddle point (i.e. $S'(z_0) = 0$ and $S''(z_0) \neq 0$),

(b) in a neighborhood of point z_0 the contour γ passes through both sectors in which $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$ (Fig. 162).

Theorem 2 *Suppose conditions* (a) *and* (b) *are met. Then, as* $\lambda \to +\infty$, *the following asymptotic expansion is valid:*

$$F(\lambda) \equiv \int_\lambda f(z) e^{\lambda S(z)}\, dz \sim e^{\lambda S(z_0)} \sum_{n=0}^{\infty} c_n \lambda^{-n-1/2}. \tag{45.6}$$

This expansion can be differentiated with respect to λ *term-by-term any number of times. The principal term in the asymptotic expansion has the form*

$$F(\lambda) = \sqrt{-\frac{2\pi}{\lambda S''(z_0)}}\, e^{\lambda S(z_0)} \left[f(z_0) + O\, \frac{1}{\lambda} \right], \quad (\lambda \to +\infty). \tag{45.7}$$

The choice of the branch of the root in (45.7) and the formula for the expansion coefficients in (45.6) are given below.

Proof of Theorem 2. Suppose U is a small neighborhood of point z_0, $\gamma_0 = \gamma \cap U$, and γ_1 and γ_2 are the remaining arcs of γ. Let us split the integral $F(\lambda)$ into three terms: $F(\lambda) = F_0(\lambda) + F_1(\lambda) + F_2(\lambda)$, where $F_j(\lambda)$ is an integral of the type (45.1) taken along arc γ_j, $j = 0, 1, 2$. Since $\max\limits_{z \in \gamma} \operatorname{Re} S(z)$ is attained only at point

$z_0 \in \gamma_0$, we can show, just as we did in the proof of Theorem 2 of Sec. 43, that the following estimate holds for $F_1(\lambda)$ and $F_2(\lambda)$:

$$|F_j(\lambda)| \leqslant c\,|e^{\lambda(S(z_0)-\delta)}| \quad (\lambda > 0),\ j = 1,\ 2, \tag{45.8}$$

where c and δ are positive constants.

Let us find the asymptotic expansion for $F_0(\lambda)$. If U is small, then there is a neighborhood V of point $\zeta = 0$ and a function $z = \varphi(\zeta)$ such that (i) $S(\varphi(\zeta)) = S(z_0) - \zeta^2$, $\zeta \in V$, and (ii) the function $\varphi(\zeta)$ is regular in V and maps V onto U in a one-to-one manner, with $\varphi(0) = z_0$.

This follows from Corollary 2 in Sec. 32. Substituting $\varphi(\zeta)$ for z in $F_0(\lambda)$, we obtain

$$F_0(\lambda) = e^{\lambda S(z_0)} \int\limits_{\tilde{\gamma}} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta. \tag{45.9}$$

Here $g(\zeta) = f(\varphi(\zeta))\,\varphi'(\zeta)$, and $\tilde{\gamma}$ is the image of contour γ_0. For V we can take the circle $|\zeta| < \rho$ of a small radius $\rho > 0$; we can also assume that $\varphi(\zeta)$ is regular in the closed circle $|\zeta| \leqslant \rho$.

The level curve $\operatorname{Re}(-\zeta^2) = 0$ consists of two straight lines $\xi \pm \eta = 0$ $(\zeta = \xi + i\eta)$ and divides V into four sectors. Suppose D_1 is the sector containing the interval l_1: $(0, \rho)$ and D_2 is the sector containing the interval l_2: $(-\rho, 0)$. Curve γ_0, by hypothesis, consists of two curves γ_{01} and γ_{02} (with a common point z_0); these curves lie in the different sectors in which $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$. Hence, point $\zeta = 0$ partitions curve $\tilde{\gamma}$ into two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ that lie in sectors D_1 and D_2, respectively. Suppose C_1 is the arc of circle $|\zeta| = \rho$ that lies in D_1 and connects the end points of the curves l_1 and $\tilde{\gamma}_1$. By Cauchy's theorem,

$$\int\limits_{\tilde{\gamma}_1} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta = \int\limits_0^{\rho} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta + \int\limits_{C_1} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta. \tag{45.10}$$

