

THE CELL REYNOLDS NUMBER MYTH

H. DOYLE THOMPSON*, BRENT W. WEBB† AND JOE D. HOFFMAN*

School of Mechanical Engineering, Purdue University, West Lafayette, Indiana 47907, U.S.A.

SUMMARY

The commonly accepted linear stability analysis for the forward-time centred-space (FTCS) algorithm applied to the transport equation has led to the concept of a cell Reynolds number restriction on the spatial grid size. This paper shows where the commonly accepted original analysis is in error and presents the correct stability restrictions, which are restrictions on the time step, not the spatial grid size. There is no cell Reynolds number restriction. The results are confirmed by numerical computations for the two-dimensional driven cavity problem.

INTRODUCTION

The study of the numerical stability of finite difference algorithms usually starts with the analysis of the forward-time centred-space (FTCS) algorithm applied to a model equation such as the one-dimensional transport equation

$$f_t = -uf_x + \alpha f_{xx} \quad (1)$$

Such a didactic study provides insights into some of the sources of numerical instability, illustrates the relationship between the convective term uf_x and the diffusion term αf_{xx} , and can be used to demonstrate such concepts as artificial viscosity and numerical diffusion. Furthermore, in spite of its limitations, the FTCS algorithm is still used for many practical calculations. Therefore, owing to its educational, historical and practical value, it is important that the stability analysis for the FTCS algorithm be correct and accurate.

Unfortunately, there remains in the literature and among practitioners the idea that there is a cell Reynolds number restriction that must be satisfied for the FTCS algorithm to be numerically stable. That myth is almost as hard to dispel as the flat earth theory in Columbus's time or the idea that the sun revolved around the earth in the days of Galileo.

The concept of a cell Reynolds number restriction evolved from the 1964 work of Fromm.¹ He used the now classical von Neumann stability analysis technique² to determine necessary stability limits on the two-dimensional vorticity transport equation, and mistakenly arrived at the conclusion that

$$0 \leq C \leq 2d \leq 1 \quad (2)$$

where C is the Courant number and d is the diffusion number, defined by

$$C = u\Delta t/\Delta x \quad \text{and} \quad d = \alpha\Delta t/\Delta x^2 \quad (3)$$

*Professor

†Research Assistant

Equation (2) corresponds to the well-known criteria

$$\Delta t \leq \Delta x^2 / 2\alpha \quad (4)$$

and

$$R_c \equiv u\Delta x / \alpha \leq 2 \quad (5)$$

where R_c is the so called cell Reynolds number. Equations (2)–(5) are actually the one-dimensional equivalent of Fromm's two-dimensional result. Their derivation contains an important error, but in the two-dimensional setting there is also a second (distinct) error in Fromm's analysis. That second error is corrected by the recent work of Hindmarsh, Gresho and Griffiths³ who appear to be the first to have derived the correct stability limits for the two- and three-dimensional cases. To illustrate the error that led to the cell Reynolds number concept, it is sufficient to consider only the one-dimensional case.

Equation (4) is a correct result but, as is shown below, equation (5) is both overly restrictive and very misleading since it requires a spatial grid size limitation for stability where in reality none exists. As early as 1968 Hirt⁴ arrived at the correct criteria for the one-dimensional case using an entirely different approach to the stability analysis. That result received little notice, however, and the cell Reynolds number myth has propagated and amplified throughout the numerical analysis community. Examples in the literature include the works of Marshall,⁵ Noye,⁶ Olson and Tuann⁷ and Torrance *et al.*⁸ The result also appears in educational texts and reference books, e.g. those of Roache,⁹ Mitchell and Griffiths¹⁰ and Lapidus and Pinder.¹¹

Many practitioners know that the cell Reynolds number limit can be violated (excessively) and still obtain stable numerical results, and many have suspected that something is not quite right. Roache⁹ even went so far as to show that the result obtained by considering the geometric aspects of the stability ellipse obtained from the von Neumann analysis gives a result that is not consistent with the cell Reynolds number restriction, but he did not pursue the discrepancy.

