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Workshop in Autrans (August 2022): Labourie's conjecture

Introduction to minimal surfaces & harmonic maps

Let M, N be two complete Riemannian manifolds, and let $f: M \to N$ be a smooth map. We would like to have a way of selecting, inside the homotopy class of f, of a representative $u: M \to N$ with good geometric properties. If $\dim M = 1$, then special representatives are for example geodesics, which are stationary pts of the energy functional: $\forall \alpha : [0, L] \to N$ let

$$E(\alpha) := \frac{1}{2} \int_0^L \|\alpha'(t)\|^2 dt$$

Looking at the Euler-Lagrange equation of E, one obtains:

$$\alpha \text{ stationary pt} \Leftrightarrow \nabla_{\alpha'} \alpha' \equiv 0$$

Let's generalize this to maps between manifolds:

Given $f: M \to N$, consider $E := f^* TN \to M$, endowed with the metric $\langle \cdot, \cdot \rangle_N$ and with the pull back D of the Levi-Civita connection ∇^N of N.

$df \in \Gamma(M, T^*M \otimes E)$, and $T^*M \otimes E$ has a Riemannian metric $g_M^* \otimes \langle \cdot, \cdot \rangle_N$ and a connection $\nabla := \nabla^M \otimes D$, satisfying

$$\nabla_X (\alpha \otimes s) = \nabla_X^M \alpha \otimes s + \alpha \otimes D_X s \quad \forall X \in \Gamma(TM)$$

Then we define $\nabla^2 f := \nabla . df$.

<u>Exercise</u> $\nabla^2 f \in \Gamma(T^*M \otimes T^*M \otimes E)$ is symmetric in T^*M, i.e.

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X) \qquad \forall X, Y \in \Gamma(TM)$$

<u>Proof</u> Based on the fact that ∇^M & D are torsion-free

Then we set $\Delta f := \mathrm{tr}_{g_M}(\nabla^2 f) = C^1_1 C^1_1(g^*_M \otimes \nabla^2 f)$

$$= \sum_{i=1}^m (\nabla^2 f)(e_i, e_i) \qquad \text{with } (e_i)_i \text{ local } g_M\text{-orthonormal frame}$$

<u>Def 1</u> $f: M \to N$ is harmonic if $\Delta f = 0$

From now on, assume M to be <u>compact</u>.

On the other hand, we can define $E(f) = E(f, g_M, g_N) := \frac{1}{2}\int_M \|df\|^2 dvol_M$

<u>Theorem</u> (1st variation formula of E) Let $f_t: M \to N$ be a smooth homotopy of maps, $t \in (-\varepsilon, \varepsilon)$, and let

$$V(\cdot) = \frac{d}{dt} f_t(\cdot)\Big|_{t=0} \in \Gamma(E).$$

Then $$dE_f(V) := \frac{d}{dt}E(f_t)\Big|_{t=0} = -\int_M \langle \Delta f, V\rangle_N \, dvol_M$$

In particular from here we see that Def 1 is equivalent to:

<u>Def 2</u> A map $f: M \to N$ is harmonic if it is a critical pt of the energy functional, i.e.

$\forall\, (f_t)_{t\in(-\varepsilon,\varepsilon)}$ smooth homotopy $\frac{d}{dt}E(f_t)\Big|_{t=0} = 0.$

<u>Proof of Thm</u> (sketch) As often happens in these cases, the relation follows from an application of Stokes. The relation that we need here is the following:

$$\langle df(\cdot), V\rangle_N \in \Gamma(T^*M) \Rightarrow \overset{\text{Hodge } * \text{ wrt } g_M}{*}(\langle df(\cdot), V\rangle_N) \in \Gamma(\Lambda^{m-1}T^*M), \text{ where } m = \dim M$$

Then $$d(*(\langle df(\cdot), V\rangle_N)) = \Big(\underbrace{\mathrm{tr}_{g_M}(\langle \nabla^2 f, V\rangle_N)}_{=\,\langle \Delta f, V\rangle_N} + \mathrm{tr}_{g_M}(\langle df, \nabla_\cdot V\rangle)\Big) dvol_M$$

In particular $$\int_M \mathrm{tr}_{g_M}(\langle df, \nabla_\cdot V\rangle_N)\, dvol_M = -\int_M \langle \Delta f, V\rangle \, dvol_M \quad (\star)$$

Let's see now how to conclude: let $\tilde{\nabla}$ be the Levi-Civita connection of $M \times \mathbb{R}$ wrt the metric $g_M + dt^2$, and set $\hat{\nabla} := \tilde{\nabla} \otimes D$ connection on $T^*(M \times \mathbb{R}) \otimes E$. We will need the following: $\forall X \in \Gamma(TM)$ $\forall \sigma \in \Gamma(T^*(M \times \mathbb{R}) \otimes E)$

$$\hat{\nabla}_X \hat{\nabla}_{\partial t} \sigma = \hat{\nabla}_{\partial t} \hat{\nabla}_X \sigma \qquad (\text{I})$$

Then

$\langle\langle \cdot, \cdot \rangle\rangle$ scalar prod on $T^*M \otimes E$

$$dE_f(V) = \frac{1}{2} \int_M \frac{d}{dt} \|df_t\|^2 \Big|_{t=0} dvol_M = \int_M \langle\langle \hat{\nabla}_{\partial t} df_t |_{t=0}, df \rangle\rangle \, dvol_M$$

$$= \int_M tr_{g_M} (\langle \nabla_{\partial t} df_t |_{t=0}, df \rangle_N) \, dvol_M$$

by (I) $\hat{\nabla}_{\partial t} df_t |_{t=0} = \hat{\nabla}_\cdot \frac{d}{dt} f_t |_{t=0}$
$= \hat{\nabla}_\cdot V = \nabla_\cdot V$

when applied to vector fields that are indep of t

$$= \int_M tr_{g_M} (\langle \nabla_\cdot V, df \rangle_N) \, dvol_M$$

by (*) $$= - \int_M \langle V, \Delta f \rangle_N \, dvol_M$$

$\square$

There is another characterization:

<u>Def 3</u> $f: M \to N$ is harmonic if $d_\nabla^*(df) := * d_\nabla * (df) = 0$, where d_∇ is the covariant exterior differential on $\Omega^0(M; E)$

(in fact $d_\nabla^*(df) = -\Delta f$) (We will not need this characterization, but it is worth to mention it)

Since $d_\nabla(df) = 0$, this is the same as saying that df is both exact and coexact, and hence $(d_\nabla d_\nabla^* + d_\nabla^* d_\nabla)(df) = 0$, which is the Hodge-theory definition of harmonic forms.

