① Distributivities

$$\begin{tikzcd}
T^b B \arrow[r, bend left, "T^b B\eta"] & T^b B T \arrow[l, bend left, "\chi'"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & BT \arrow[d, "\chi"] \\
B \arrow[ur, "B\eta"] \arrow[r, "\eta B"'] & TB
\end{tikzcd}
\qquad T^b$$

$\Leftrightarrow$

$$\begin{tikzcd}
TTB \arrow[r, "T\lambda"] \arrow[d, "T^{\#}B \,=\, \mu B"'] & TBT \arrow[d, "\lambda T"] \\
 & BTT \arrow[d, "B_\mu"] \\
TB \arrow[r, "\lambda"'] & BT
\end{tikzcd}
\quad \left.\right\} \bar{B}^T T^{\#}$$

$\Leftrightarrow$

$$\begin{tikzcd}[column sep=large]
 & & \mathbb{C} \arrow[dll] \arrow[d, "B"] \arrow[drr, "T^{\#}"] & & \\
 & & \mathbb{C} \arrow[dll, bend right, "T^b B\eta"'] \arrow[ddll, "T^b"'] \arrow[ddrr, "T^{\#}"] \arrow[r, phantom, "\xrightarrow{\lambda}"] & & \\
\mathbb{C}_T \arrow[urr, bend right, "\chi'"'] \arrow[d, "\bar{B}_T \,=\, T^b B T_b"'] \arrow[ddrr, "T_b"] & & & & \mathbb{C}^T \arrow[d, "\bar{B}^T"] \arrow[ddll, "T_{\#}"'] \\
\mathbb{C}_T \arrow[ddrr, "T_b"'] & & & & \mathbb{C}^T \arrow[ddll, "T_{\#}"] \\
 & & \mathbb{C} \arrow[d, "B"] \arrow[r, phantom, "\cong"] & & \\
 & & \mathbb{C} & &
\end{tikzcd}$$

$\eta B T_b$, $\quad BT_b\varepsilon' \circ \lambda T_b$

$$TX \xrightarrow{\xi} X \quad \longmapsto \quad TBX \xrightarrow{\lambda} BTX \xrightarrow{B\xi} BX$$

$\Leftrightarrow$

$$\begin{tikzcd}
BT_b \arrow[r, "\eta B T_b"] \arrow[dr, bend right, "B\eta T_b"'] & TBT_b \arrow[d, "\lambda T_b"] \\
 & BT_b T^b T_b \arrow[ul, "BT_b\varepsilon"]
\end{tikzcd}$$

$\Leftrightarrow$

$$\begin{tikzcd}
TTB \arrow[r, "\mu B"] \arrow[d, "T\lambda"'] & TB \arrow[dd, "\lambda"] \\
TBT \arrow[d, "\lambda T"'] & \\
BTT \arrow[r, "B_\mu"'] & BT
\end{tikzcd}$$

$$\begin{tikzcd}
 & TB \arrow[d, "\lambda"] \\
B \arrow[ur, "\eta"] \arrow[r, "B\eta"'] & BT
\end{tikzcd}$$

② Lifting

$$\begin{tikzcd} \mathbb{C}^T \arrow[r, "\overline{B}"] & \mathbb{C}^T \\ \mathbb{C} \arrow[r, "B"'] \arrow[u, squiggly, "\xi"] & \mathbb{C} \end{tikzcd} \quad \overset{\gamma}{\rightsquigarrow} \quad \begin{tikzcd} \mathrm{Alg}(\overline{B}) \arrow[d, bend left, "U"] \\ \mathrm{Alg}(B) \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

