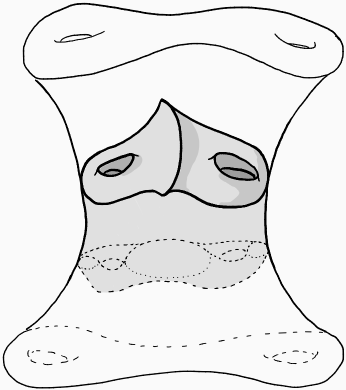
Constant Gaussian curvature surfaces in hyperbolic 3-manifolds

Filippo Mazzoli (University of Virginia)

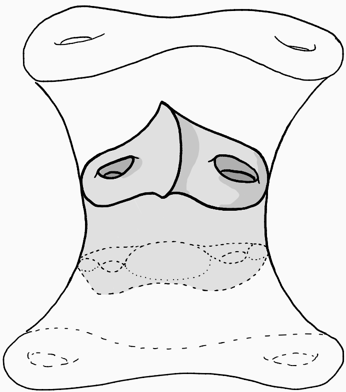
November 3^{rd}, 2020

Pangolin Seminar



M quasi-Fuchsian

Sketch



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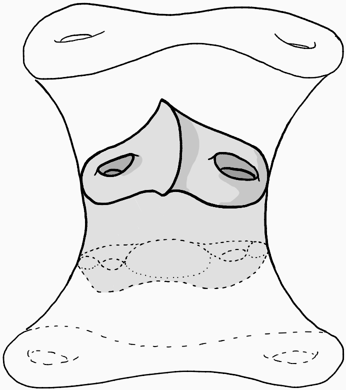

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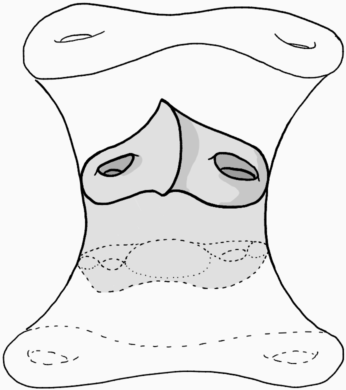

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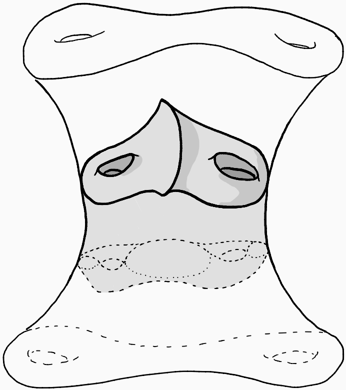

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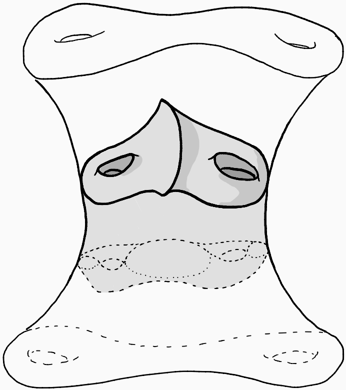

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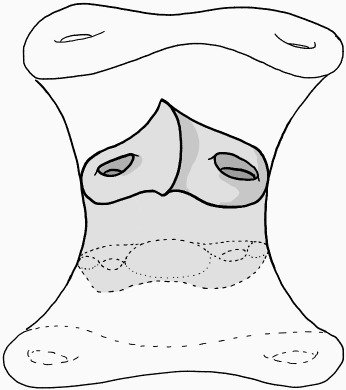

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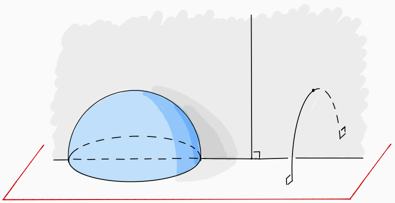
Preliminaries

