A para-hyperKähler structure on the space of GHMC AdS^3-manifolds

Heidelberg, Differential Geometry Seminar

June 10^{th}, 2021

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(joint work with A. Seppi & A. Tamburelli)

Σ closed surface of genus ≥ 2

Lorentzian metrics with sec $\equiv -1$ s.t. ...

Geometric objects : AdS^3-structures on $\Sigma \times \mathbb{R}$

$\mathcal{GH}(\Sigma)$: deformation space of AdS^3-str's on $\Sigma \times \mathbb{R}$

isotopy classes

Q: Does $\mathcal{GH}(\Sigma)$ carry some natural structure?

(natural : invariant by the action of the mapping class group)

Thm (M-Seppi-Tamburelli) The space $\mathcal{GH}(\Sigma)$ carries a natural para-hyperKähler structure

Inspiration: Donaldson's work on the space of almost Fuchsian manifolds

Every ingredient has a description in terms of AdS^3-geometry...

Def (M, g, I) pseudo- Kähler manifold if

- g pseudo- Riemannian metric on M

- I integrable almost complex structure ($I \in \mathrm{End}(TM)$ s.t. $I^2 = -\mathbb{1}$ induced by cplx str on M)

such that

- $g(Iu, v) = -g(u, Iv)$

- $\omega = g(\cdot, I\cdot)$ is a symplectic structure (closed non-degenerate 2-form)

same

Def $(M, g, \overset{P}{\cancel{I}})$ para-Kähler manifold if

- g pseudo-Riemannian metric on M
- $\overset{P}{\cancel{I}}$ integrable almost para-complex structure ($\overset{P}{\cancel{I}} \in \mathrm{End}(TM)$ s.t. $P^2 = +\mathbb{1}$ ~~$I^2 = -\mathbb{1}$ induced by cplx str~~ on M)
  the ± 1-eigenspaces have the same dim and integrable

such that

- $g(\overset{P}{\cancel{I}}u, v) = -g(u, \overset{P}{\cancel{I}}v)$
- $\omega = g(\cdot, \overset{P}{\cancel{I}}\cdot)$ is a symplectic structure (closed non-degenerate 2-form)

If (M, g, P) is para-Kähler, then $g(P\cdot, P\cdot) = -g(\cdot,\cdot)$

$\Rightarrow$ g has neutral signature

<u>Def</u> (M, g, I, J, K) para-hyperKähler if

- (M, g, I) pseudo-Kähler
- $(M, g, J), (M, g, K)$ para-Kähler
- I, J, K satisfy para-quaternionic relations $\quad IJ = K \quad I^2 = -\mathbb{1}$, $J^2 = \mathbb{1}$, $K^2 = \mathbb{1}$

$\omega_I = g(\cdot, I\cdot)$, $\omega_J = g(\cdot, J\cdot)$, $\omega_K = g(\cdot, K\cdot)$ sympl forms

Anti-de Sitter geometry in dim 3

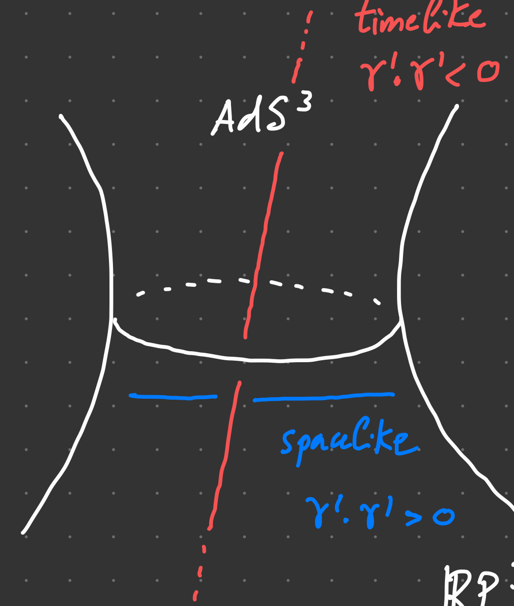
$$\mathrm{AdS}^3 := \left(\mathbb{P}\mathrm{SL}_2\mathbb{R},\ -\tfrac{1}{8}B_{\mathrm{Kill}}\right)$$