In addition, the misconception has led to some unusual schemes that misleadingly appear to have merit. For example Chien¹² developed an elaborate scaling procedure for the FTCS algorithm which he claimed had the main advantage of eliminating the stability limit on the grid size. Careful analysis shows, however, that Chien's scaling does not change the stability limits of the algorithm, and that the apparent improvement, shown by numerical experiment, is simply a manifestation of the fact that the assumed cell Reynolds number limit never really existed.

Rigal¹³ clearly states that there is not a cell Reynolds number stability limit, but unfortunately his two-dimensional limit, although necessary, is not sufficient for stability. Leonard¹⁴ appears to have been the first to publish the correct one-dimensional result. Clancy¹⁵ independently obtained the correct one-dimensional result. More recently, Hindmarsh, Gresho and Griffiths³ have more thoroughly analysed the stability of the general multidimensional case. Whereas Leonard,¹⁴ Clancy¹⁵ and Hindmarsh *et al.*³ clearly state that Fromm's original work is in error, they each resort to other means (e.g. the stability ellipse approach discussed by Roache⁹) to obtain the correct limits. It is believed that the following derivation is the first which shows where Fromm went wrong and brings all the analysis methods that depend on the von Neumann approach into agreement.

CORRECTING FROMM'S ANALYSIS

The von Neumann stability analysis is based on the representation of numerical errors at a specific time level by an infinite Fourier series, and the examination of the relative magnitudes of the various Fourier components of the errors at successive time steps. The complex amplification factor, G , for the FTCS algorithm applied to equation (1) is

$$G = 1 + 2d(\cos \theta - 1) - iC \sin \theta \quad (6)$$

where $i = \sqrt{-1}$ and θ is a parameter that can take on any value between $\pm \infty$. The von Neumann analysis requires that for numerical stability the magnitude of G be no greater than unity for any value of θ . Multiplying G by its complex conjugate \bar{G} , the stability requirement can be expressed as

$$G\bar{G} = [1 + 2d(\cos \theta - 1)]^2 + C^2(1 - \cos^2 \theta) \leq 1 \quad (7)$$

The stability criteria (i.e. restrictions on the permissible values of d and C) result from examination of equation (7) for all values of θ . This can be done by several methods, but each must produce the same result. Fromm¹ chose to infer the stability criteria from examination of the critical points (maxima and minima) of the function $G\bar{G}$. In that examination he also chose $\cos \theta$ as the independent variable. Thus, substituting x for $\cos \theta$, the problem becomes one of finding the values of C and d for which

$$G\bar{G} = [1 + 2d(x - 1)]^2 + C^2(1 - x^2) \leq 1 \quad (8)$$

in the interval $-1 \leq x \leq 1$.

At $x = 1$, $G\bar{G} = 1$, and no limitation is present. At $x = -1$, $G\bar{G} = (1 - 4d)^2$. Thus, one obvious limitation is $0 \leq d \leq 1/2$ which leads to the time step limitation given by equation (4). A second limitation can be deduced from the behaviour of the function $G\bar{G}$. As illustrated in Figure 1, there are three different types of behaviour for the function $G\bar{G}$ in the range $-1 \leq x \leq 1$. They are that $G\bar{G}$ exhibits a maximum in the interval (curve (a)), a minimum in the interval (curve (b)), or neither a maximum nor minimum in the interval (curve (c)). Case 1 (curve (a)) represents unstable solutions, since the maximum value would be greater than either end point, and thus exceed unity. Case 2 (curve (b)) represents a set of stable solutions provided that $0 \leq d \leq 1/2$ and case 3 (curve (c)) represents another set of stable solutions for $0 \leq d \leq 1/2$.

Using well-known methods to solve for the maximum of the function $G\bar{G}$, the results in Table I are obtained.