$$FU\alpha \xrightarrow{\varepsilon_\alpha} \alpha$$

$$\begin{tikzcd}
TBTX \arrow[rr, bend left, "T\gamma"] \arrow[dr, "TB\xi"] \arrow[dd, "\lambda T"'] & & TTBX \arrow[ll, bend left, "T\lambda"] \arrow[rr, "TT\alpha"] \arrow[dddd, dotted, "\mu"] & & TTX \arrow[dl, "T\eta"'] \arrow[dddd, "\mu"] \\
& TBX \arrow[rr, "T\alpha"'] \arrow[d, "\lambda"] \arrow[dd, bend left, "\overline{B}^T\xi"] & & TX \arrow[dd, "\eta"] & \\
BTTX \arrow[r, "B\mu"] \arrow[dd, "B\mu"'] & BTX \arrow[d, "B\xi"] \arrow[rr, phantom, "\alpha \in \mathbb{C}^T(\overline{B}\xi{,}\eta) \Rightarrow"] & & & \\
& BX \arrow[rr, "\alpha", "T_\#\alpha \in \mathbb{C}(T^\#\xi{,}T^\#\eta)"'] & & X & \\
BTX \arrow[ur, "B\xi"] \arrow[rr, bend left, "\gamma"] & & TBX \arrow[ll, bend left, "\lambda"] \arrow[rr, "T\alpha"'] & & TX \arrow[ul, "\eta"']
\end{tikzcd}$$

CONDUCTION UP-TO & ABSTRACT INTERPRETATION

- Notation

$$\begin{tikzcd} C \arrow[loop left, "b"] \arrow[r, bend left, "\alpha"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

C, A complete lattices

$b : C \to C$ monotone map

$f = \gamma \circ \alpha : C \to C$ closure operator.

- Main goals

A) Abstract Interpretation
+ Fix-point completeness

$$\alpha(\mu b) = \mu(\alpha b \gamma)$$

B) Up-to techniques
+ f is b-sound

$$\nu(bf) \subseteq \nu(b)$$

(the other inclusion is trivial)

② Sufficient Conditions

Abstract Interpretation

Backward-Completeness

$fb = fbf$

$\Updownarrow$

Kleisli-Law

$bf \leq fb$

$\Updownarrow$

Kleisli-Lifting

$$\begin{tikzcd} A \arrow[r, "\alpha b \gamma"] & A \\ C \arrow[u, "\alpha"] \arrow[r, "b"'] & C \arrow[u, "\alpha"'] \end{tikzcd}$$

Coinduction Up-to

Forward-Completeness

$bf = fbf$

$\Updownarrow$

E-M-Law

$fb \leq bf$

f is b-compatible

$\Updownarrow$

E-M-Lifting

$$\begin{tikzcd} A \arrow[r, "b'"] \arrow[d, "\gamma"'] & A \arrow[d, "\gamma"] \\ C \arrow[r, "b"'] & C \end{tikzcd}$$

We give the def of b' below..

③ Theorems

+ Cousot POPL 1977

Kleisli-Lifting $\Longrightarrow$ Fix-point Completeness

PROOF

We assume b to be Scott Continuous

$$b(\bigsqcup_n d_n) = \bigsqcup_n b\, d_n \quad \text{for all directed } d_n$$

So by the Kleene Theorem, we have

$$\mu b = \bigsqcup_n b^n(\bot_C)$$

We have that $\alpha(\bigsqcup_n b^n(\bot)) = \bigsqcup_n \alpha b^n(\bot)$

since α is left-adjoint.

$$\mu(\alpha b \gamma) = \bigsqcup_n (\alpha b \gamma)^n(\bot_A)$$

by induction on n : $\alpha b^n(\bot_C) = (\alpha b \gamma)^n(\bot_A)$

$n=0$ $\quad \alpha(\bot_C) = \bot_A$ ✓

$n+1$ $\quad \alpha b \gamma (\alpha b \gamma)^n(\bot_A) \overset{IH}{=} \alpha b \gamma \alpha b^n(\bot_C) =$

$\overset{\text{Kleisli Lifting}}{=} \alpha b b^n(\bot_C) = \alpha b^{n+1}(\bot_C)$ ✓

+ Sangiorgi Pous 2013

f is b-compatible $\Longrightarrow$ f is b-sound

PROOF $\quad x \le bfx \Rightarrow fx \le fbfx \le bffx \le bfx$

$\Longrightarrow$ by Knaster Tarski $\quad x \le fx \le \nu b$.