$\mathbb{H}^n \cong (H^n, h)$, with

$$H^n := \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid t > 0\},$$

$$h = \frac{|\mathrm{d}x|^2 + \mathrm{d}t^2}{t^2}.$$



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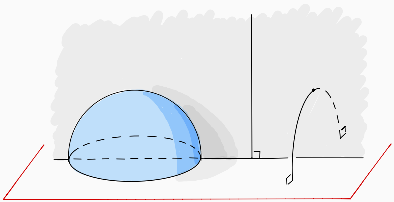

$$\partial_\infty \mathbb{H}^2 \cong \mathbb{R} \cup \infty = \mathbb{RP}^1, \quad \partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \infty = \mathbb{CP}^1$$

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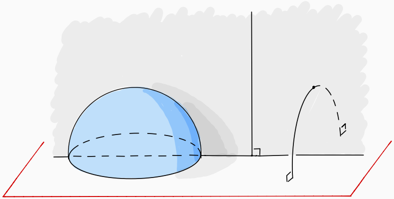

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$$\mathsf{Iso}^+(\mathbb{H}^2) \cong \mathsf{Aut}(\mathbb{RP}^1) = \mathbb{P}\,\mathsf{SL}_2\,\mathbb{R},$$
$$\mathsf{Iso}^+(\mathbb{H}^3) \cong \mathsf{Aut}(\mathbb{CP}^1) = \mathbb{P}\,\mathsf{SL}_2\,\mathbb{C},$$

where

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot z := \frac{\alpha z + \beta}{\gamma z + \delta}.$$

A **hyperbolic manifold** is a $M = {}^{\mathbb{H}^n}\!/_{\Gamma}$, with $\Gamma \overset{\text{d. t.}}{<} \mathrm{Iso}^+(\mathbb{H}^n)$.

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Examples for $n = 2$

Let Σ be a oriented surface of genus ≥ 2. $\forall c$ conformal str

$$(\widetilde{\Sigma}, \tilde{c}) \cong H^2, \qquad \text{(Unif. thm)}$$

and $\exists \rho : \pi_1\Sigma \to \Gamma \overset{\text{d. t.}}{<} \mathrm{Aut}(H^2) = \mathbb{P}\,\mathrm{SL}_2\,\mathbb{R}$.

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and $\exists \rho : \pi_1\Sigma \to \Gamma \overset{\text{d. t.}}{<} \mathrm{Aut}(H^2) = \mathbb{P}\,\mathrm{SL}_2\,\mathbb{R}$. Therefore

$$(\Sigma, c) \text{ Riemann surface} \overset{\text{Unif. thm}}{\leftrightsquigarrow} (\Sigma, h) \text{ hyperbolic surface}$$

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$\mathcal{T} \cong_{\text{homeo}} \mathbb{R}^{6(g-1)}$ is Kähler, distances d_{WP}, $d_{\mathcal{T}}$, $d_{\mathbf{Th}}$...

$$\mathcal{M} := {}^{\mathcal{T}}\!/_{\mathcal{MCG}}. \qquad \text{(Moduli space)}$$

Quasi-Fuchsian manifolds

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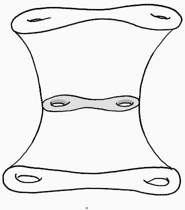
$$M_0 := {}^{\mathbb{H}^3}\!\big/_{\rho_0(\pi_1\Sigma)}. \qquad \text{(Fuchsian mfd)}$$



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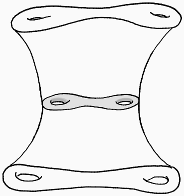

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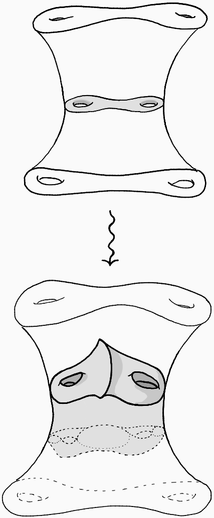
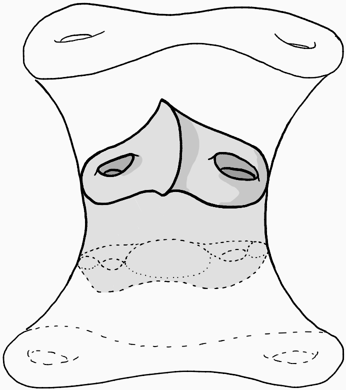


Now $\rho_0 \rightsquigarrow (\rho_t)_t$, with $\rho_t(\pi_1\Sigma)$ discrete and ρ_t injective. Then

$$M_t := {}^{\mathbb{H}^3}\!/_{\rho_t(\pi_1\Sigma)}$$

is a **quasi-Fuchsian mfd** (for t small).