In conclusion we have

Thm Let M, N be Riemannian mflds, with M cpt. Then the following are equivalent:

$$dE_f \equiv 0 \Leftrightarrow \Delta f = 0 \Leftrightarrow d_\nabla^*(df) = 0 \Leftrightarrow df \text{ is a harmonic } E\text{-valued 1-form}$$

this is the only one that really needs cpt!

can be replaced with $d(E_K)_f = 0$
$\forall K \subset M$ cpt subset

Let's focus now on a special class of harmonic maps, namely isometric harmonic maps:

Def An harmonic map $f: M \to N$ is isometric if $f^* g_N = g_M$

Rmk isometric harmonic maps are always immersions, and $\dim M = m \leq n = \dim N$

Since $f^* g_N = g_M$, g_M coincides with the first fundamental form of f, i.e. $I_f := g_M$.
Furthermore we have:

$$(\nabla^2 f)(X,Y) = (\nabla_X(df))Y = \nabla_X(df(Y)) - df(\nabla_X Y)$$
$$= D^N_X(f_* Y) - f_*(\nabla^M_X Y) = \nabla^N_{f_* X}(f_* Y) - f_*(\nabla^M_X Y)$$

This is saying that $\nabla^2 f$ measures the difference between the Levi-Civita connection of I_f
This already has a name, it coincides with the second fundamental form $I\!I_f := \nabla^2 f$

$$I_f \in \Gamma(\mathrm{Sym}(T^*M \otimes T^*M)) \qquad I\!I_f \in \Gamma(\mathrm{Sym}(T^*M \otimes T^*M) \otimes E)$$

In particular $\Delta f = \mathrm{tr}_{I_f} I\!I_f = \dim M \cdot H_f$, where H_f is the mean curvature of f
This shows:

Lemma $f: M \to N$ isometric harmonic $\Leftrightarrow$ f is a minimal immersion

Let's focus now on the special case $\dim M = 2$.

Lemma If $\dim M = 2$, then $E(f, g_M, g_N) = E(f, e^{2\varphi} g_M, g_N)$ $\forall \varphi \in \mathcal{C}^\infty(M;\mathbb{R})$.

Proof Let $h_M := e^{2\varphi} g_M$ for some $\varphi \in \mathcal{C}^\infty(M;\mathbb{R})$. Then

$$\|df\|^2_{h_M^* \otimes g_N} = \mathcal{C}_1^1 \mathcal{C}_1^1 (h_M^* \otimes \langle df(\cdot), df(\cdot)\rangle_N)$$
$$= e^{-2\varphi} \mathcal{C}_1^1 \mathcal{C}_1^1 (g_M^* \otimes \langle df(\cdot), df(\cdot)\rangle_N)$$
$$= e^{-2\varphi} \|df\|^2_{g_M^* \otimes g_N}$$

$$d\mathrm{vol}_{h_M} = \sqrt{\det(e^{2\varphi} g_M)}\, dx^1 \wedge \dots \wedge dx^n$$
$$= e^{m\varphi} \sqrt{\det(g_M)}\, dx^1 \wedge \dots \wedge dx^n$$

So if $\dim M = m = 2$, then $\|df\|^2_{h_M^* \otimes g_N} = \|df\|^2_{g_M^* \otimes g_N}$ $\square$

This means that we can replace (M, g_M) with X Riemann surface and have a well-defined energy!

Alert: Δf is NOT conformally invariant, but $\Delta_{g_M} f = 0 \Leftrightarrow \Delta_{e^{2\varphi} g_M} f = 0$ in $\dim M = 2$

Corollary Let $f : X \to N$ harmonic. Then f minimal $\Leftrightarrow$ f conformal

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We now discuss an equivalent characterization of harmonic maps, which is well-adapted to the case $m=2$.

Recall that, if X is a Riemann surface, then $\begin{cases} T^*_{\mathbb{C}} X \simeq T^{1,0} X \oplus T^{0,1} X \\ T^*_{\mathbb{C}} X \otimes T^*_{\mathbb{C}} X \simeq T^{2,0} X \oplus T^{1,1} X \oplus T^{0,2} X \end{cases}$

for all (U, z) holomorphic coordinates.

If S is a symmetric 2-tensor, then $S = S^{2,0} + S^{1,1} + S^{0,2}$

$$= a \cdot dz \otimes dz + b(dz \otimes d\bar{z} + d\bar{z} \otimes dz) + c\, d\bar{z} \otimes d\bar{z}$$

$dz^2 := dz \otimes dz$, $|dz|^2 := dz \otimes d\bar{z} + d\bar{z} \otimes dz$, $d\bar{z}^2 := dz \otimes dz$

Similarly $\alpha = \alpha^{1,0} + \alpha^{0,1} \in T^{1,0}X \oplus T^{0,1}X$. Take for example

$$df = d^{1,0}f + d^{0,1}f \in \Gamma(T^{1,0}X \otimes E^{\mathbb{C}} \oplus T^{0,1}X \otimes E^{\mathbb{C}})$$

So $f^*g_N = (f^*g_N)^{(2,0)} + (f^*g_N)^{(1,1)} + (f^*g_N)^{(0,2)}$

<u>Prop</u> Let $f : X \to N$ be a smooth map

- f harmonic $\Rightarrow$ $\mathrm{Hopf}(f)$ is holomorphic
- f minimal $\Leftrightarrow$ $\mathrm{Hopf}(f) \equiv 0$

<u>Proof</u> $\mathrm{Hopf}(f) = (f^*g_N)^{(2,0)} = \langle df(\partial_z), df(\partial_z)\rangle_N \, dz^2$