Lorentzian manifold with curvature -1

$$\mathrm{Isom}_0(\mathrm{AdS}^3) = \mathbb{P}\mathrm{SL}_2\mathbb{R} \times \mathbb{P}\mathrm{SL}_2\mathbb{R}$$

left/right action

$$\partial\mathrm{AdS}^3 \simeq \mathbb{R}P^1 \times \mathbb{R}P^1$$



AdS^3 and para-complex numbers

$$\mathbb{B} = \mathbb{R} \oplus \tau\mathbb{R}, \quad \tau^2 = 1$$

$$SL_2\mathbb{B} := \{ A \in M_{2\times 2}(\mathbb{B}) \mid \det(A) = 1 \}$$

$$SL_2\mathbb{R} \times SL_2\mathbb{R} \xrightarrow{\sim} SL_2\mathbb{B}$$

$$(A,B) \longmapsto \underbrace{\frac{1+\tau}{2}}_{\tau\cdot(\ ) = (\ )} A + \frac{1-\tau}{2} B$$

$$\tau(\ ) = -(\ )$$

Therefore $\mathrm{Isom}_0(AdS^3) \simeq PSL_2\mathbb{B}$

$\mathbb{H}^3$ $\quad$ $\mathbb{C}$

$$\mathbb{RP}^1 \times \mathbb{RP}^1 \xrightarrow{\sim} \mathbb{BP}^1 = \partial AdS^3$$

$\mathbb{B}^2 \setminus \{0\} / \sim$ $\quad$ $\mathbb{CP}^1$ $\quad$ $\partial\mathbb{H}^3$

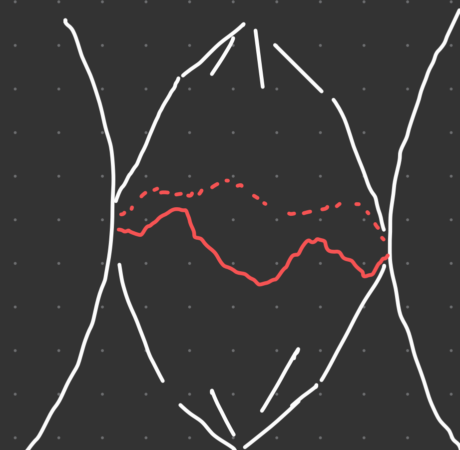
first evidence for para-cplx structures

GHMC AdS^3-manifolds

Let $\rho_\ell, \rho_r : \pi_1\Sigma \longrightarrow \mathbb{P}SL_2\mathbb{R}$ be Fuchsian repr's

$\rho_\ell \times \rho_r : \pi_1\Sigma \longrightarrow \mathbb{P}SL_2\mathbb{R} \times \mathbb{P}SL_2\mathbb{R}$

acts freely & prop. disc. on Ω_φ



$M_{\rho_\ell, \rho_r} := \Omega_\varphi / \rho_\ell \times \rho_r(\pi_1\Sigma)$ GHMC AdS^3-mfld (Mess '07)

Three interpretations of $\mathcal{GH}(\Sigma)$:

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{M}} \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma) \qquad\qquad \mathcal{GH}(\Sigma) \xrightarrow{\mathcal{F}} T^*\mathcal{T}(\Sigma)$$

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{C}} \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$$

The structure arises from the interplay of these interpretations

Mess' homeomorphism

$$\mathcal{M} : \mathcal{GH}(\Sigma) \longrightarrow \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \subset \mathcal{X}(\pi_1 \Sigma, \mathbb{P}SL_2 \mathbb{B})$$

$$[M_{\rho_l, \rho_r}] \longmapsto ([\rho_l], [\rho_r]) \qquad H^1(\Sigma ; \underbrace{\mathfrak{sl}_2 \mathbb{R} \times \mathfrak{sl}_2 \mathbb{R}}_{\mathfrak{sl}_2 \mathbb{B}}{}_{Ad})$$

We find:

- two symplectic structures

$$\mathcal{M}^*(\Omega_{WP} \oplus \Omega_{WP}), \qquad \mathcal{M}^*(\Omega_{WP} \oplus (-\Omega_{WP}))$$

- a para-cplx structure $\mathbf{J} \in \mathrm{End}(T\,\mathcal{GH}(\Sigma))$

$$\mathbf{J} := \mathcal{M}^*(\mathbb{1}, -\mathbb{1})$$

some evidence...