Since $\operatorname{Re}(-\zeta^2) < 0$ on C_1, there is a positive constant δ_1 such that $\operatorname{Re}(-\zeta^2) \leqslant -\delta_1$ on C_1, and the integral taken along $\tilde{\gamma}_1$ is equal to the sum of the integral taken along the segment $[0, \rho]$ and a term of the order of $O(e^{-\lambda\delta_1})$ $(\lambda \to +\infty)$. Applying the same line of reasoning to the integral taken along the arc $\tilde{\gamma}_2$, we find that

$$e^{-\lambda S(z_0)} F_0(\lambda) = \int\limits_{-\rho}^{\rho} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta + O(e^{-\lambda\delta'}) \quad (\lambda \to +\infty), \tag{45.11}$$

where δ' is a positive constant. On the right-hand side of (45.11) we have an integral taken along a segment, i.e. a Laplace integral, (43.1), with $S = -\zeta^2$. Further $\max\limits_{-\rho \leqslant \zeta \leqslant \rho} S(\zeta)$ is attained only at point $\zeta = 0$, with $S''(0) \neq 0$. Applying Theorem of Sec. 43, we arrive at expansion (45.6). The proof of the theorem is complete.

Let us select a branch of the root in (45.7) (the choice depends, of course, on the orientation of γ). In proving Theorem 2 it was found that γ can be deformed into a contour γ' which in a neighborhood of the saddle point z_0 coincides with the path of steepest descent l: $\operatorname{Im} S(z) = \operatorname{Im} S(z_0)$ on l and $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$ for $z \in l$, $z \neq z_0$. Let us show that

$$\arg \sqrt{\frac{1}{-S''(z_0)}} = \varphi_0, \tag{45.12}$$

where φ_0 is the angle between the tangent to l at point z_0 and the positive direction of the real axis.

It is sufficient to consider the case with $f(z) \equiv 1$ and $S(z) = az^2/2$, since the principal term in the asymptotic expansion is expressed only in terms of $f(z)$, $S(z)$, and $S''(z)$ at the saddle point. The path of steepest descent l, which passes through the saddle point $z = 0$, is the straight line (see Sec. 45.2) on which $\operatorname{Im} S(z) \equiv 0$ and while $\operatorname{Re} S(z)$ is negative for $z \neq 0$. We write its equation in the form $z = e^{i\varphi_0}\rho$, $-\infty < \rho < \infty$. Then $S(z) = -|a|\rho^2/2$ for $z \in l$. The integral along l is equal to

$$\int\limits_l e^{\lambda a z^2/2}\, dz = e^{i\varphi_0} \int\limits_{-\infty}^{\infty} e^{-\lambda|a|\rho^2/2}\, d\rho = e^{i\varphi_0} \sqrt{\frac{2\pi}{|a|}},$$

which proves formula (45.12).

The proof of Theorem 2 leads to

Theorem 3 *Suppose* $\max\limits_{z\in\gamma} \operatorname{Re} S(z)$ *is attained only at the initial point a of contour γ, with $S'(a) = 0$ and $S''(a) \neq 0$. Then, as $\lambda \to +\infty$, the following asymptotic expansion is valid:*

$$F(\lambda) \equiv \int\limits_{\gamma} f(z)\, e^{\lambda S(z)}\, dz \sim e^{\lambda S(a)} \sum_{n=0}^{\infty} a_n \lambda^{-\frac{n+1}{2}}. \tag{45.13}$$

This expansion can be differentiated with respect to λ term-by-term any number of times. The principal term in the asymptotic expansion has the form

$$F(\lambda) = \frac{1}{2} \sqrt{-\frac{2\pi}{\lambda S''(a)}}\, e^{\lambda S(a)} [f(a) + O(\lambda^{-1})] \quad (\lambda \to +\infty). \tag{45.14}$$

The choice of the branch of the root in (45.14) is the same as in (45.10).

Corollary 2 *Suppose* $\max\limits_{z\in\gamma} \operatorname{Re} S(z)$ *is attained at a finite number of points* $z_1, z_2, \ldots, z_m$, *which are either the end points of the path of integration or saddle points on the contour that obey condition (b) of Theorem 2. Then the asymptotic behavior of* (45.1) *as* $\lambda \to +\infty$ *is determined by the sum of the contributions from all points* $z_1, z_2, \ldots, z_m$.