$\mathrm{Hopf}(f)$ holomorphic $\Leftrightarrow$ $\partial_{\bar{z}}(\langle df(\partial_z), df(\partial_z)\rangle_N) \equiv 0$

$\Leftrightarrow$ $2\langle \nabla_{\partial_{\bar{z}}}(df(\partial_z)), df(\partial_z)\rangle_N \equiv 0$

$$\langle \nabla_{\partial_{\bar{z}}}(df(\partial_z)), df(\partial_z)\rangle_N = \langle (\nabla_{\partial_{\bar{z}}} df)(\partial_z) + df(\nabla_{\partial_{\bar{z}}}\partial_z), df(\partial_z)\rangle_N$$

$$= \langle (\nabla^2 f)(\partial_{\bar{z}}, \partial_z), df(\partial_z)\rangle_N + \langle df(\cancel{\nabla_{\partial_{\bar{z}}}\partial_z}), df(\partial_z)\rangle_N$$

$$\nabla^2 f(\partial_z, \partial_{\bar{z}}) = \frac{1}{4}\nabla^2 f(\partial_x - i\partial_y, \partial_x + i\partial_y) = \frac{1}{4}\left(\nabla^2 f(\partial_x, \partial_x) + \nabla^2 f(\partial_y, \partial_y) - i\,\cancel{\nabla^2 f(\partial_y, \partial_x)} + i\,\cancel{\nabla^2 f(\partial_x, \partial_y)}\right)$$

If $g_M = e^{2\varphi}|dz|^2$, then $\Delta f = e^{-2\varphi}(\nabla^2 f(\partial_x, \partial_x) + \nabla^2 f(\partial_y, \partial_y))$

So $\partial_{\bar{z}}(\langle df(\partial_z), df(\partial_z)\rangle_N) = e^{-2\varphi}\langle \Delta f, df(\partial_z)\rangle_N$ Hence f harmonic $\Rightarrow$ $\mathrm{Hopf}(f)$ holomorphic

Let's see now the other statement: $f^*g_N = (f^*g_N)^{(1,1)} + \mathrm{Hopf}(f) + \overline{\mathrm{Hopf}(f)}$

$$= \frac{1}{2}\mathrm{tr}_{g_M}(f^*g_N) + \mathrm{traceless}_{g_M}(f^*g_N)$$

In particular f^*g_N is conformal to g_M $\Leftrightarrow$ $\text{traceless}_{g_M}(f^*g_N) = 0$ $\Leftrightarrow$ $\underset{\substack{\uparrow \\ (2,0)\text{ type}}}{\text{Hopf}(f)} + \underset{\substack{\uparrow \\ (0,2)\text{ type}}}{\overline{\text{Hopf}(f)}} = 0$

$\Rightarrow \text{Hopf}(f) = 0$ $\square$

Existence of harmonic maps

We now want to talk about the following result:

Thm (Eells-Sampson) Let M, N be Riemannian manifolds, with M cpt & N complete and $\sec_N \leq 0$. Then every smooth map $f: M \to N$ is homotopic to an energy-minimizing harmonic map $\hat{f}: M \to N$.

Let's see the general strategy: we saw that $dE_f(V) = -\int_M \langle V, \Delta f \rangle_N \, d\text{vol}_M \quad \forall V \in \Gamma(f^*N)$

In particular the L^2-gradient of E is equal to $-\Delta f$. If we want to minimize the energy, then it is reasonable to try to flow in the direction of $-\text{grad}\, E$. In particular we would like to study the non-linear parabolic PDE

$$\begin{cases} \partial_t f_t = \Delta f_t \\ f_0 = f \end{cases} \qquad \text{Heat flow}$$

Assume for a second that there exists a solution $(f_t)_{t \in [0,\infty)}$ for all positive time. Then

$$\frac{d}{dt} E(f_t) = -\int_M \langle \Delta f_t, \partial_t f_t \rangle_N \, d\text{vol}_M = -\int_M \|\partial_t f_t\|^2 d\text{vol}_M \Rightarrow t \mapsto E(f_t) \text{ is decreasing}$$

$$\Rightarrow \quad E(f_t) - E(f_0) = -\int_0^t \left(\int_M \|\partial_u f_u\|^2 d\text{vol}_M \right) du$$

$\int_0^t \|\partial_u f_u\|_{L^2}^2 \, du = E(f_0) - E(f_t)$ so $t \mapsto \int_0^t \|\partial_u f_u\|_{L^2}^2 \, du$ increasing & bounded above by $E(f_0)$

$$\Rightarrow \quad \exists \lim_{t\to\infty} \int_0^t \|\partial_u f_u\|_{L^2}^2 \, du = \int_0^\infty \|\partial_u f_u\|_{L^2}^2 \, du \leq E(f_0)$$

In particular $\exists (t_n)_n$ sequence s.t. $t_n \to +\infty$ & $\|\partial_t f_{t_n}\|_{L^2}^2 \to 0$. (it suggests the fact that f_t stabilizes...)

Questions

1. How do we see that $(f_t)_t$ is defined for all time?
2. Is it true that $\exists \lim_{t\to\infty} f_t = f_\infty$, with f_∞ harmonic?

The main point is to determine uniform continuity of the maps $(f_t)_t$, when they exist.

Local existence comes from the theory of non-linear parabolic PDE's (i.e. the linearized of $\partial_t - \Delta$ is a parabolic operator).

Assume that $(t_n)_n$ is s.t. $t_n \to \bar{t} \in \mathbb{R}$. Is it true that $(f_{t_n})_n$ converges uniformly (up to subsequence)?