To describe the complex structure, we need <u>maximal surfaces</u>:

$\Sigma \overset{i}{\hookrightarrow} M$ spacelike surface, $\quad h = i^* G_M$ Riemannian metric,
1st fundam form

n timelike normal vector field

$BV := \nabla_V n$ shape operator, $\quad \mathrm{II}(\cdot,\cdot) = h(\cdot, B\,\cdot)$ 2nd fundam form

$\Sigma \overset{i}{\hookrightarrow} M$ is a maximal surface if $\mathrm{tr}(B) \equiv 0$

h, $\mathrm{I\!I} = h(\cdot, B\cdot)$ satisfy constraints:

$$K_h = -1 - \det B\,, \quad \mathrm{I\!I} = \mathrm{Re}\, q\,,\ q \in QD(\Sigma, [h]) \qquad \text{(Gauss-Codazzi)}$$

<u>Thm</u> (Barbot-Béguin-Zeghib) Every M has a unique maximal surface, and the map

$$\begin{aligned} \mathcal{F} : \mathcal{GH}(\Sigma) &\longrightarrow T^*\mathcal{T}(\Sigma) \\ [M] &\longmapsto ([h], q) \end{aligned} \qquad \text{is a diffeo}$$

Through $\mathcal{F} : \mathcal{GH}(\Sigma) \longrightarrow T^*\mathcal{T}(\Sigma)$ we find:

- two symplectic structures

$$\mathcal{F}^* \mathrm{Re}\, \omega^{\mathbb{C}}_{cot}, \qquad \mathcal{F}^* \mathrm{Im}\, \omega^{\mathbb{C}}_{cot}$$

- a complex structure $\mathbf{I} \in \mathrm{End}(T\mathcal{MGH}(\Sigma))$

$$\mathbf{I} := \mathcal{F}^*(i\text{-multipl in } T^*\mathcal{T}(\Sigma) \cong QD)$$

other ingredients...

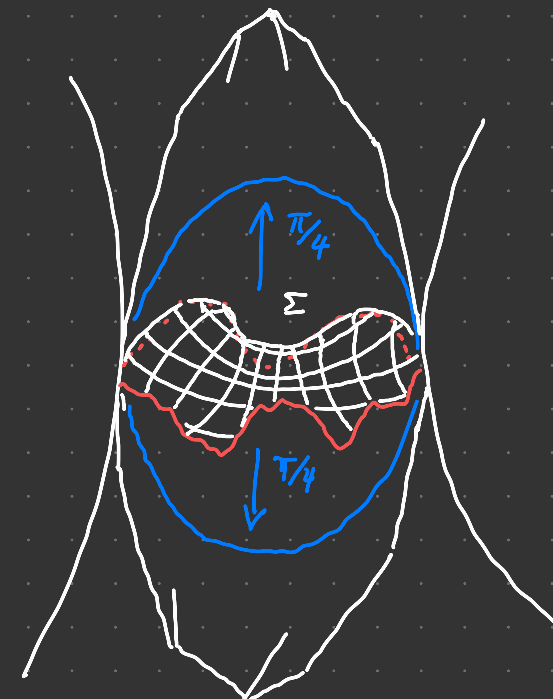
One last map...

(Σ, h, n) maximal surface,

$$\Sigma^{\pm} := \exp\left(\pm \frac{\pi}{4} n(\Sigma)\right)$$

constant curvature surfaces

$(h^+, h^-) \in \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ hyp metrics



$$\begin{aligned} \mathcal{C} : \mathcal{GH}(\Sigma) &\longrightarrow \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \\ [M] &\longmapsto (h^+, h^-) \end{aligned}$$

diffeo (Krasnov-Schlenker '07)

$$\mathrm{Hopf}\big((\Sigma, h) \longrightarrow (\Sigma, h^{\pm})\big) = \pm i q$$

$$\big((\Sigma, h) \longrightarrow (\Sigma, h_{\ell,r})\big) = \pm q$$

Let's summarize:

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{M}} \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$$

$\mathbf{J} := \mathcal{M}^*(\mathbb{1},-\mathbb{1})$ para-cplx

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{F}} T^*\mathcal{T}(\Sigma)$$