Sketch



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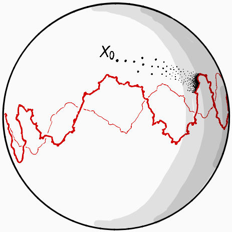

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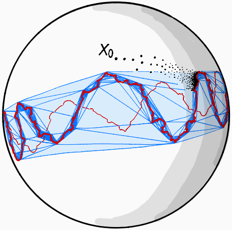

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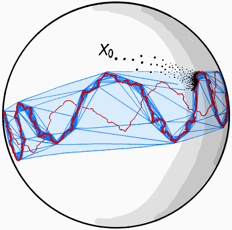

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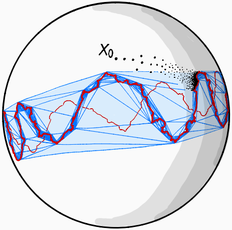

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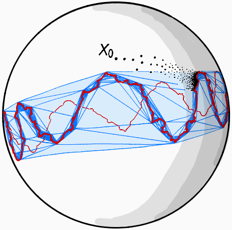

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$C(M)$ is either a hyperbolic surface (if M Fuchsian), or is homeo $\Sigma \times [-1, 1]$.

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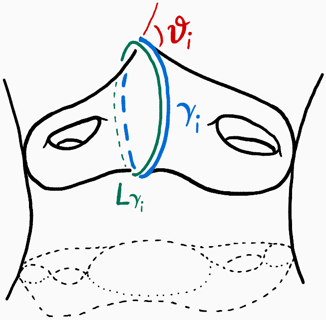

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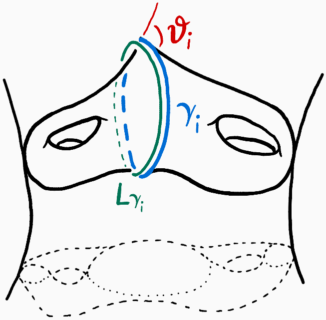

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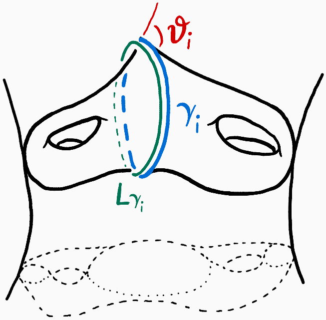
hyperbolic length

$$L_\mu(m) := \sum_i \boldsymbol{\vartheta}_i \cdot L_{\gamma_i}(m)$$



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- $c_\infty^\pm$ conformal structures on $\partial_\infty M$;
- $q_\infty^\pm$ Schwarzian at infinity: given $f_\pm : \Omega(\Gamma)^\pm \to \Delta$ uniformization map, define

$$\tilde{q}_\infty^\pm := \left(\left(\frac{f''_\pm}{f'_\pm}\right)' - \frac{1}{2}\left(\frac{f''_\pm}{f'_\pm}\right)^2\right) \mathrm{d}z^2 .$$

∂CM vs $\partial_\infty M$

On ∂CM	On $\partial_\infty M$
$m^\pm \in \mathcal{T}$	$c_\infty^\pm \in \mathcal{T}$
$\mu^\pm \in \mathcal{ML}$	$\mathcal{F}_\infty^\pm = \mathrm{Hor}(q_\infty^\pm) \in \mathcal{MF}$
$L_\mu(m)$	$\mathrm{ext}_{\mathcal{F}_\infty}(c_\infty)$

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Conjecture (Thurston)

$$\text{Is} \quad \mathcal{QF}(\Sigma) \ni M \mapsto (m^+, m^-) \in \mathcal{T}^\natural \times \mathcal{T}^\natural \quad \text{a homeo ?}$$