Here is where $\sec_N \leq 0$ is needed. Recall the classical Bochner's formula:

$$\forall f \in \mathcal{C}^\infty(M,\mathbb{R}) \qquad \frac{1}{2}\Delta\|\nabla f\|^2 = \langle \nabla f, \nabla \Delta f\rangle + \|\nabla^2 f\|^2 + Ric_{g_M}(\nabla f, \nabla f)$$

Eells-Sampson generalized this to the case $f \in \mathcal{C}^\infty(M,N)$:

$$\frac{1}{2}\Delta\|\nabla f\|^2 - \langle \nabla f, \nabla \Delta f\rangle_N = \|\nabla^2 f\|^2 + Ric^*_{g_M}(\langle \nabla f \otimes \nabla f\rangle_N) - R_{g_N}(\langle \nabla f \otimes \nabla f\rangle_M, \langle \nabla f \otimes \nabla f\rangle_M)$$

where $\langle \nabla f \otimes \nabla f\rangle_N = tr_{g_N^*}(\nabla f \otimes \nabla f) \in \Gamma(T^*M \otimes T^*M)$

$\langle \nabla f \otimes \nabla f\rangle_M = tr_{g_M}(\nabla f \otimes \nabla f) \in \Gamma(TN \otimes TN)$

Apply this to f_t that satisfies $\partial_t f_t = \Delta f_t$:

$$\begin{aligned} LHS: \frac{1}{2}\Delta\|\nabla f_t\|^2 - \langle \nabla f_t, \nabla \partial_t f_t\rangle_N &= \frac{1}{2}\Delta\|\nabla f_t\|^2 - \langle \nabla f_t, \hat{\nabla}_{\partial_t} \nabla f_t\rangle \\ &= \frac{1}{2}\left(\Delta\|\nabla f_t\|^2 - \partial_t\|\nabla f_t\|^2\right) \\ &= \frac{1}{2}(\Delta - \partial_t)\|\nabla f_t\|^2 \end{aligned}$$

Since M cpt, $\exists C>0$ s.t. $Ric_{g_M^*}(\cdot,\cdot) \geq -C g_M^*(\cdot,\cdot)$, so

$$RHS \geq \underbrace{\|\nabla^2 f_t\|^2}_{\geq 0} - C \underbrace{tr_{g_M}(\langle \nabla f_t \otimes \nabla f_t \rangle_N)}_{\|\nabla f_t\|^2} - \underbrace{R_{g_N}(\langle \dots \rangle, \langle \dots \rangle)}_{\geq 0}$$

$$\geq -C \|\nabla f_t\|^2$$

Therefore the energy density $e_t := \|\nabla f_t\|^2$ satisfies $\frac{1}{2}(\Delta - \partial_t) e_t \geq -C e_t$ for any $(f_t)_t$ satisfying $\partial_t f_t = \Delta f_t$.

If we now apply the maximum principle, we get that

$$-\frac{1}{2} \partial_t e_t(x_{max}) \geq \frac{1}{2}(\Delta - \partial_t) e_t \Big|_{x_{max}} \geq -C \max_M e_t \Rightarrow \partial_t e_t(x_{max}) \leq C \cdot \max_M e_t$$

In particular this "implies" (there is work here) that $\max_M e_t \leq \max_M e_0 \cdot e^{2Ct}$

This is sufficient to show that $(f_t)_t$ is defined for all $t \in \mathbb{R}$, since it gives us

$$\|\nabla f_t\|_\infty \leq \|\nabla f_0\|_\infty e^{2Ct}$$

So $(f_t)_t$ are equicontinuous on $[0,T]$ $\forall T>0$. However, this does not guarantee the existence of $\lim_{t\to\infty} f_t$, since this bound gets worse as t increases.

Here we need something more sophisticated: subsolutions of the heat equation satisfy a generalized maximum principle, which in fact implies:

$$\exists C = C(M, g_M) > 0 \text{ s.t. } \quad \|\nabla f_t\|_\infty \leq \underline{C\, E(f_0)} \quad \forall t \text{ where the flow is defined}$$

this is independent of t!

In fact, the hypothesis $sec_N \leq 0$ tells us also that $\frac{d^2}{dt^2} E(f_t) \geq 0$ (E is convex). The combination of these facts allow to prove that

$$\exists \lim_{t \to \infty} f_t = \hat{f} \quad , \text{with } \hat{f} \text{ harmonic}$$

(Here there is also more analysis involved, but now it reduces to more standard elliptic regularity theory...)

What does Donaldson-Corlette's thm say?

Let X be a Riemann surface and let G be a reductive cplx algebraic group. For simplicity let's take $G = GL(n,\mathbb{C})$.

<u>Def</u> $E \to X$ vector bundle, ∇ connection on E. (E,∇) is flat if $d_\nabla^2 \equiv 0$

<u>Fact</u> (E,∇) flat $\Leftrightarrow$ $(E,\nabla) \cong (X \times_\rho \mathbb{C}^n, \hat{\nabla})$ where
$\operatorname{rk} E = n$

- $\rho : \pi_1(X) \to GL_n(\mathbb{C})$ representation
- $X \times_\rho \mathbb{C}^n = \tilde{X} \times \mathbb{C}^n \big/ (\tilde{x},v) \sim (\gamma\tilde{x}, \rho(\gamma)v)$
- $\hat{\nabla}$ is the connection obtained from the trivial one on $\tilde{X} \times \mathbb{C}^n$ and going to the quotient

Let now h be a Hermitian metric on (E,∇). Any connection can be decomposed into $\nabla = \nabla_h + \Psi_h$, where

- ∇_h is a unitary connection compatible with h
- $\Psi_h \in \Omega^1(X; \operatorname{End}(E))$ is s.t. $\Psi_h^{*h} = \Psi_h$ adjoint wrt h

Let now $\mathbb{X}_G := G/K = GL_n(\mathbb{C})/U(n)$. The choice of a Hermitian metric describes a reduction of the structure group, and induces a ρ-equivariant classifying map $f : \tilde{X} \to \mathbb{X}_G$.

$$\rightsquigarrow \qquad df \in \Omega^1(\tilde{X}, f^* T\mathbb{X}_G), \text{ and } T\mathbb{X}_G \cong \mathrm{End}_{sym}(E)$$

Under this identification one sees that $\Psi_h = df$, and ∇_h = pull-back of the Levi-Civita of $\mathbb{X}_G$.