Remark 1. If all the points z_j at which $S'(z_j) = 0$ are simple saddle points, the asymptotic behavior of (45.1) is determined by formulas (45.3), (45.6) and (45.13), with the principal term in the asymptotic expansion determined by formulas (45.4), (45.7) and (45.14). The asymptotic behavior can also be determined when there are saddle points of multiplicity greater than unity among the z_j (e.g. see Evgrafov [2], Fedoryuk [1], and Lavrent'ev and Shabat [1]).

45.5 Examples

Example 2. Let us establish the asymptotic behavior of the Airy-Fock function

$$\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + tx\right) dt$$

as $x \to +\infty$. We first transform this integral into

$$\operatorname{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3}+tx\right)} dt \tag{45.15}$$

(the function $\sin(t^3/3 + tx)$ is odd in t and the integral of this function taken along the real axis is zero). The integral in (45.15) is conditionally convergent; we will transform it so as to make it absolutely convergent.

Let us take the straight line l_{η_0}: $-\infty < \xi < \infty$, $\eta = \eta_0$ in the complex $\zeta = \xi + i\eta$ plane, which line is parallel to the real axis. On this line

$$\operatorname{Re} S(\zeta, x) = -\xi^2\eta_0 + \frac{\eta_0^3}{3} - x\eta_0, \tag{45.16}$$

where $S(\zeta, x) = i(\zeta^3/3 + x\zeta)$. Hence, $\int\limits_{l_{\eta_0}} e^{S(\zeta, x)}\, d\zeta$ is absolutely convergent if $\eta_0 > 0$. It was found that the integral (45.15) is equal to the integral taken along the straight line l_{η_0} for any positive η_0, i.e.

$$\operatorname{Ai}(x) = \frac{1}{2\pi}\int\limits_{l_{\eta_0}} e^{S(\zeta, x)}\, d\xi. \tag{45.17}$$

For every fixed $x > 0$ the function $S(\zeta, x)$ has two saddle points, $\zeta_1(x) = i\sqrt{x}$ and $\zeta_2(x) = -i\sqrt{x}$. For l_{η_0} we select the straight line that passes through the saddle point $\zeta_1(x)$, i.e. we put $\eta_0 = \sqrt{x}$. We substitute $\xi\sqrt{x}$ for ξ in the integral in (45.17), so as to bring it to the (45.1) type. Then

$$\operatorname{Ai}(x) = \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{\infty} e^{x^{3/2}\tilde{S}(\tilde{\xi})}\, d\tilde{\xi}, \quad \tilde{S}(\tilde{\xi}) = i\left[\frac{(\tilde{\xi}+i)^3}{3} + \tilde{\xi} + i\right]. \tag{45.18}$$

The path of integration contains the saddle point $\tilde{\xi} = 0$ of $\tilde{S}(\tilde{\xi})$. Further, for real $\tilde{\xi}$'s we have

$$\operatorname{Re}\tilde{S}(\tilde{\xi}) = -\tilde{\xi}^2 - \frac{2}{3}, \tag{45.19}$$

so that $\max \operatorname{Re}\tilde{S}(\tilde{\xi})$ is attained in the integration path only at the saddle point $\tilde{\xi} = 0$. This is a simple saddle point since $\tilde{S}''(0) = -2 \neq 0$.

Thus, the integral in (45.18) meets all the conditions of Theorem 2 except one, namely, the path of integration is an infinite straight line. Let us partition this straight line into three parts: the rays $(-\infty, -1)$ and $(1, \infty)$ and the segment $[-1, 1]$. In view of (45.19) we have

$$\left|\int_1^\infty e^{x^{3/2}\tilde{S}(\tilde{\xi})}\, d\tilde{\xi}\right| \leqslant e^{-(2/3)x^{3/2}}\int_1^\infty e^{-x^{3/2}\tilde{\xi}^2}\, d\tilde{\xi}.$$

By Lemma 1 of Sec. 43, the integral on the right-hand side is of the order of $O(e^{-x^{3/2}})$ $(x \to +\infty)$ since $-\tilde{\xi}^2 \leqslant -1$ for $\tilde{\xi} \geqslant 1$, so that the integral taken along the ray $1 \leqslant \tilde{\xi} < \infty$ is exponentially small compared with $e^{-2/3x^{3/2}}$ as $x \to +\infty$. The integral taken along the ray $-\infty \leqslant \tilde{\xi} \leqslant 1$ can be estimated in the same manner.