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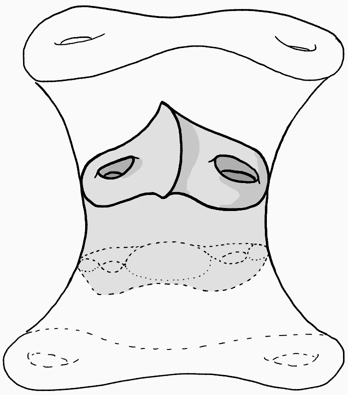

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k-SURFACES

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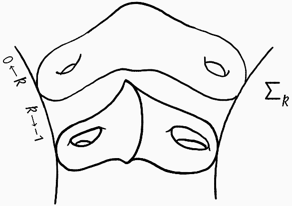
Theorem (Labourie '91)

Let M be q-Fuchsian. Then $M \setminus CM$ is foliated by k-surfaces $\Sigma_k = \Sigma_k^+ \sqcup \Sigma_k^-$, with $k \in (-1, 0)$.



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Fact : III_k has constant curvature too!

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$$(\Sigma_k, I_k) \xleftarrow{id} (\Sigma_k, II_k) \xrightarrow{id} (\Sigma_k, III_k)$$

are harmonic and

$$q_k := +\,\mathrm{Hopf}(id\colon II_k \to I_k) \doteq -\,\mathrm{Hopf}(id\colon II_k \to III_k) \in \mathcal{QD}(\Sigma_k, [II_k]).$$

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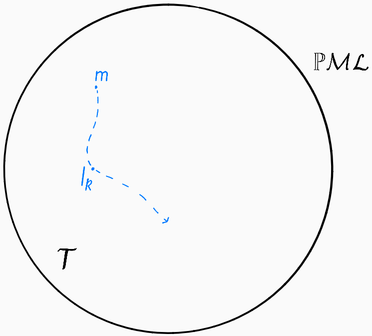
Rmk: All statements are ok up to multiply by the correct function of k...

Fix a component of $M \setminus CM$.

$$l_k \xrightarrow{k \to -1} m$$



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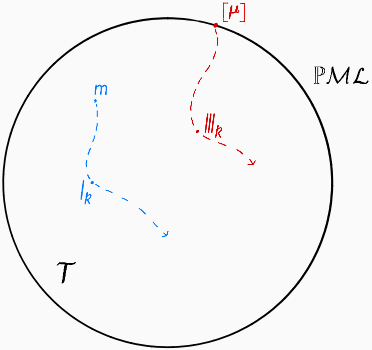
$$|||_k \xrightarrow{k \to -1} [\mu] \qquad \text{(Belraouti '17)}$$



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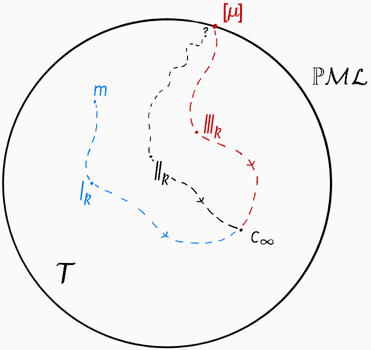

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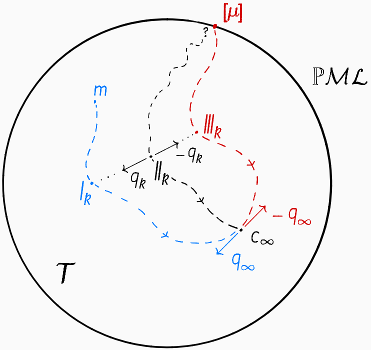
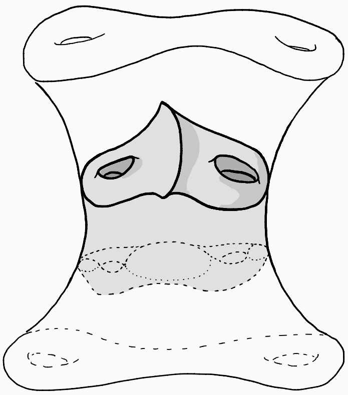
$\hat{q}_k \xrightarrow{k\to 0} q_\infty$ (Quinn '20)