<u>Def</u> A Hermitian metric h on a flat bundle (E,∇) is harmonic if one of the following equivalent conditions is satisfied:

i) h is a critical pt of the energy functional

$$E(h) = \frac{1}{2} \int_M \|\Psi_h\|^2 \, d\mathrm{vol}_M$$

ii) $d^*_{\nabla_h} \Psi_h \equiv 0$

iii) The classifying map $f : \tilde{X} \to \mathbb{X}_G$ is harmonic

<u>Thm</u> (Corlette) Let M cpt Riemannian manifold, let $\mathbb{X} = G/K$ be a symmetric space of non-cpt type and let $\rho : \pi_1 M \to G$ be a group homomorphism. Then $\exists f : \tilde{M} \to \mathbb{X}$ ρ-equivariant harmonic map $\Leftrightarrow$ ρ is reductive, i.e. (when $G = GL_n\mathbb{C}$) completely reducible

For what happens here, let me assume $G = SL_n\mathbb{C}$ instead of $GL_n\mathbb{C}$

Lemma 1 Let $\rho : \pi_1(X) \to SL_n\mathbb{C}$ be a representation. Then $\exists$ sequence $f_j : \tilde{X} \to \mathbb{X}_n$ of smooth functions s.t.

i) f_j is energy minimizing

ii) the family (f_j) is uniformly Lipschitz

iii) $\Delta f_j \to 0$ in L^2

$\mathbb{X}_n := SL_n\mathbb{C}/SU(n)$

Lemma 2 Let $\rho : \pi_1(X) \to SL_n\mathbb{C}$ be a irreducible representation, and let $f_j : \tilde{X} \to \mathbb{X}_n$ be uniformly Lipschitz. Then $f_j(\tilde{p})$ is bounded.

Proof Let $h_j := f_j(\tilde{p}) \in \mathbb{X}_n = \{ A \in M(n\times n;\mathbb{C}) \mid \bar{A}^t A = I, A > 0 \text{ definite}, \det A = 1 \}$

in particular every $A \in \mathbb{X}_n$ determines a hermitian scalar product $\langle A\cdot,\cdot\rangle_{\mathbb{C}^n}$

Assume by contradiction that $\exists (\varepsilon_j)_j$ s.t. $\varepsilon_j h_j \to h_\infty \neq 0$ (up to subsequence)

$\det h_\infty = \lim_{j\to\infty} \varepsilon_j^n \underbrace{\det h_j}_{=1} = 0$, so $V := \operatorname{Ker} h_\infty$ is a proper subspace

Claim ρ preserves V

Proof Let $\gamma \in \pi_1(X)$ & $v \in V$.

$d_{\mathbb{X}_n}(f_j(\tilde{p}), \rho(\gamma) f_j(\tilde{p})) = d_{\mathbb{X}_n}(f_j(\tilde{p}), f_j(\gamma\cdot\tilde{p})) \leq M \cdot d(\tilde{p}, \gamma\cdot\tilde{p})$ in particular it is unif bounded in $j \in \mathbb{N}$

Therefore $\forall w \in V$ $\exists B = B_w > 0$ s.t. $|\langle h_j v, w\rangle_{\mathbb{C}^n} - \langle \overline{\rho(\gamma^{-1})}^t h_j \rho(\gamma^{-1}) v, w\rangle_{\mathbb{C}^n}| \leq B$ $\forall j$

$\Rightarrow$ $|\langle \varepsilon_j h_j v, w\rangle_{\mathbb{C}^n} - \langle \varepsilon_j h_j \rho(\gamma^{-1}) v, \rho(\gamma^{-1}) w\rangle_{\mathbb{C}^n}| \leq \varepsilon_j B$ $\forall j$

$\downarrow j \to \infty$

$\langle h_\infty v, w\rangle = \langle h_\infty \rho(\gamma^{-1}) v, \rho(\gamma^{-1}) w\rangle$ $\forall w \in \mathbb{C}^n$ $\Rightarrow$ $\underbrace{h_\infty v}_{=0} = \underbrace{\overline{\rho(\gamma^{-1})}^t}_{\text{invertible}} h_\infty \rho(\gamma^{-1}) v$

$\Rightarrow h_\infty \rho(\gamma^{-1}) v = 0 \Rightarrow \rho(\gamma^{-1}) v \in \operatorname{Ker} h_\infty = V$ $\square$

This concludes the proof of the proposition $\square$

Parametrizing Teich(Σ) via harmonic maps

Let X be a Riemann surface, and let $\mathrm{Teich}(\Sigma) := \{h \text{ hyperbolic metric on } \Sigma\} / \mathrm{Diffeo}_0(\Sigma)$

By Eells-Sampson we know that $\forall h$ hyperbolic metric $\exists f: X \to (\Sigma, h)$ harmonic map isotopic to id_Σ (fix a homeo $X \simeq \Sigma$ once and for all).

There are a few other properties that hold in this setting:

Theorem (Hartman, '67) Let M, N be Riemannian mflds s.t. M cpt & N complete with $\sec_N < 0$.
Let $f: M \to N$ be a harmonic map. Then f is the unique harmonic map in its homotopy class if and only if $f(M)$ is not a pt nor a geodesic.

Theorem (Sampson '78) Let X, Y be two Riemannian cpt surfaces with same genus, and let $f: X \to Y$ be a harmonic map of degree 1. Then $\det(df) > 0$ everywhere.

In particular $f: X \to (\Sigma, h)$ is unique, and it is an orientation-preserving diffeo!

We saw before that $\mathrm{Hopf}(f)$ is a holomorphic quadratic differential, and that

$$f^*h = \underbrace{\mathrm{Hopf}(f)}_{(2,0)} + \underbrace{(f^*h)^{(1,1)}}_{\substack{\uparrow \\ \text{in the conformal} \\ \text{class of } X}} + \underbrace{\overline{\mathrm{Hopf}(f)}}_{(0,2)}$$

Let w be a local holomorphic chart on (Σ, h), so that $h = \rho \cdot |dw|^2$. Then

$$f^*h = \rho |d^{1,0}f + d^{0,1}f|^2 = \rho \left(d^{1,0}f \cdot \overline{d^{0,1}f} + |d^{1,0}f|^2 + |d^{0,1}f|^2 + d^{0,1}f \, \overline{d^{1,0}f} \right)$$
$$= \underbrace{\rho f_z \overline{f_{\bar z}} \, dz^2}_{\mathrm{Hopf}(f)} + \rho \left(|f_z|^2 + |f_{\bar z}|^2 \right) |dz|^2 + \underbrace{\rho \overline{f_z} f_{\bar z} \, d\bar z^2}_{\overline{\mathrm{Hopf}(f)}}$$