$\mathbf{I} := \mathcal{F}^*(i\,\cdot)$ cplx

$\mathbf{K} := \mathcal{C}^*(\mathbb{1},-\mathbb{1})$ para-cplx

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{C}} \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$$

Let's summarize:

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{M}} \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$$

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$\mathcal{M}^*(\Omega_{WP}\oplus(-\Omega_{WP}))$

$\mathcal{M}^*(\Omega_{WP}\oplus\Omega_{WP})$

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{F}} T^*\mathcal{T}(\Sigma)$$

$\mathbf{I} := \mathcal{F}^*(i\cdot)$ cplx

$\mathcal{F}^*\operatorname{Re}\omega_{cot}$

$\mathcal{F}^*\operatorname{Im}\omega_{cot}$

$\mathcal{C}^*(\Omega_{WP}\oplus\Omega_{WP})$

$\mathcal{C}^*(\Omega_{WP}\oplus(-\Omega_{WP}))$

$\mathbf{K} := \mathcal{C}^*(\mathbb{1},-\mathbb{1})$ para-cplx

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$\mathcal{M}^*(\Omega_{WP}\oplus(-\Omega_{WP}))$

ω_K

$\mathcal{F}^*\mathrm{Re}\,\omega_{cot}$

$\mathcal{M}^*(\Omega_{WP}\oplus\Omega_{WP})$

$\mathcal{F}^*\mathrm{Im}\,\omega_{cot}$

ω_I

ω_J

$\mathcal{C}^*(\Omega_{WP}\oplus\Omega_{WP})$

$\mathcal{C}^*(\Omega_{WP}\oplus(-\Omega_{WP}))$

$\mathbf{K} := \mathcal{C}^*(\mathbb{1}, -\mathbb{1})$ para-cplx

$$\mathcal{GH}(\Sigma) \xrightarrow{\mathcal{C}} \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$$

Theorem (M-Seppi-Tamburelli)

The ingredients above $\mathbf{I}, \mathbf{J}, \mathbf{K}, \omega_{\mathbf{I}}, \omega_{\mathbf{J}}, \omega_{\mathbf{K}}$ are part of a natural para-hyperKähler structure $(\mathcal{GH}(\Sigma), g, \mathbf{I}, \mathbf{J}, \mathbf{K})$, with

$$\omega_{\mathbf{I}} = g(\cdot, \mathbf{I}\cdot), \quad \omega_{\mathbf{J}} = g(\cdot, \mathbf{J}\cdot), \quad \omega_{\mathbf{K}} = g(\cdot, \mathbf{K}\cdot)$$

we recover

$\mathcal{T}(\Sigma) \xhookrightarrow{\iota} T^*\mathcal{T}(\Sigma)$ 0-section $\Rightarrow$ $(\mathcal{T}(\Sigma), \iota^*g, \iota^*\mathbf{I}, \iota^*\omega_{\mathbf{I}})$

Weil-Petersson Kähler structure

Comments and remarks

- Comparison with almost-Fuchsian structures $\mathcal{AF}(\Sigma) \subset \mathcal{X}(\pi_1\Sigma, \mathbb{P}SL_2\mathbb{C})$

Donaldson '03 : $\mathcal{AF}(\Sigma)$ admits a natural hyperkähler structure

$\mathcal{AF}(\Sigma) \subset \mathcal{X}(\pi_1\Sigma, \mathbb{P}SL_2\mathbb{C})$	$\mathcal{GH}(\Sigma) \subset \mathcal{X}(\pi_1\Sigma, \mathbb{P}SL_2\mathbb{B})$
g Riemannian metric	g neutral signature
I, J, K cplx str's	I cplx, J, K para-cplx
g does not extend to $\mathcal{QF}(\Sigma)$	structure defined on $\mathcal{GH}(\Sigma)$

Thank you for your attention!

- GHMC AdS^3-manifolds have holonomy in $\mathcal{X}(\pi_1\Sigma, PSL_2\mathbb{B})$ and they are Anosov

  $\hookleftarrow$ rank 2 Lie Group

- Geometric transitions:

$\mathbb{H}^3$	HP^3	AdS^3
$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$	$\mathbb{R} \oplus \sigma\mathbb{R}$	$\mathbb{B} = \mathbb{R} \oplus \tau\mathbb{R}$

Question: Is there a transition between the hyperKähler structure of $\mathcal{AF}(\Sigma)$ and the para-hyperKähler structure of $\mathcal{GH}(\Sigma)$?