The asymptotic behavior of the integral taken along the segment $[-1, 1]$ can be established via Theorem 2; the principal term in the asymptotic expansion is calculated by formula (45.7). We have $\tilde{S}(0) = -2/3$ and $\tilde{S}''(0) = -2$. What is left is to select a branch of the root in (45.7). We have

$$\tilde{S}(\tilde{\zeta}) - \tilde{S}(0) \sim -\tilde{\xi}^2 \quad (\tilde{\zeta} \to 0),$$

where $\tilde{\zeta} = \tilde{\xi} + i\tilde{\eta}$. For this reason the path of steepest descent l, which passes through the saddle point $\tilde{\zeta} = 0$, has the same tangent as the path of steepest descent l_0 corresponding to $-\tilde{\zeta}^2$. The equation of l_0 has the form $\tilde{\zeta} = \rho$, $-\infty < \rho < \infty$, i.e. $\varphi_0 = 0$ in (45.12). The final asymptotic formula is

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} [1 + O(x^{-3/2})] \quad (x \to +\infty). \tag{45.20}$$

In the case at hand we can calculate all the coefficients in the asymptotic series (see Fedoryuk [1]). The asymptotic behavior of the Airy-Fock function as $x \to -\infty$ will be established in Example 4. □

Example 3. Let us establish the asymptotic behavior for real $x \to \infty$ of the integral

$$F(x) = \int_{-\infty}^{\infty} e^{-\frac{t^{2n}}{2n} + ixt}\, dt, \tag{45.21}$$

where n is a positive integer. Since

$$F(-x) = \overline{F(x)} \tag{45.22}$$

for real x's, it is sufficient to establish the asymptotic behavior of the integral (45.21) as $x \to +\infty$. By changing the variable from $x^{-1/(2n-1)} t$ to t, we transform the integral $F(x)$ to the form (45.1):

$$F(x) = x^{1/(2n-1)} \Phi(\lambda), \quad \Phi(\lambda) = \int_{-\infty}^{\infty} e^{\lambda S(t)}\, dt, \tag{45.23}$$

where

$$\lambda = x^{2n/(2n-1)}, \quad S(t) = -\frac{t^{2n}}{2n} + it. \tag{45.24}$$

The saddle points of $S(t)$ are determined from the equation $t^{2n-1} = i$ and are

$$t_k = e^{i\varphi_k}, \quad \varphi_k = \frac{(\pi/2) + 2k\pi}{2n-1}, \quad 0 \leqslant k \leqslant 2n-2. \tag{45.25}$$

Hence,

$$S(t_k) = \left(1 - \frac{1}{2n}\right) it_k, \quad \mathrm{Re}\, S(t_k) = \left(\frac{1}{2n} - 1\right) \sin \varphi_k. \tag{45.26}$$

For this reason $\mathrm{Re}\, S(t_k) < 0$ if point t_k lies in the upper half-plane of the complex t plane and $\mathrm{Re}\, S(t_k) > 0$ if t_k lies in the lower half-plane.

Integral (45.21) tends to zero as $x \to +\infty$ by the Riemann-Lebesgue lemma (see Sec. 44). Hence, the points t_k lying in the lower half-plane contribute nothing to the asymptotic behavior of $\Phi(\lambda)$, since the absolute value of the integrand $|e^{\lambda S(t_k)}| = e^{\lambda\, \mathrm{Re}\, S(t_k)}$ at such a point exponentially grows as $x \to +\infty$. For this reason the asymptotic behavior of $\Phi(\lambda)$ is determined only by those points t_k that lie in the upper half-plane $\mathrm{Im}\, t > 0$.

