STOCHASTIC CALCULUS FOR FINANCE @ IBS - CHEAT SHEET

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Abstract. This document is a quick-reference "cheat sheet" for the Stochastic Calculus for Finance course within the Risk Management module that I assist in teaching @ IBS.

Contents

1. Random variables and σ-algebras

Definition 1. *Given a set Ω, one defines $\mathcal{F} \subset \mathcal{P}(\Omega)$ σ-algebra if:*

(1) $\emptyset \in \mathcal{F}$,
(2) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
(3) $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

Exercise 1. *(Additional exercise left in class) Is it true that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras then $\mathcal{F}_1 \cup \mathcal{F}_2$ is σ-algebra? If yes, prove it. If no, exhibit a counterexample.*

2. Finite dimensional asset markets

2.1. **Introduction.** In a finite dimensional asset market, we model the payoff of the market with a matrix:

$$A \in \mathbb{R}^{m\times n}$$

where n is the number of different assets in the market, and m refers to the number of different states that the market can have. The prices are usually stored in:

$$S \in \mathbb{R}^n.$$

When given a new asset with payoff $b \in \mathbb{R}^m$, and you are asked if this can be perfectly hedged, mathematically you just need to verify if:

$$b \in \mathrm{Span}(A),$$

where $\mathrm{Span}(A) := \mathrm{Span}(A_1, \dots, A_n)$, and where $A_1, \dots, A_n$ are the columns of A.

2.2. **Definition of arbitrage.**

(1) **First type:** $\exists x$ such that $S \cdot x \leq 0$, $Ax \geq 0$ and $Ax \neq 0$.
(2) **Second type:** $\exists x$ such that $S \cdot x < 0$, $Ax = 0$.

Theorem 1. *there is no type 2 arbitrage if and only if $\exists \psi$ such that $S = A^T\psi$.*

Theorem 2. *there is no type 1 or 2 if and only if $\exists \psi >> 0$ such that $S = A^T\psi$.*

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Remark 1. *let us define* $R_f := \frac{1}{\mathbb{I}\cdot\psi}$ *where:*

$$\mathbb{I} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and define $q := R_f\psi$. *Notice that* $q \cdot \mathbb{I} = 1$.
The fair price of the new asset b *is then:*

$$\psi \cdot b = \frac{1}{R_f} q \cdot b = \frac{1}{R_f}\mathbb{E}_q\left[b\right]$$

The idea is: the price is fair when:

$$\mathbb{E}_q\left[\textit{price} \cdot R_f - \textit{payoff}\right] = 0$$

being q *the risk-neutral probability.*

3. Infinite dimensional asset markets

3.1. Random variables and conditional expectation.

Definition 2. *given* $(E, \mathcal{E})$ *and* $(F, \mathcal{F})$ *two measurable spaces, we say that* $X : E \to F$ *is measurable if* $X^{-1}(A) \in \mathcal{E}$ *for all* $A \in \mathcal{F}$.

Theorem 3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Given* $\mathcal{G} \subset \mathcal{F}$ *another* σ*-algebra contained in* $\mathcal{F}$*, and given* $X : \Omega \to \mathbb{R}$ *a random variable with* $\mathbb{E}\left[|X|\right] < +\infty$ *there exists a unique random variable* Y *such that:*

$$\mathbb{E}\left[XZ\right] = \mathbb{E}\left[YZ\right],$$

for all Z *that is* $\mathcal{G}$*-measurable.*

Definition 3. *we call the random variable* Y *from the previous theorem* $\mathbb{E}\left[X|\mathcal{G}\right]$.

Proposition 1. *The following properties hold:*

(1) ***Tower property:*** *if* $\mathcal{G} \subset \mathcal{H}$ *then:*

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{H}\right]|\mathcal{G}\right] = \mathbb{E}\left[X|\mathcal{G}\right].$$

References