On ∂CM	$-1 \leftarrow k$	On ∂M_k	$k \to 0$	On $\partial_\infty M$
$m \in \mathcal{T}$		I_k & c_k		$c_\infty \in \mathcal{T}$
$\mu \in \mathcal{ML}$		III_k & $\mathcal{F}_k = \mathrm{Hor}(q_k)$		$\mathcal{F}_\infty \in \mathcal{MF}$
$L_\mu(m)$		$\int H_k$ & $\mathrm{ext}_{\mathcal{F}_k}(c_k)$		$\mathrm{ext}_{\mathcal{F}_\infty}(c_\infty)$



M quasi-Fuchsian

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Volumes

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Let $M \in \mathcal{QF}(\Sigma)$. Notion of V^* related to the duality $\mathbb{H}^3 \leftrightarrow \mathrm{d}\mathbb{S}^3$.

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Given $N \subset M$ convex with regular ∂,

$$V^*(N) := V(N) - \frac{1}{2} \int_{\partial N} H \, \mathrm{d}a \,,$$
$$W(N) := \frac{(V + V^*)(N)}{2} = V(N) - \frac{1}{4} \int_{\partial N} H \, \mathrm{d}a \,.$$

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$t \mapsto W(N_t)$ grows linearly in t (not exponential as V or $-V^*$).

Renormalized volume: motivated by AdS/CFT correspondence (Witten, Graham). Krasnov and Schlenker studied it for convex co-compact hyp 3-manifolds, highlighted its geometric properties.

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Dual volume of the convex core: $V_C^*(M) := V(M) - \frac{1}{2} L_\mu(m)$.

$$\mathrm{d}V_C^* = -\frac{1}{2}\,\mathrm{d}L_\mu(\dot{m})$$ (Krasnov-Schlenker '09, M '18)

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Proposition (M)

$$\mathrm{d}W_k(\dot{M}) = -\operatorname{Re}\langle q_k, \dot{c}_k\rangle = -\frac{1}{2}\,\mathrm{d}\,\mathrm{ext}_{\mathcal{F}_k}(\dot{c}_k),$$

$$\mathrm{d}V_k^*(\dot{M}) = -\frac{1}{2}\,\mathrm{d}L_{\mathrm{III}_k}(\dot{I}_k).$$

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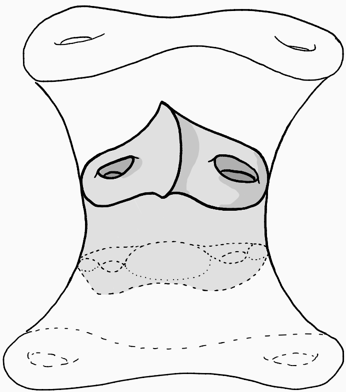
No need to study equidistant foliations!

Proof : $\mathrm{d}W_k = \mathrm{d}\,\mathrm{ext}_{\mathcal{F}_k}(\dot{c}_k) \to \mathrm{d}\,\mathrm{ext}_{\mathcal{F}_\infty}(\dot{c}_\infty) = \mathrm{d}V_R$.

$W_k(M) = 0 = V_k(M)$ for every M Fuchsian.



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There exists a time-dep Hamiltonian vector field $(X_k)_k$ on $(T^*\mathcal{T}, \omega_{\text{cot}})$ whose integral curves are precisely the γ_E's.

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Analogous to Moncrief's flow for CMC-foliations in constant curvature MGHC spacetimes of dimension 3.

Hamiltonian flow: proof

$$w_k(E) := V(E_k) - \frac{1}{4}\int_{\Sigma_k} H_k \,\mathrm{d}a_k + \frac{1}{2}L_\mu(m)$$

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$$\begin{array}{rccc} \mathrm{Sch}: & \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T} \\ & E & \longmapsto & (c_\infty, q_\infty) \end{array} \qquad \begin{array}{rccc} \mathrm{Th}: & \mathcal{E}(\Sigma) & \longrightarrow & \mathcal{T} \times \mathcal{ML} \\ & E & \longmapsto & (m, \mu) \end{array}$$

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Using k-surfaces, Labourie defined

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Ψ_k and Φ_k are homeos for every k.