Since there are no saddle points of $S(t)$ in the integration path, we must deform this contour into a saddle contour. For $|t| \to \infty$ we have $S(t) \sim -t^{2n}/2n$, i.e. $\mathrm{Re}\, S(t) \to -\infty$ as t tends to ∞ in the sectors $|\arg t| < \pi/2n$ and $|\arg t - \pi| < \pi/2n$, which contain the real axis. Moreover, on each straight line $\mathrm{Im}\, t = c$ (c is a constant) we have $\mathrm{Re}\, S(t) \sim -\frac{1}{2n} (\mathrm{Re}\, t)^{2n}$ ($\mathrm{Re}\, t \to \pm\infty$). Hence the integral of the type (45.23) taken along the straight line $\mathrm{Im}\, t = c$ is absolutely convergent. It is easy to show that

$$\Phi(\lambda) = \int_{\mathrm{Im}\, t = c} e^{\lambda S(t)}\, dt$$

for any value of c. Of course, there are other contours γ equivalent to the real axis besides the straight lines $\mathrm{Im}\, t = c$; for instance, for γ we can take any simple infinite curve with the rays $\arg t = \alpha$, $|\alpha| < \pi/2n$, and $\arg(-t) = \beta$, $|\beta| < \pi/2n$, as asymptotes. But the saddle contour is among the straight lines parallel to the real axis.

We select the straight line $\mathrm{Im}\, t = \mathrm{Im}\, t_0$ that passes through the saddle point $t_0 = e^{i\pi/2(2n-1)}$ as the integration path n $\Phi(\lambda)$. There is also another saddle point that lies on this line, namely, $t = -\bar{t}_0$. We wish to show that $\max \mathrm{Re}\, S(t)$ is attained on the straight line l: $\mathrm{Im}\, t = \mathrm{Im}\, t_0$ only at the saddle points t_0 and $-\bar{t}_0$. We have $t = \xi + i\eta_0$ and $\eta_0 = \mathrm{Im}\, t_0$ on l, so that $\mathrm{Re}(it) = -\eta_0 = \mathrm{const}$. The points at which $\mathrm{Re}\,[(\xi + i\eta_0)^{2n}]$ is extremal can be determined from the equation

$$0 = \frac{d}{d\xi} \mathrm{Re}\,[(\xi + i\eta_0)^{2n}] = 2n\, \mathrm{Re}\,(\xi + i\eta_0)^{2n-1},$$

whence

$$(\xi + i\eta_0)^{2n-1} = iy \tag{45.27}$$

at a point of extremum, with y a real number. Suppose $y > 0$. Then

$$\xi + i\eta_0 = y^{1/(2n-1)} e^{i\varphi_k}, \quad e^{i\varphi_k} = \sqrt[2n-1]{i}.$$

This implies that at points of extremum we have

$$\xi = \eta_0 \cot \varphi_k, \quad \xi + i\eta_0 = \frac{\eta_0}{\sin \varphi_k} e^{i\varphi_k}$$

$$-\mathrm{Re}\,(\xi + i\eta_0)^{2n} = -\mathrm{Re}\left[\frac{\eta_0^{2n}}{(\sin \varphi_k)^{2n}} e^{i2n\varphi_k}\right] = \frac{\eta_0^{2n}}{(\sin \varphi_k)^{2n-1}},$$

since $e^{i(2n-1)\varphi_k}=i$. But since $\varphi_k=\frac{\pi/2+2k\pi}{2n-1}$, the $\max_k (\sin\varphi_k)^{-2n+1}$ is attained at $k=0$. This value of k corresponds to the point of extremum $\xi+i\eta_0=\frac{\eta_0}{\sin\varphi_0}e^{i\varphi_0}=t_0$.

If $y<0$, then we have

$$(-\xi+i\eta_0)^{2n-1}=-iy,$$

and the point $-\xi+i\eta_0$ lies on the straight line l. Just as in the case with $y>0$, we can prove that $\max_{t\in l} \operatorname{Re} S(t)$ is attained at the point $-\bar{t}_0$.

