SCforF

Finite dimensional asset markets

Introduction

In a finite dimensional asset market, we model the payoff of the market with a matrix:

$$A \in \mathbb{R}^{m \times n}$$

where n is the number of different assets in the market, and m refers to the number of different states that the market can have.
The prices are usually stored in:

$$S \in \mathbb{R}^{n}.$$

When given a new asset with payoff $b \in \mathbb{R}^m$, and you are asked if this can be perfectly hedged, mathematically you just need to verify if:

$$b \in \mathrm{Span}(A),$$

where $\mathrm{Span}(A) := \mathrm{Span}(A_1, \ldots, A_n)$, and where $A_1, \ldots, A_n$ are the columns of A.

Definition of arbitrage

First type:
$\exists x$ such that $S \cdot x \leq 0$, $Ax \geq 0$ and $Ax \neq 0$.

Second type:
$\exists x$ such that $S \cdot x < 0$, $Ax = 0$.

Theorem: there is no type 2 arbitrage if and only if $\exists \psi$ such that $S = A^T \psi$.

Theorem: there is no type 1 or 2 if and only if $\exists \psi >> 0$ such that $S = A^T \psi$.

Remark: let us define $R_f := \frac{1}{\mathbb{1} \cdot \psi}$ where:

$$\mathbb{1} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and define $q := R_f \psi$. Notice that $q \cdot \mathbb{1} = 1$.

The fair price of the new asset b is then:

$$\psi \cdot b = \frac{1}{R_f} q \cdot b = \frac{1}{R_f} \mathbb{E}_q \left[b \right]$$

The idea is: the price is fair when:

$$\mathbb{E}_q\left[\text{price} \cdot R_f - \text{payoff}\right] = 0$$

being q the risk-neutral probability.

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Random variables and conditional expectation

Definition: given $(E, \mathcal{E})$ and $(F, \mathcal{F})$ two measurable spaces, we say that $X : E \to F$ is measurable if $X^{-1}(A) \in \mathcal{E}$ for all $A \in \mathcal{F}$.

Theorem: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Given $\mathcal{G} \subset \mathcal{F}$ another σ-algebra contained in $\mathcal{F}$, and given $X : \Omega \to \mathbb{R}$ a random variable with $\mathbb{E}\left[|X|\right] < +\infty$ there exists a unique random variable Y such that:

$$\mathbb{E}\left[XZ\right] = \mathbb{E}\left[YZ\right],$$

for all Z that is $\mathcal{G}$-measurable.

Definition: we call the random variable Y from the previous theorem $\mathbb{E}\left[X|\mathcal{G}\right]$.

Properties:

- Tower property: if $\mathcal{G} \subset \mathcal{H}$ then:

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{H}\right]|\mathcal{G}\right] = \mathbb{E}\left[X|\mathcal{G}\right].$$