- Underlying process in both constructions: infinite dimensional simplectic reduction

(X,ω) symplectic manifold, G Lie group $\curvearrowright (X,\omega)$

Q: Is X/G symplectic? Not really...

$\mu : X \longrightarrow \mathfrak{g}^*$ moment map

Ad^*- equivariant $\rightsquigarrow$ $\mu^{-1}(0)/G$ has a symplectic structure

$\langle d\mu(\cdot), X \rangle = \omega(V_X, \cdot)$

Idea (Donaldson): Describe the deformation space as a symplectic quotient...

Ex $\mathcal{T}(\Sigma)$ can be constructed as follows:

Ingredient: $\mathcal{J}(\mathbb{R}^2) = \{\text{linear cplx str's on } \mathbb{R}^2\} \simeq \mathbb{H}^2$ is Kähler and $SL_2(\mathbb{R})$ preserves the structure (with $\mu : \mathcal{J}(\mathbb{R}^2) \to \mathfrak{sl}_2\mathbb{R}^*$)

$$\rightsquigarrow \quad X := \mathcal{J}(\Sigma) = \Gamma\left(\mathrm{Frame}(\Sigma, p) \times_{SL_2\mathbb{R}} \mathcal{J}(\mathbb{R}^2) \longrightarrow \Sigma\right)$$
$$= \{ J \in \mathrm{End}(T\Sigma) \mid J^2 = -\mathbb{1} \}$$

has an induced formal Kähler structure

Let ρ be an area form on Σ

$$g(\dot{s}, \dot{s}') = \int_\Sigma \hat{g}(\dot{s}(p), \dot{s}'(p))\, \rho(p)$$

$$\omega(\dot{s}, \dot{s}') = \int_\Sigma \hat{\omega}(\dot{s}(p), \dot{s}'(p))\, \rho(p)$$

$G = \mathrm{Symp}_0(\Sigma, \rho) \curvearrowright \mathcal{J}(\Sigma)$ preserving the structure

$$\mu : \mathcal{J}(\Sigma) \longrightarrow \mathrm{Lie}(\mathrm{Symp}_0(\Sigma, \rho))^*$$
$$J \longmapsto (K_{h_J} + 1)\rho$$

$$\mathrm{Teich}(\Sigma) \simeq \mu^{-1}(0) \big/ \mathrm{Symp}_0(\Sigma, \rho)$$

where $h_J = \rho(\cdot, J\cdot)$

First step: existence of a para-hyperkähler structure on $\underbrace{T^*\mathcal{J}(\mathbb{R}^2)}_{\ni (J,\phi)}$, invariant by the action of $SL_2\mathbb{R}$

Rmk In Donaldson's work, $T^*_{<1}\mathcal{J}(\mathbb{R}^2)$ carries a hyperkähler structure, again invariant by $SL_2\mathbb{R}$

μ_I, μ_J, μ_K moment maps for the action of $\mathrm{Symp}_0(\Sigma,\rho)$

$$\mu_I, \mu_J, \mu_K : T^*\mathcal{J}(\Sigma) \longrightarrow \mathrm{Lie}(\mathrm{Symp}_0(\Sigma,\rho))^*$$

$$\rightsquigarrow \qquad \mathcal{GH}(\Sigma) = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) \Big/ \mathrm{Symp}_0(\Sigma,\rho)$$

Geometric interpretation of the moment maps:

$$(\mu_J + i\mu_K)(J,\sigma) = 0 \iff \sigma = \mathrm{Re}\,\phi \,,\ \phi \in QD(\Sigma, J)$$

$g = \rho(\cdot, J\cdot)$ Riemannian metric on Σ with area form ρ

$h = \left(1 + \sqrt{1 + \|\sigma\|_g^2}\right) g$ $\quad B := h^{-1}\sigma$ (i.e. $\sigma = h(\cdot, B\cdot)$)

Then $(J,\sigma) \in \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) \iff (h, B)$ satisfies Gauss-Codazzi equations of a maximal surface in AdS^3