We have therefore established that $\max_{t\in l} \operatorname{Re} S(t)$ is attained only at the saddle points t_0 and $-\bar{t}_0$. Following the line of reasoning developed in Example 1, we can easily show that the asymptotic behavior of $\Phi(\lambda)$ is determined by the sum of the contributions from these two saddle points, in spite of the fact that the curve is infinite. From (45.25) and (45.26) we find that

$$S(t_0)=i\left(1-\frac{1}{2n}\right)e^{i\varphi_0},\quad S''(t_0)=-i(2n-1)e^{-i\varphi_0};$$
$$S(-\bar{t}_0)=\overline{S(t_0)},\quad S''(-\bar{t}_0)=\overline{S''(t_0)}; \tag{45.28}$$
$$\varphi_0=\frac{\pi}{2(2n-1)}.$$

Asymptotically the integral $\Phi(\lambda)$ is equal to a sum of expressions of the (45.7) type; what remains to be specified is the branches of $\sqrt{-1/S''(t)}$ in these formulas.

For small values of $|t-t_0|$ we have

$$S(t)-S(t_0)\sim\frac{1}{2}S''(t_0)(t-t_0)^2,$$

so that the equation of the path of steepest descent l_0, which passes through point t_0, has the form

$$t=t_0+\rho e^{i\psi_0}+O(\rho^2)\ (\rho\to 0),\quad \psi_0=-\frac{\pi}{4}+\frac{\varphi_0}{2},$$

which follows from (45.28). Hence, in (45.7) we must have

$$\sqrt{-\frac{1}{S''(t_0)}}=\frac{1}{|\sqrt{S''(t_0)}|}e^{i\psi_0},$$

so that the contribution from point t_0 to the integral $\Phi(\lambda)$ is

$$V(t_0)=e^{i\lambda S(t_0)}\sqrt{\frac{2\pi}{\lambda(2n-1)}}e^{i\psi_0}[1+O(\lambda^{-1})].$$

Similar considerations can be applied to the saddle point at $\bar{t}_0$, but a more simple approach is to employ the fact that $F(x)$ assumes real values for real x's. Indeed, the function $e^{-t^{2n}/2n}\sin tx$ is odd in t, and the integral of this function taken along the real axis is equal to zero. Hence,

$$F(x)=\int_{-\infty}^{\infty}e^{-t^{2n}/(2n)}\cos tx\,dt$$

for real x's.

The principal term in the asymptotic expansion is of the form $\Phi(\lambda)\approx V(t_0)+V(-\bar{t}_0)$, and, since the values of $\Phi(\lambda)$ must be real, we conclude that $V(-\bar{t}_0)\sim\overline{V(t_0)}$. Hence, the contribution from the saddle point $-\bar{t}_0$ to the asymptotic behavior of $\Phi(\lambda)$ is equal to

$$V(-\bar{t}_0)=\overline{V(t_0)}[1+O(\lambda^{-1})].$$

The final result is, as $x\to+\infty$.

$$F(x)=Ae^{-ax^{2n/(2n-1)}}x^{-(n-1)/(2n-1)}\times\left[\cos\left(bx^{2n/(2n-1)}-\frac{\pi}{4}+\frac{\varphi_0}{2}\right)+O(x^{-2n/(2n-1)})\right]. \tag{45.29}$$

Here

$$A=2\sqrt{\frac{2\pi}{2n-1}},\quad a=\left(1-\frac{1}{2n}\right)\sin\varphi_0,$$
$$b=\left(1-\frac{1}{2n}\right)\cos\varphi_0,\ \text{and}\ \varphi_0=\frac{\pi}{2(2n-1)}.$$

We have found that the integral (45.21) exponentially decreases as $x\to\pm\infty$ and has an infinitude of real zeros.

45.6 The saddle-point method and the method of stationary phase

In Sec. 44 we considered integrals of the type

$$F(\lambda)=\int_a^b f(x)e^{i\lambda S(x)}dx$$

taken along a finite segment $[a, b]$ on which $S(x)$ assumes only real values. We found the principal term in the asymptotic expansion for the case where $S(x)$ has only one stationary point x_0, $a<x_0<b$, $S''(x_0)\neq 0$.