Let now $g = \sigma |dz|^2$ be a metric in the conformal class of X

$$e_f := \frac{1}{2}\|df\|^2 = \frac{1}{2}\mathrm{tr}_g(f^*h) = \frac{\rho\circ f}{\sigma}\left(|f_z|^2 + |f_{\bar z}|^2\right)$$

$\mathrm{Hopf}(f) \otimes \overline{\mathrm{Hopf}(f)}$
traceless

Hence $\quad f^*h = \mathrm{Hopf}(f) + e_f \cdot g + \overline{\mathrm{Hopf}(f)}$

$$\|\mathrm{Hopf}(f)\|^2 := \frac{(\rho\circ f)^2}{\sigma^2}|f_z|^2|f_{\bar z}|^2$$

Using Wolf's notation, we also set $\mathcal{H} := \|d^{1,0}f\|^2 = \frac{\rho\circ f}{\sigma}|f_z|^2$, $\mathcal{L} := \|d^{0,1}f\|^2 = \frac{\rho\circ f}{\sigma}|f_{\bar z}|^2$

$$\|\mathrm{Hopf}(f)\|^2 = \mathcal{H}\cdot\mathcal{L}, \quad e_f = \mathcal{H} + \mathcal{L}, \quad \mathcal{J} := \det(\mathrm{Jac}(f)) = \frac{\rho\circ f}{\sigma}\left(|f_z|^2 - |f_{\bar z}|^2\right) = \mathcal{H} - \mathcal{L}$$

By the aforementioned results $\mathcal{J} > 0$, so $\mathcal{H} > \mathcal{L} \geqslant 0$.

The Beltrami differential of f is $\nu_f := \frac{f_{\bar z}}{f_z}\,\partial_z \otimes d\bar z = \frac{f_{\bar z}\cdot \overline{f_z}}{|f_z|^2}\,\partial_z \otimes d\bar z = \mathcal{H}^{-1} g^{-1}\,\overline{\mathrm{Hopf}(f)}$

From the equation $\Delta f = 0$ & $K(h) = -1$, we see that the following relations hold:

$$\begin{cases} \Delta \log \mathcal{H} = 2(\mathcal{H} - \mathcal{L}) + 2K(g) \\ \Delta \log \mathcal{L} = 2(\mathcal{L} - \mathcal{H}) + 2K(g) \quad \text{where } \mathcal{L} \neq 0. \end{cases}$$

We are free to choose $g = \sigma|dz|^2$ of curvature -1. We also set $u := \log \mathcal{H}$, and notice that $\mathcal{L} = \mathcal{L}\mathcal{H}\cdot\mathcal{H}^{-1} = e^{-u}\|\mathrm{Hopf}(f)\|^2$. In particular, the system above becomes

$$\begin{cases} \frac{1}{2}\Delta u = e^{u} - e^{-u}\|\mathrm{Hopf}(f)\|^2 - 1 \\ \frac{1}{2}\Delta\left(\log\|\mathrm{Hopf}(f)\| - u\right) = e^{-u}\|\mathrm{Hopf}(f)\|^2 - e^{u} - 1 \quad \text{where } \mathrm{Hopf}(f) \neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{2}\Delta u = e^{u} - e^{-u}\|\mathrm{Hopf}(f)\|^2 - 1 \\ \frac{1}{2}\Delta \log\|\mathrm{Hopf}(f)\|^2 = -2 \quad \text{where } \mathrm{Hopf}(f) \neq 0. \end{cases}$$

We now have all the elements to state and prove:

<u>Thm</u> (Wolf) Let X be a Riemann surface. Then the map

$$\Phi : \mathrm{Teich}(\Sigma) \longrightarrow QD(X)$$
$$[h] \longmapsto \mathrm{Hopf}(f_h), \quad f_h : X \to (\Sigma, h) \text{ harmonic diffeo} \sim id.$$

is a well-defined global diffeomorphism.

<u>Proof</u> STEP 0) well defined: let $h' := u^* h$ for some $u \in \mathrm{Diff}_0(\Sigma)$. Then $u : (\Sigma, h') \to (\Sigma, h)$ is an isometry isotopic to id_Σ. If $f_h : X \to (\Sigma, h)$ is the unique harmonic diffeo $\sim id$, then $u^{-1} \circ f_h : X \to (\Sigma, h')$ is also a harmonic diffeo $\sim id$ $\Rightarrow u^{-1} \circ f_h = f_{u^* h}$

Therefore $(f_{u^*h})^*(u^* h) = (u^{-1} \circ f_h)^*(u^* h) = (u \circ u^{-1} \circ f_h)^* h = f_h^* h$

This implies in particular that $\mathrm{Hopf}(f_{u^*h}) = (f_{u^*h})^*(u^* h)^{(2,0)} = (f_h)^* h^{(2,0)} = \mathrm{Hopf}(f_h)$

So Wolf's map Φ is well defined.

STEP 1) Φ is injective

Let $[h], [h'] \in \mathrm{Teich}(\Sigma)$ be such that $\mathrm{Hopf}(f_h) = \mathrm{Hopf}(f_{h'}) = q$

Recall that

$$f_h^* h = \mathrm{Hopf}(f_h) + e_{f_h} \cdot g + \overline{\mathrm{Hopf}(f_h)}$$
$$= q + e_{f_h} \cdot g + \bar{q}$$

On the other hand, we know that $\mathcal{H} \cdot \mathcal{L} = \|q\|^2 = \mathcal{H}' \mathcal{L}'$, where $\mathcal{H} = \mathcal{H}_{f_h}$, $\mathcal{H}' = \mathcal{H}_{f_{h'}}, \dots$

If we have $\mathcal{H} = \mathcal{H}'$, then $\mathcal{H}\mathcal{L} = \mathcal{H}'\mathcal{L}'$ implies $\mathcal{L} = \mathcal{L}'$, and so $e_{f_h} = \mathcal{L} + \mathcal{H} = \mathcal{L}' + \mathcal{H}' = e_{f_{h'}}$

This proves that $f_h^* h = f_{h'}^* h' \Rightarrow u := f_{h'} \circ f_h^{-1} : (\Sigma, h) \to (\Sigma, h')$ is an isometry isotopic to $id_\Sigma \Rightarrow [h] = [h'] \in \mathrm{Teich}(\Sigma)$.