From the works of Quinn and Belraouti, Φ_k and Ψ_k "converge" to Sch and Th, resp.

We change Th and Ψ_k^* into

$$\hat{\mathsf{Th}}: \begin{array}{ccc} \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T} \\ E & \longmapsto & (m, \mathrm{d}(L_\mu)_m) \end{array} \qquad \hat{\Psi}_k: \begin{array}{ccc} \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T} \\ E & \longmapsto & \left(I_k, \mathrm{d}\left(L_{\mathrm{III}_k}\right)_{I_k}\right) \end{array}$$

Corollary (M)

For every $k, k' \in (-1, 0)$

$$\mathsf{Sch}^*\,\omega_{\mathrm{cot}} = \Phi_k^*\omega_{\mathrm{cot}} = \hat{\Psi}_{k'}^*\omega_{\mathrm{cot}} = \hat{\mathsf{Th}}^*\omega_{\mathrm{cot}}.$$

Generalization of results of Krasnov-Schlenker '09, Bonsante-Mondello-Schlenker '15.

Theorem (M)

The image of the map

$$\begin{array}{rccc} F_k : & \mathcal{QF}(\Sigma) & \longrightarrow & T^*(\mathcal{T} \times \overline{\mathcal{T}}) \\ & M & \longmapsto & (c_k^+, c_k^-; q_k^+, q_k^-) \end{array}$$

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is a Lagrangian submanifold of $(T^*(\mathcal{T} \times \overline{\mathcal{T}}), \omega_{\mathrm{cot}})$.

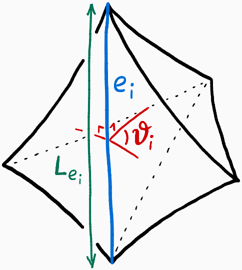
Proof : the pullback of the Liouville form λ on $T^*(\mathcal{T} \times \overline{\mathcal{T}})$ by F_k is equal to $\mathrm{d}W_k$ (Schläfli formulae).

$$F_k^*\omega_{\mathrm{cot}} = \mathrm{d}\Big(F_k^*\lambda\Big) = \mathrm{d}(\mathrm{d}W_k) = 0.$$

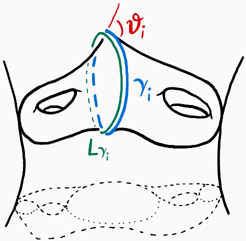
Thank you for your attention!

Polyhedra vs Convex cores

Polyhedron



Convex core



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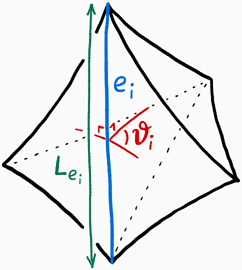

hyp metric on $\partial P \setminus V$

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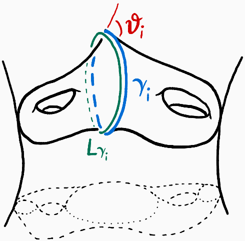

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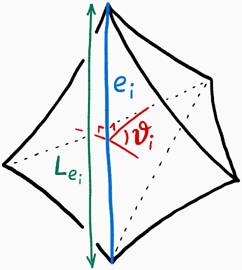

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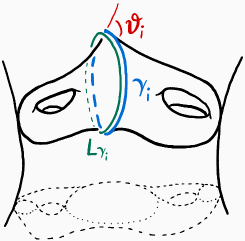

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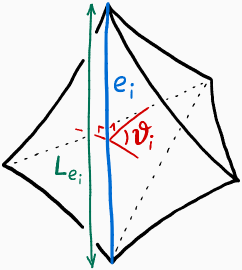

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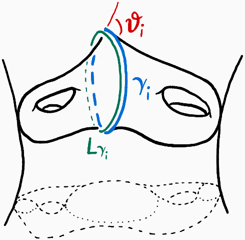