Suppose that now the functions $f(x)$ and $S(x)$ are the values of two functions $f(z)$ and $S(z)$ regular in a neighborhood of the segment $[a, b]$. Then we can show that the following asymptotic expansion is valid:

$$F(\lambda) \sim e^{i\lambda S(b)} \sum_{n=0}^{\infty} b_n (i\lambda)^{-n-1}$$

$$- e^{i\lambda S(a)} \sum_{n=0}^{\infty} a_n (i\lambda)^{-n-1} + e^{i\lambda S(x_0)} \sum_{n=0}^{\infty} c_n \lambda^{-n-\frac{1}{2}} \quad (\lambda \to +\infty). \qquad (45.30)$$

Here the coefficients a_n and b_n are determined via (44.5). In other words, the asymptotic behavior of the integral $F(\lambda)$ as $\lambda \to +\infty$ is determined by the contributions from the saddle point and the end points a and b of the path of integration.

To prove this proposition, it is sufficient to deform the segment $[a, b]$, the path of integration, into a contour γ such that $\max_{z \in \gamma} \operatorname{Re}(iS(z))$ is attained only at the points $z = a$, $z = b$, and

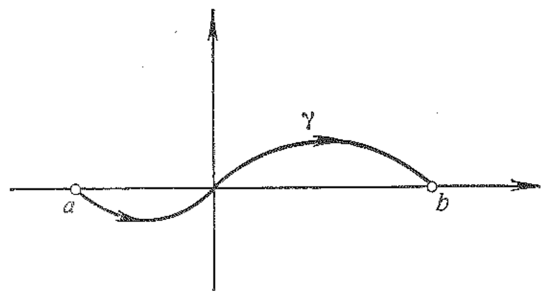

Fig. 163

$z = x_0$. The asymptotic behavior of the integral taken along γ is described by (45.30), in view of Corollary 2 and Theorems 1 and 2.

Let us restrict our discussion to the case of a quadratic phase, $S(x) = x^2$; the general case can be investigated similarly. We have $\operatorname{Re}(iz^2) < 0$ in the first and third quadrants. We replace the initial path of integration, $[a, b]$, by the contour γ depicted in Fig. 163. Then $\operatorname{Re}(iz^2) < 0$ everywhere on γ except the end points a and b and the saddle point $z = 0$. The contour satisfies the conditions of Corollary 2.

Thus, the method of stationary phase is a particular case of the saddle-point method if the integrand is a regular function.

Remark 2. Formula (45.30) is also valid if $f(x)$ and $S(x)$ are infinitely differentiable on the segment $[a, b]$ (see Fedoryuk [1]).

Example 4. Let us establish the asymptotic behavior of the Airy-Fock function (45.15) as $x \to -\infty$. Substituting t for $t\sqrt{|x|}$, we obtain

$$\operatorname{Ai}(x) = \frac{\sqrt{|x|}}{2\pi} F(\lambda), \quad F(\lambda) = \int_{-\infty}^{\infty} e^{\lambda S(t)}\, dt, \qquad (45.31)$$

where $\lambda = |x|^{3/2}$ and $S(t) = i(t^3/3 - t)$. The saddle points $t_{1,2} = \pm 1$ of $S(t)$ lie in the path of integration. Let us deform the integration contour in a way such that $\operatorname{Re} S(t)$ is attained only at the saddle points t_1 and t_2.

On the real axis in the complex t plane we have $\operatorname{Re} S(t) = 0$. Let us see where $\operatorname{Re} S(t)$ is negative. Putting $t = \xi + i\eta$, we find

