CFT : 40+ !

&

Andrea's Fest !

Andrea : The Best Fisherman !!!

... since many years. Thank you !

Two good Italian friends !

Playing with random matrices and free probability in noisy many-body systems

(A small — possibly entertaining — exercice)

Denis Bernard (CNRS & LPENS, Paris)
(following work done with Ludwig Hruza)

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LPENS
LABORATOIRE DE PHYSIQUE DE L'ÉCOLE NORMALE SUPÉRIEURE
ENS
ÉCOLE NORMALE SUPÉRIEURE
PSL
RESEARCH UNIVERSITY PARIS
Agence Nationale de la Recherche
ANR

(Standard) random matrices :

with impact on modeling many physical phenomenons (say random surfaces, chaotic systems, etc)

— Basic examples :

« Wigner's matrices » | $M = (M_{ij})$ hermitian, NxN
with Gaussian entries $\mathbb{E}[M_{ij}M_{ji}] = N^{-1}$

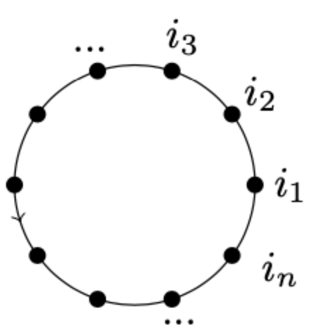
« Haar orbits » | $M = UDU^\dagger$ with U unitary Haar distributed, D diagonal
with HCIZ integral as cumulant generating function $\mathbb{E}[e^{tr(AM)}] = \int dU\, e^{tr(AUDU^\dagger)}$

— Properties : | - U(1) invariance
- Unstructured matrices (invariance under permutation, in law)

—> Expectation values are « topological »

$$\mathbb{E}[M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1}] = N^{1-n}\,\kappa_n + \cdots$$

(if indices are distincts)



—> Look at structured matrices ?...

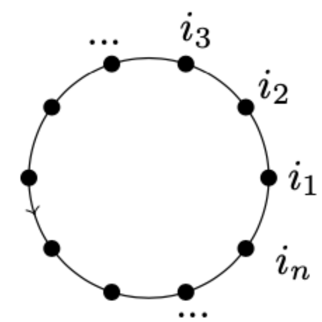
Ensemble of « structured » random matrices :

—> Originates from noisy many-body systems (more later…)

— Ensemble of large NxN matrices M with measure E
with three defining properties :

- U(1) invariance : $M_{jk} \equiv_{\text{in law}} e^{i\theta_j} M_{jk} e^{-i\theta_k}$

- Scaling of the loop expectation values :



$$\mathbb{E}[M_{i_1 i_n} \cdots M_{i_3 i_2} M_{i_2 i_1}] \sim N^{1-n}$$

- Factorisation of product of loops:

$$\mathbb{E}[M_{j_1 j_n} \cdots M_{j_2 j_1} \cdot M_{i_1 i_p} \cdots M_{i_2 i_1}] = \mathbb{E}[M_{j_1 j_n} \cdots M_{j_2 j_1}]\, \mathbb{E}[M_{i_1 i_p} \cdots M_{i_2 i_1}]$$

— The matrix structure is coded into the n-point (loop) cumulants $(x_k = i_k/N, \text{distincts})$

$$g_n(x_1, x_2, \cdots, x_n) := \lim_{N\to\infty} N^{n-1}\, \mathbb{E}[M_{i_1 i_2} M_{i_2, i_3} \cdots M_{i_n i_1}]^c$$

— Importance (or dominant role) of the loop expectation values !

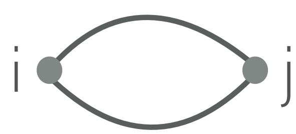
Cumulants and non-crossing partitions (I) :

Let us compute moment via the cumulant expansion

Recall the moment-cumulant formula : $\mathbb{E}[X_1 \cdots X_n] = \sum_{\pi \in P(n)} \mathbb{E}_\pi[X_1 \cdots X_n]^c$

— N=1 : $\mathbb{E}[M_{ii}] = g_1(x)$

— N=2 : $\mathbb{E}[M_{ij}M_{ji}] = \mathbb{E}[M_{ij}M_{ji}]^c + \mathbb{E}[M_{ij}]^c\mathbb{E}[M_{ji}]^c = N^{-1}\,[g_2(x,y) + \delta(x-y)g_1(x)g_1(y)]$

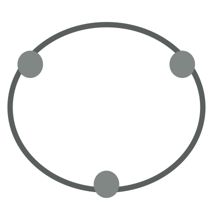


$\delta_{ij}\mathbb{E}[M_{ii}] \qquad \delta_{ij}\mathbb{E}[M_{jj}] \qquad (with\ \delta_{ij} \rightsquigarrow N^{-1}\delta(x-y)\ )$

— N=3 :