The statement follows for $x = \nu bf$ ✓

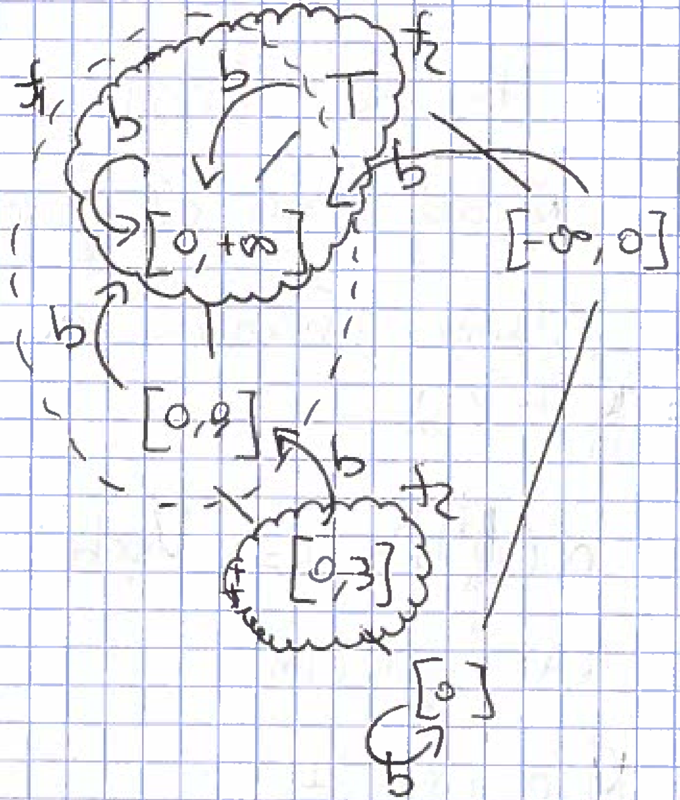
④ Journal of ACM '2000

$$fb = fbf \iff b^{-1}f = fb^{-1}f$$

Under the assumption that b^{-1} is the right adjoint of b.

<u>Example</u>

$b = \lambda x.\, x^2$ $\longrightarrow$



f_1 is Forward Complete but not Backward Complete

f_2 is Backward Complete but not ~~Backward~~ Forward Complete

(5) Categorical Question for Dusko.

$$\begin{tikzcd} C \arrow[loop left, "B"] \arrow[loop right, "T"] \end{tikzcd}$$

B - Functor
D - Monad

1) IF B has a right adjoint A

Then

$$\exists \bar{B},\quad \begin{tikzcd} Kl(T) \arrow[r, "\bar{B}"] & Kl(T) \\ C \arrow[u] \arrow[r, "B"'] & C \arrow[u] \end{tikzcd} \iff \exists \bar{A},\quad \begin{tikzcd} EM(T) \arrow[r, "\bar{A}"] \arrow[d] & EM(T) \arrow[d] \\ C \arrow[r, "A"'] & C \end{tikzcd}$$

2) MOST IMPORTANT

ASSUME $\exists\, \bar{B}$

$$\begin{tikzcd} EM(T) \arrow[r, "\bar{B}"] \arrow[d, "U"'] & EM(T) \arrow[d, "U"] \\ C \arrow[r, "B"] & C \end{tikzcd}$$

THEN

$$\begin{tikzcd} Alg(\bar{B}) \arrow[d, Rightarrow, "U"] \\ Alg(B) \end{tikzcd}$$

HAS A LEFT ADJOINT

$$\begin{tikzcd} Alg(\bar{B}) \arrow[d, shift left=2, "U"] \\ Alg(B) \arrow[u, shift left=2, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$