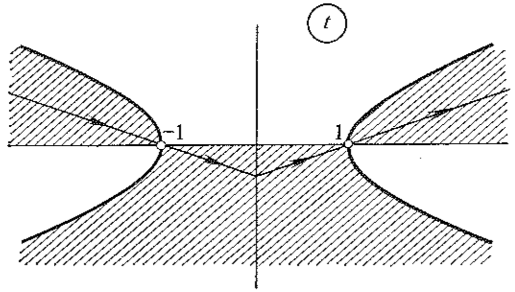

Fig. 164

that the equation $\operatorname{Re} S(t) = 0$ has the form $\eta(\xi^2 - \eta^2/3 - 1) = 0$. For this reason the curve representing $\operatorname{Re} S(t) = 0$ consists of the real axis ($\eta = 0$) and the hyperbola $\xi^2 - \eta^2/3 - 1 = 0$. This curve is depicted in Fig. 164 (the region in which $\operatorname{Re} S(t)$ is negative is hatched). We replace the path of integration by γ (Fig. 164). Since $t \sim |t| e^{i\pi/6}$ on γ as $\operatorname{Re} t \to +\infty$, we conclude that $\operatorname{Re} S(t) \sim -(1/3)|t|^3$ and the function $|e^{\lambda S(t)}|$ decreases exponentially as $\operatorname{Re} t \to +\infty$, $t \in \gamma$. The same is true when $\operatorname{Re} t \to -\infty$, $t \in \gamma$. It was found that the integral (45.31) is equal to the integral taken along γ:

$$F(\lambda) = \int_{\gamma} e^{\lambda S(t)}\, dt.$$

On γ we have $\operatorname{Re} S(t) < 0$ everywhere except at the saddle points t_1 and t_2, at which $\operatorname{Re} S(t) = 0$. In view of Corollary 2, the asymptotic behavior of $F(\lambda)$ is determined by the sum of the contributions from the saddle points t_1 and t_2. We have

$$S(t_{1,2}) = \mp \frac{2}{3} i, \quad S''(t_{1,2}) = \pm 2i,$$

and the final result is

$$\mathrm{Ai}\,(x) = \frac{1}{\sqrt{\pi}}\,|\,x\,|^{-1/4}\left[\cos\left(\frac{2}{3}\,|\,x\,|^{3/2} \cancel{+} - \frac{\pi}{4}\right) + O\,(\,|\,x\,|^{-3/2}\,)\right]$$
$$(x \to -\infty). \quad \square$$

46 Laplace's Method of Contour Integration

46.1 Laplace's contour transformation Let us start with a second-order linear homogeneous differential equation with linear coefficients

$$(a_0 z + a_1)\, w'' + (b_0 z + b_1)\, w' + (c_0 z + c_1)\, w = 0, \qquad (46.1)$$

where the a_j, b_j, and c_j are constants. We will look for the solution to this equation in the form

$$w\,(z) = \int\limits_C e^{\zeta z} v\,(\zeta)\, d\zeta, \qquad (46.2)$$

where $v\,(\zeta)$ is the unknown function, and C is an integration contour that is independent of z. Below we give a formal derivation of the solution. Differentiation under the integral sign yields

$$w'\,(z) = \int\limits_C e^{\zeta z}\,\zeta v\,(\zeta)\, d\zeta, \quad w''\,(z) = \int\limits_C e^{\zeta z}\zeta^2 v\,(\zeta)\, d\zeta.$$

If we now integrate by parts, we find that

$$zw\,(z) = \int\limits_C v\,(\zeta)\, de^{\zeta z} = v\,(\zeta)\, e^{\zeta z}\,\Big|_C - \int\limits_C e^{\zeta z} v'\,(\zeta)\, d\zeta,$$

where the first term on the right-hand side is taken at the end points of contour C. Similarly,

$$zw'\,(z) = \zeta v\,(\zeta)\, e^{\zeta z}\,\Big|_C - \int\limits_C e^{\zeta z}\,(\zeta v\,(\zeta))'\, d\zeta,$$

$$zw''\,(z) = \zeta^2 v\,(\zeta)\, e^{\zeta z}\,\Big|_C - \int\limits_C e^{\zeta z}\,(\zeta^2 v\,(\zeta))'\, d\zeta.$$

Suppose contour C and the function $v\,(\zeta)$ are selected in a way such that the sum of the terms without the integrals vanishes:

$$(a_0\zeta^2 + b_0\zeta + c_0)\, v\,(\zeta)\, e^{\zeta z}\,|_C = 0. \qquad (46.3)$$

Then Eq. (46.1) takes the form

$$\int\limits_C e^{\zeta z}\,[(a_1\zeta^2 + b_1\zeta + c_1)\, v\,(\zeta) - a_0\,(\zeta^2 v\,(\zeta))' - b_0\,(\zeta v\,(\zeta))' - c_0 v'\,(\zeta)]\, d\zeta = 0.$$