$\delta_{ij} \qquad \delta_{ij}N^{-1} \qquad \delta_{ij} \qquad \delta_{jk} \qquad \delta_{ki}$

$$\mathbb{E}[M_{ij}M_{jk}M_{ki}] = \mathbb{E}[M_{ij}M_{jm}M_{ki}]^c + \mathbb{E}[M_{ij}]^c\mathbb{E}[M_{jk}M_{ki}]^c + \circlearrowleft + \mathbb{E}[M_{ij}]^c\mathbb{E}[M_{jk}]^c\mathbb{E}[M_{ki}]^c$$
$$= N^{-2}\,[g_3(x_1,x_2) + \delta(x_1-x_2)g_1(x_1)g_2(x_2,x_3) + \circlearrowleft$$
$$+\,\delta(x_1-x_2)\delta(x_2-x_3)g_1(x_1)g_1(x_2)g_1(x_3)]$$



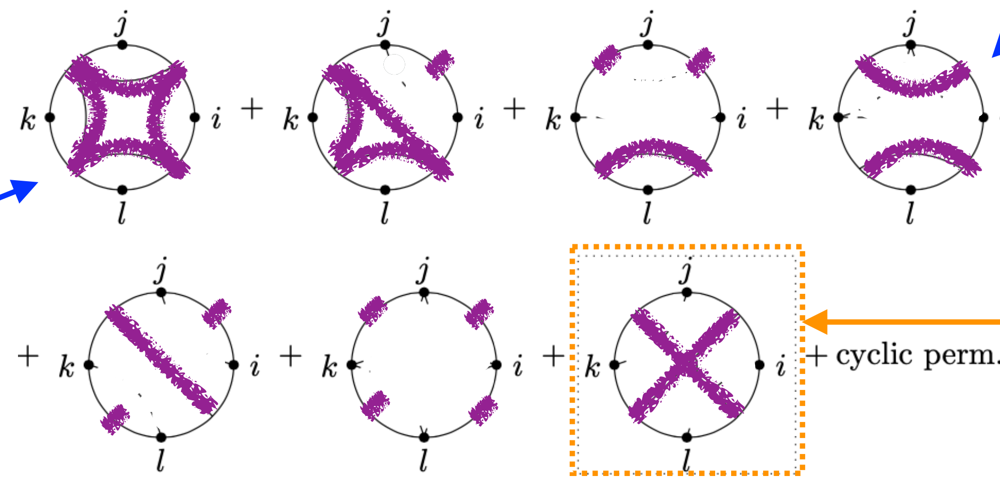
Cumulants and non-crossing partitions (II) :

— N=4 : the role of non-crossing partitions...

$\mathbb{E}[M_{ij}M_{jk}]^c\mathbb{E}[M_{kl}M_{li}]^c$

$\delta_{ik}N^{-2}$

$\mathbb{E}[M_{ij}M_{jk}M_{kl}M_{li}] =$



cumulants
(partition of edges)

$\mathbb{E}[M_{ij}M_{kl}]^c\mathbb{E}[M_{jk}M_{li}]^c$

$\delta_{ij}\delta_{jk}\delta_{kl}N^{-2}$

negligeable !

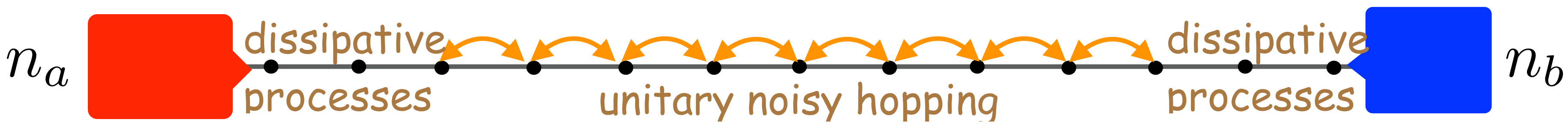
$$= N^{-3}\,[g_4(x_1,x_2,x_3,x_4) + \delta(x_1,x_2)g_1(x_1)g_3(x_2,x_3,x_4) + \circlearrowleft$$
$$+\ \delta(x_1,x_3)g_2(x_1,x_2)g_2(x_3,x_4) + \circlearrowleft$$
$$+ \cdots \text{"only non} - \text{crossing partitions"}]$$

— Only the **non-crossing partitions** contribute.
Generalisation to higher order expectations

$\Longrightarrow$ **Free probability !**

Origin in /Relation with Q-SSEP :

— Q-SSEP = Quantum Symmetric Simple Exclusion Process :



$$dH_t = \sqrt{D}\, \sum_j \left(c_{j+1}^\dagger c_j \, dW_t^j + c_j^\dagger c_{j+1} \, d\overline{W_t^j} \right)$$

+ boundary terms…
+ injection/extraction…

— Stochastic many-body quantum system (quadratic but noisy) :