The result for case 1 was obtained by requiring that $G\bar{G}$ have a maximum in the interval $-1 \leq x \leq 1$. Case 2 represents situations for which a minimum exists. Case 3 represents situations

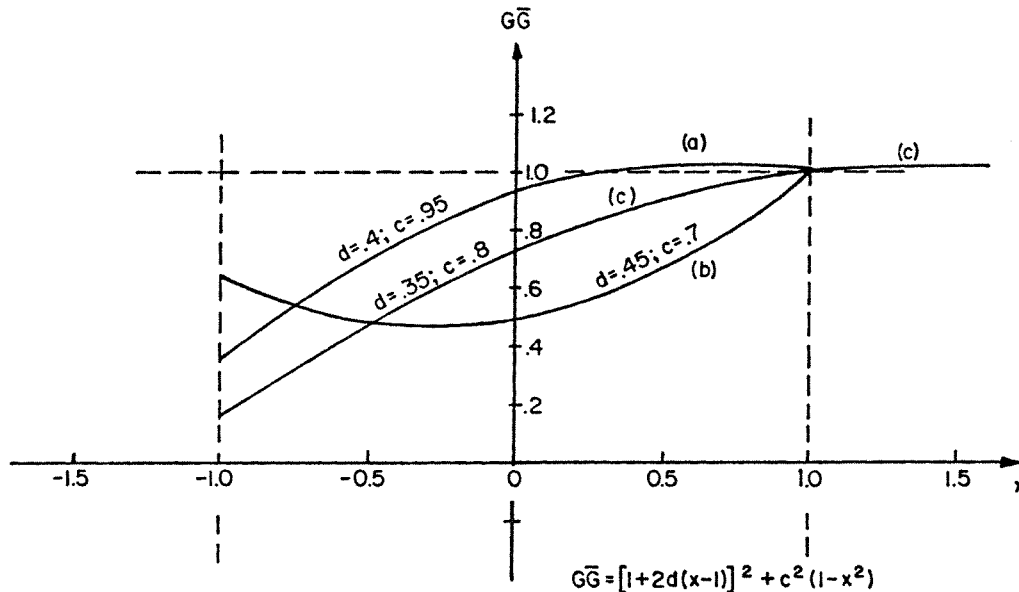


Figure 1. $G\bar{G}$ as a function of x for various combinations of C and d

Table I

Case	Curve	Criteria	Comments
1	(a)	$C^2 > 2d$	unstable
2	(b)	$0 \leq C \leq 2d \leq 1$	stable
3	(c)	$0 \leq C^2 \leq 2d \leq 1$	stable

for which a maximum occurs at $x > 1$. Since there are no values of θ for which $x = \cos \theta > 1$, the values of $G\bar{G}$ at $x > 1$ are of no interest. It is an unreal set of solutions that was introduced by changing the independent variable from θ to $\cos \theta$.

Fromm's error was failing to recognize the set of stable solutions represented by case 3. He thereby obtained overly restrictive and misleading conditions given by equation (2).

The correct condition for stability is the union of stable solutions represented by cases 2 and 3, namely

$$0 \leq C^2 \leq 2d \leq 1 \quad (9)$$

which reduces to equation (4) plus the restriction $R_c C \leq 2$ or

$$\Delta t \leq 2\alpha/u^2 \quad (10)$$

Equation (10) is a restriction on the time step and is independent of the time step restriction given by equation (4). Notice that neither equation (4) nor equation (10) contains a restriction on the spatial grid size.

Equations (4) and (5) are sufficient for stability, since they are more conservative than the sufficient and necessary conditions, equations (4) and (10). However, the adoption of equations (4) and (5) as necessary conditions leads to incorrect conclusions such as the cell Reynolds number criterion and the commonly held belief that it is not possible to compute stable solutions at high Reynolds numbers using the FTCS algorithm without a small spatial grid.

The mesh refinements necessitated by equation (5) are definitely more costly in terms of computer storage and computation time than the reduction in time step prescribed by equation (10). Those who have stabilized the algorithm at higher Reynolds number by mesh refinements have done so because the smaller grid size also requires a smaller time step due to the limitation in equation (4). The conclusive stabilizing factor is the reduction in time step, not the refinement of the spatial mesh.