So it is sufficient to prove

<u>Claim</u> $\mathcal{H} = \mathcal{H}'$

<u>Proof</u> Let $u := \log \mathcal{H}$, $u' := \log \mathcal{H}'$. We know that

$$\frac{1}{2}\Delta u = e^{u} - e^{-u}\|q\|^2 - 1 \quad \& \quad \frac{1}{2}\Delta u' = e^{u'} - e^{-u'}\|q\|^2 - 1$$

$$\Rightarrow \frac{1}{2}\Delta(u-u') = (e^{u} - e^{u'}) + \|q\|^2(e^{-u'} - e^{u})$$

Take $p_+ \in \Sigma$ s.t. $u(p_+) - u'(p_+) = \max_{\Sigma}(u-u')$. If $u(p_+) - u'(p_+) \geq 0$, then

$$0 \geq \frac{1}{2}\Delta(u-u')\Big|_{p^+} = \underbrace{\left(e^{u(p^+)} - e^{u'(p^+)}\right)}_{>0} + \|q\|^2 \underbrace{\left(e^{-u'(p^+)} - e^{-u(p^+)}\right)}_{>0} > 0 \quad \underline{\text{absurd}}$$

So $u \leq u'$ everywhere. However, we can repeat the exact same argument to $u'-u$ and get $u' \leq u \Rightarrow u \equiv u' \Rightarrow \mathcal{H} = \mathcal{H}'$, as desired $\square$

This concludes STEP 1.

STEP 2) Φ is <u>continuous</u> & proper.

follows from the continuous dep of sol's of PDE $\Delta f = 0$

<u>Claim</u>: $\|\mathrm{Hopf}(f_n)\|_{L^1} \to \infty \Leftrightarrow E(f_n) \to \infty$

<u>Proof</u> We will prove a comparison between $\|\mathrm{Hopf}(f_n)\|_{L^1} = \int_{\Sigma} \mathcal{H}\cdot\mathcal{L}\, d\mathrm{vol}_g$ & $E(f_n) = \int (\mathcal{H}+\mathcal{L})\, d\mathrm{vol}_g$. Recall that $\mathcal{J} = \mathcal{H} + \mathcal{L}$, so

$$\int_{\Sigma} \mathcal{H}\, d\mathrm{vol}_g - \underbrace{\int_{\Sigma} \mathcal{J}\, d\mathrm{vol}_g}_{-2\pi\chi(\Sigma)} = \int_{\Sigma} \mathcal{L}\, d\mathrm{vol}_g = \int_{\Sigma} \underbrace{\frac{\mathcal{L}^{1/2}}{\mathcal{H}^{1/2}}}_{\|\nu_f\|} \cdot \underbrace{(\mathcal{L}\mathcal{H})^{1/2}}_{\|\mathrm{Hopf}(f_n)\|} d\mathrm{vol}_g \leq \underbrace{\int_{\Sigma} \|\mathrm{Hopf}(f_n)\|\, d\mathrm{vol}_g}_{= \|\mathrm{Hopf}(f_n)\|_{L^1}}$$

Beltrami diff < 1

We saw that $\nu_f = \mathcal{H}^{-1} g^{-1} \mathrm{Hopf}(f_h)$, which implies $\|\mathrm{Hopf}(f_h)\| = \mathcal{H} \cdot \|\nu_f\| < \mathcal{H}$, so

$$\|\mathrm{Hopf}(f_h)\|_{L^1} \leq \int_\Sigma \mathcal{H}\, d\mathrm{vol}_g = \underbrace{\int_\Sigma (\mathcal{H} - \mathcal{L})\, d\mathrm{vol}_g}_{-2\pi\chi(\Sigma)} + \int_\Sigma \mathcal{L}\, d\mathrm{vol}_g$$

Combining these inequalities we get

$$\int_\Sigma \mathcal{H} d\mathrm{vol}_g + 2\pi\chi(\Sigma) + \int_\Sigma \mathcal{L}\, d\mathrm{vol}_g \leq 2 \cdot \|\mathrm{Hopf}(f_h)\|_{L^1} \leq \int_\Sigma \mathcal{H}\, d\mathrm{vol}_g + \left(-2\pi\chi(\Sigma) + \int_\Sigma \mathcal{L}\, d\mathrm{vol}_g\right)$$

$$E(f_h) + 2\pi\chi(\Sigma) = \qquad\qquad = \int_\Sigma e_{f_h}\, d\mathrm{vol}_g - 2\pi\chi(\Sigma)$$

$$= E(f_h) - 2\pi\chi(\Sigma)$$

$$\Longrightarrow E(f_h) + 2\pi\chi(\Sigma) \leq \|\mathrm{Hopf}(f_h)\|_{L^1} \leq E(f_h) - 2\pi\chi(\Sigma)$$

This concludes the proof of the claim $\square$

So the properness of Φ is equiv to the properness of
$$\mathrm{Teich}(\Sigma) \longrightarrow \mathbb{R}$$
$$[h] \longmapsto E(f_h)$$

We need to prove that $B := \{[h] \in \mathrm{Teich}(\Sigma) \mid E(f_h) \leq K\}$ is cpt $\forall K > 0$.