—> « Coherences » (alias off-diagonal two-point functions)

$$G_{ij} = \langle c_j^\dagger c_i \rangle_t = \mathrm{Tr}(\rho_t \, c_j^\dagger c_i) \quad \Longrightarrow$$

random quantum states

Large « structured » random matrices

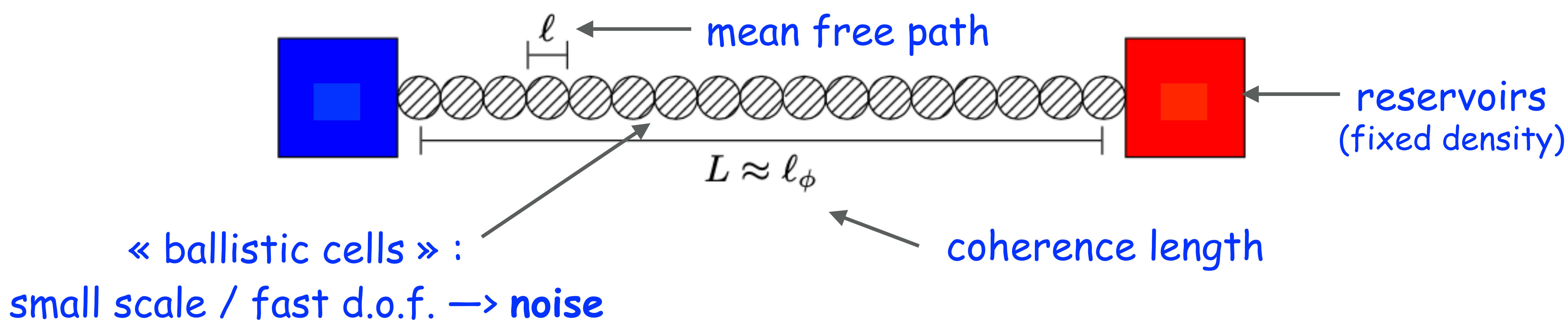
— Claim : At any time (within the measure E induced by the above SDE, and hence in particular in the infinite time steady measure :

The matrix of coherences G belongs to the above 'stuctured' RMT class.

⟶ Many (physical) consequences : (e.g. long range correlation, volume law for entanglement mutual information, …

Application to /Relation with mesoscopic physics :

— « Mesoscopic » (diffusive + coherent) systems out-of-equilibrium



- Transport and its fluctuations (cf. classical Macroscopic Fluctuation Theory).
- Fluctuations of quantum coherent effects at mesoscopic scales & out-of-equilibrium.

— Conjecture :

If the fast dynamics on the small scale degrees of freedom is ergodic,
If the mean dynamics follows the classical 'macroscopic fluctuation theorem',
Then
The matrix of two point functions belongs to the above 'structured' RMT class.

Thank you !

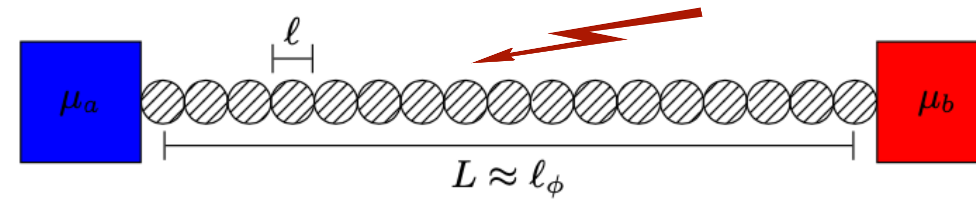
Bon Anniversaire

Andrea !

Emergence of free probability in noisy systems

— **Coarse-grained description** (at mesoscopic scale)

« ballistic cells »



(i) separation of time scales :

fast, closed dynamics within ballistic cells for $t < t_\ell$

—> unitary dynamics within each cells for $t < t_\ell$

(ii) ergodicity of the fast dynamics (—> noise) :

(local) Haar measure

$$\mathbb{E}_t[G_{jk}] := \frac{1}{t_\ell}\int_t^{t+t_\ell} dt'\, G_{jk}(t') = \mathrm{Tr}(\rho_t\, [c_i^\dagger c_j]_U)$$

— **Validity of U(1) sym. + MFT in mean —> the « universality three conditions »**

- U(1) invariance :

  $G_{jk} \equiv_{\text{in law}} e^{i\theta_j} G_{jk} e^{-i\theta_k}$

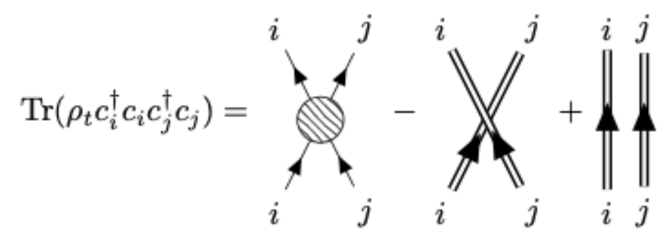
  <— local conservation and closed unitary dynamics at short time

- Scaling of the loop expectation values :

  $\mathbb{E}[G_{j_1 j_n} \cdots G_{j_3 j_2} G_{j_2 j_1}] \sim N^{1-n}$

  If some perturbation theory is valid $(H = H_0 + V)$

  <— If mean densities satisfy MFT



- Factorisation of loop expectations :

  <— closed fast dynamics / cells independence

— Validation / Violation of these three conditions ? …