It should be noted that this conclusion deals only with the stability aspects and not the accuracy of the solution. Spatial grid refinements at higher Reynolds number will undoubtedly produce a more accurate solution.

TWO-DIMENSIONAL CRITERIA

The general two-dimensional transport equation, analogous to equation (1), is

$$f_t = -uf_x - vf_y + \alpha_1 f_{xx} + \alpha_2 f_{yy} \quad (11)$$

The stability criterion from the von Neumann analysis (comparable to equation (7)) is

$$G\bar{G} = [1 + 2d_1(\cos \theta_1 - 1) + 2d_2(\cos \theta_2 - 1)]^2 + [C_1 \sin \theta_1 + C_2 \sin \theta_2]^2 \leq 1 \quad (12)$$

for all values of θ_1 and θ_2 , where

$$C_1 = \frac{u\Delta t}{\Delta x}; \quad C_2 = \frac{v\Delta t}{\Delta y}; \quad d_1 = \frac{\alpha_1 \Delta t}{\Delta x^2}; \quad d_2 = \frac{\alpha_2 \Delta t}{\Delta y^2} \quad (13)$$

Fromm¹ simplified his original analysis by assuming that $\theta_1 = \theta_2$. He did note, however, that the criteria so obtained can only be less stringent than the true criteria, that is, that simplification is under-restrictive and leads to a necessary condition that is not sufficient. That logical error has been masked heretofore by the conservative cell Reynolds number myth.

For the special case of $\theta_1 = \theta_2$, equation (12) can be written

$$G\bar{G} = [1 + 2(d_1 + d_2)(\cos \theta - 1)]^2 + (C_1 + C_2)^2(1 - \cos^2 \theta) \leq 1 \quad (14)$$

for all values of θ . By analogy to equation (7), the resulting necessary stability criteria are

$$0 \leq (C_1 + C_2)^2 \leq 2(d_1 + d_2) \leq 1 \quad (15)$$

Hindmarsh, Gresho and Griffiths³ have shown that the special condition $\theta_1 = \theta_2$ leads to a non-sufficient restriction (i.e. equation (15) is necessary but not sufficient), and have determined the correct necessary and sufficient conditions by treating θ_1 and θ_2 as independent variables. Their result is generalized for n -dimensions. The correct result for the two-dimensional case is

$$0 \leq 2(d_1 + d_2) \leq 1 \quad (16)$$

and

$$0 \leq C_1^2/2d_1 + C_2^2/2d_2 \leq 1 \quad (17)$$

In terms of a time-step limitation, equations (16) and (17) give

$$\Delta t \leq 1 / \left[\frac{2\alpha_1}{\Delta x^2} + \frac{2\alpha_2}{\Delta y^2} \right] \quad (18)$$

and

$$\Delta t \leq 1 / \left[\frac{u^2}{2\alpha_1} + \frac{v^2}{2\alpha_2} \right] \quad (19)$$

There is no limitation on the spatial grid.

The stability criteria given by equations (18) and (19) were tested using numerical experimentation for the FTCS algorithm applied to the two-dimensional driven cavity problem using the stream-function–vorticity approach. Because of the non-linearity of the problem (u and v are not constant) the demarcation between stable and unstable calculations is not as sharp as for linear problems. None the less, the result was clear in that stable solutions (not necessarily accurate solutions) are always obtained when equations (18) and (19) are locally satisfied. Furthermore, unstable results are obtained when the criteria of equation (15) are satisfied but equation (17) is violated.

Finally, the numerical results show that the local cell Reynolds number limitation given by Fromm can be violated by nearly two orders of magnitude while obtaining well-behaved, stable, reasonable solutions. Spatial grid refinements were required for enhanced accuracy, but offered no advantage with respect to the stability of the algorithm.

CONCLUSIONS

The necessary and sufficient conditions for stability of the FTCS algorithm have finally been corrected after 20 years. The correct criteria are limitations on the admissible time step and do not contain a space grid limitation. The idea of a cell Reynolds number restriction is a myth.

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