Let $\ell_h(\gamma) :=$ length of the h-geod representative of γ. We claim that $\exists D = D(g) > 0$ such that $\forall \gamma \in \pi_1(\Sigma)$

$$(\star) \quad \ell_h(\gamma) \leq D \cdot K^{1/2}\, \ell_g(\gamma)$$

hyp metric conformal to X

this is enough to conclude, since $\exists$ finitely many γ_i's s.t. $\sum_i \ell_\bullet(\gamma_i)$ is a proper function on $\mathrm{Teich}(\Sigma)$

Let's prove $(\star)$

<u>Proof</u> This reduces to the Courant–Lebesgue lemma, which provides equicontinuity on the maps f_h associated to $[h] \in B$. More precisely

Lemma $\exists\, C = C(g) > 0$ s.t. $\forall \delta > 0$ $\forall p, q \in \Sigma$: $d_{\tilde g}(p,q) < \delta < C$ we have

$$d_{\tilde h}(\tilde f_h(p), \tilde f_h(q)) \leq 4\sqrt{2}\pi K^{1/2} \left(\log \frac{1}{\delta}\right)^{-1/2}$$

lifts to univ cover $\tilde\Sigma \to \Sigma$

First let's see how to conclude from here: we select δ_0 s.t.

$$\delta_0^{1/2} < \min\left\{ \frac{sys(\Sigma, g)}{sys(\Sigma, g) + 1}, C, 1 \right\}$$

and we pick $\tilde D = \tilde D(\delta_0)$ s.t. $\left(\log \frac{1}{x}\right)^{-1/2} \leq \tilde D \cdot x \quad \forall x \in [\delta_0, \delta_0^{1/2}]$

Since $\delta_0^{1/2} < \frac{sys_g}{sys_g + 1}$ one sees that $\forall \gamma \in \pi_1(\Sigma)$ $\exists N_\gamma \in \mathbb{N}$ s.t. $\delta_0 < \frac{L_g(\gamma)}{N} < \delta_0^{1/2} < C$

and set $t_i := i \cdot \frac{L_g(\gamma)}{N_\gamma}$, $i = 0, \dots, N_\gamma$. If $\alpha : [0, L_g(\gamma)] \to \Sigma$ parametrizes a lift of the g-geodesic representative of γ, then

$$\delta_0 < d_{\tilde g}(\alpha(t_i), \alpha(t_{i+1})) = \frac{L_g(\gamma)}{N_\gamma} < \delta_0^{1/2} < C$$

If we apply the claim with $\delta = \delta_0^{1/2}$

$$\begin{aligned} d_{\tilde h}(\tilde f_h(\alpha(t_i)), \tilde f_h(\alpha(t_{i+1}))) &\leq 4\sqrt{2}\pi K^{1/2} \left(\log \frac{1}{\delta_0^{1/2}}\right)^{-1/2} \\ &= 4\sqrt{2}\pi K^{1/2} \sqrt{2} \left(\log \frac{1}{\delta_0}\right)^{-1/2} \\ &\leq 8\pi K^{1/2} \tilde D \delta_0 \\ &< 8\pi K^{1/2} \tilde D \cdot \frac{L_g(\gamma)}{N_\gamma} \end{aligned}$$

Take now the piecewise $\tilde h$-geodesic β obtained by joining $\tilde f_h(\alpha(t_i))$ & $\tilde f_h(\alpha(t_{i+1}))$ $\forall i$, whose length satisfies

$$\begin{aligned} L_{\tilde h}(\beta) &= \sum_{i=0}^{N_\gamma - 1} d_{\tilde h}(\tilde f_h(\alpha(t_i)), \tilde f_h(\alpha(t_{i+1}))) \\ &< \cancel{N_\gamma} \cdot 8\pi K^{1/2} \tilde D \frac{L_g(\gamma)}{\cancel{N_\gamma}} = \underbrace{8\pi K^{1/2} \tilde D}_{=: C(g)} L_g(\gamma) \end{aligned}$$

The proj of β on Σ is in the homotopy class of $(f_h)_* \gamma = \gamma$, so

$$L_h(\gamma) \leq L_{\tilde{h}}(\beta) < C(g) \cdot K^{1/2} L_g(\gamma)$$

Hence it is enough to prove the Lemma

<u>Proof of the Lemma</u>

First notice that, up to replacing h with $f_h^* h$, we can assume $f_h = id_\Sigma$

Now pick $p, q \in \tilde{\Sigma}$, $\delta > 0$ s.t.

$$d_{\tilde{g}}(p, q) < \delta < \min\{1, \operatorname{injrad}(g)^2\}$$

We can find coord's centered at p_0 s.t. $\tilde{g} = dr^2 + \sinh^2 r \, d\theta$ (g is hyperbolic)

If $p', q' \in \partial B(p_0, r)$, then

$$d_{\tilde{g}}(p', q') \leq \int_0^{2\pi} \left\| \frac{\partial}{\partial\theta} \right\| d\theta \leq (2\pi)^{1/2} \left(\int_0^{2\pi} \left\| \frac{\partial}{\partial\theta} \right\|^2 d\theta \right)^{1/2}$$

Let $A = A(x_0, \delta, \delta^{1/2})$ be the annulus centered at x_0 of inner radius δ and outer radius $\delta^{1/2}$, distances being measured in the g-metric.

$$K \geq E(g, h) \geq \int_\Sigma \| d(id) \|^2 dvol_g$$

$$\geq \frac{1}{2} \int_A \left(\left\| \frac{\partial}{\partial r} \right\|_h^2 + \frac{1}{\sinh^2 r} \left\| \frac{\partial}{\partial\theta} \right\|_h^2 \right) \sinh r \, d\theta \, dr$$

$$\geq \frac{1}{2} \int_A \left\| \frac{\partial}{\partial\theta} \right\|_\rho^2 d\theta \frac{dr}{\sinh r}$$

For $r < \delta^{1/2} < 1$, we have $\sinh r < 2r$, so $\displaystyle\int_\delta^{\delta^{1/2}} \frac{dr}{\sinh r} > \frac{1}{4} \log \frac{1}{\delta}$

Hence we can find some $\hat{R} \in (\delta, \delta^{1/2})$ s.t.

$$\int_0^{2\pi} \left\| \frac{\partial}{\partial\theta} \right\|_h^2 \Big|_{\hat{r}} d\theta \leq \frac{2K}{\frac{1}{4} \log \frac{1}{\delta}}$$

Since p,q are s.t. $d_{\tilde{g}}(p,q) < \delta$, then we can find p_0 s.t. $p,q \in B(p_0,\delta)$

Moreover, $\exists\, p',q' \in \partial B(p_0,\hat{R})$ s.t. $d_{\tilde{h}}(p,q) \leq d_h(p',q') \leq (2\pi)^{1/2}\left(\dfrac{2K}{\frac{1}{4}\log\frac{1}{\delta}}\right)^{1/2}$

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