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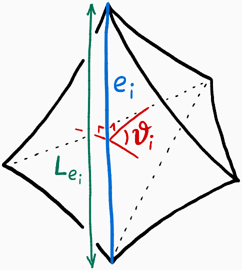

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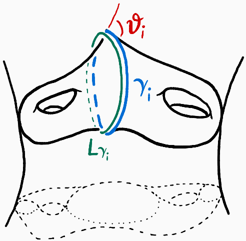

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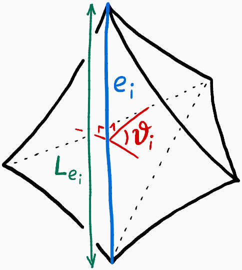

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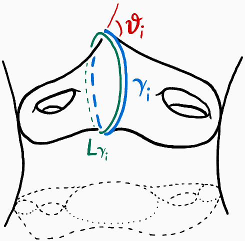

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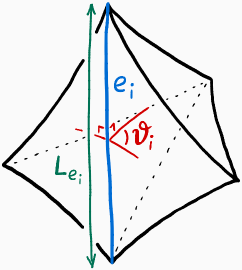

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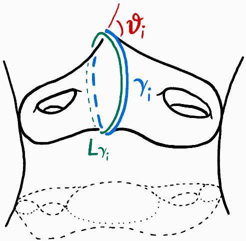

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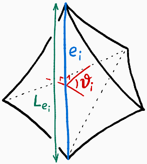
bending $\mu := \sum_i \boldsymbol{\vartheta}_i \cdot \boldsymbol{\gamma}_i \in \mathcal{ML}$
length $L_\mu(m) := \sum_i \boldsymbol{\vartheta}_i \cdot L_{\gamma_i}(m)$

Dual volume

Let P be a compact polyhedron.



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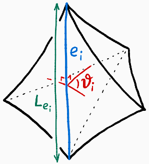

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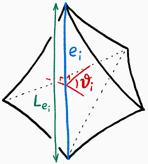

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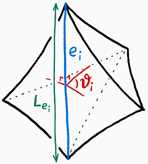

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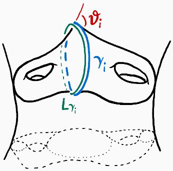
Dual volume (in M)



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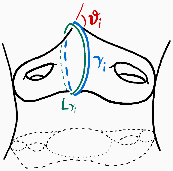

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$$\mathrm{d}V_C\,(\dot{M}) = \frac{1}{2} L_{\dot{\mu}}(m)$$

(Bonahon-Schläfli '98)

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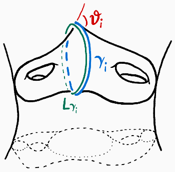

$$V_C^*(M) := V_C(M) - \frac{1}{2} L_\mu(m)$$

$$\mathrm{d}V_C(\dot{M}) = \frac{1}{2} L_{\dot{\mu}}(m) \qquad \text{(Bonahon-Schläfli '98)}$$

$$\mathrm{d}V_C^*(\dot{M}) = -\frac{1}{2} \mathbf{d}L_\mu(\dot{m}) \qquad \text{(dual B-S, Krasnov-Schlenker '09)}$$

Schläfli formulae

$$\delta V(N) = \frac{1}{2}\int_{\partial N}\left(\delta H + \frac{1}{2}(\delta I, II)\right)\mathrm{d}a\,. \qquad \text{(Rivin-Schlenker '99)}$$

Lemma (M)

$$\delta W(N) = \frac{1}{4}\int_{\partial N}\left(\left(\delta II, III - \frac{H}{2}II\right)_{II} + \frac{\delta K^e}{2K^e}H\right)\mathrm{d}a\,.$$

$$\mathrm{d}W_k\,(\dot{M}) = \frac{1}{8}\int_{\partial M_k}\left(\dot{II}_k, -2\,\mathrm{Re}\,q_k\right)_{II_k}\mathrm{d}a_{II_k} = -\,\mathrm{Re}\langle q_k, \dot{c}_k\rangle$$