Pleated Surfaces for $SO_0(2,n)$-maximal representations

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Geometry Seminar, University of Virginia

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Outline

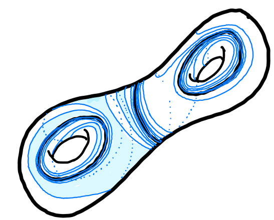
0. Motivation

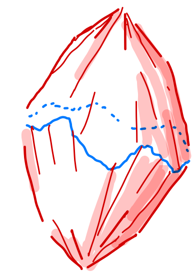
1. The geometry of $\mathbb{H}^{2,n}$

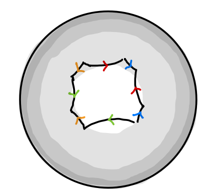
2. $SO_0(2,n+1)$-maximal representations

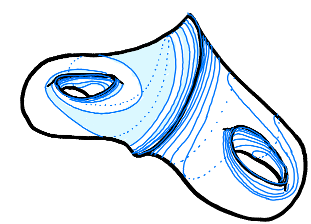
3. Pleated surfaces in $\mathbb{H}^{2,n}$

4. Some applications

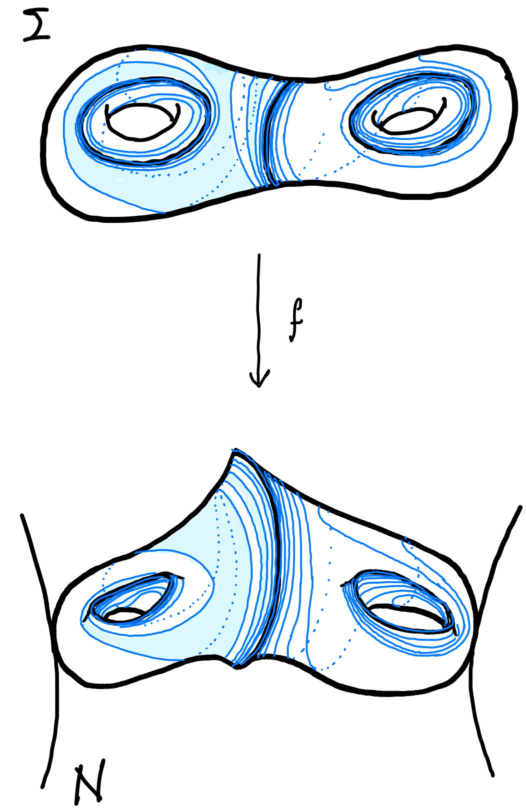








Pleated surfaces (in hyp 3-mfld N)



Def $f : (\Sigma, h) \to N$ path-isometry such that

i) (Σ, h) hyp surface, N hyp 3-mfld

ii) $\exists$ max geodesic lamination λ (closed subset foliated by geodesics) s.t. $\forall g \subset \lambda$

$f(g)$ geodesic in N

and $\forall T \subset S \setminus \lambda$

$f|_T$ totally geodesic immersion in N

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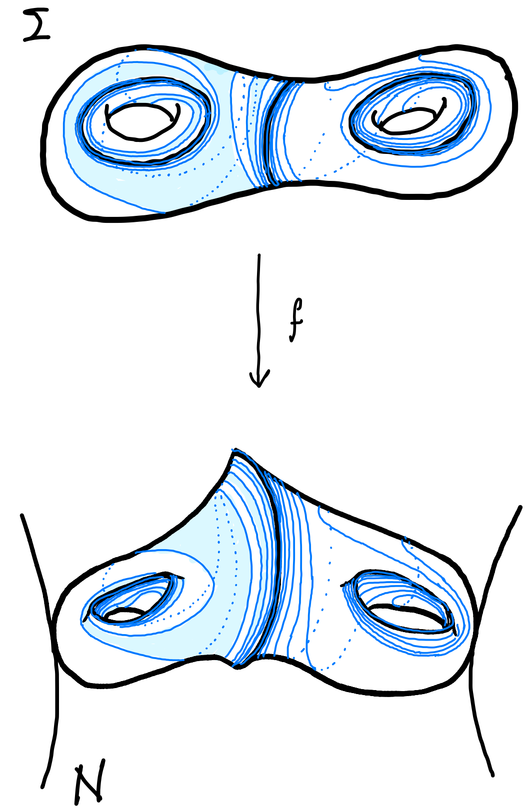

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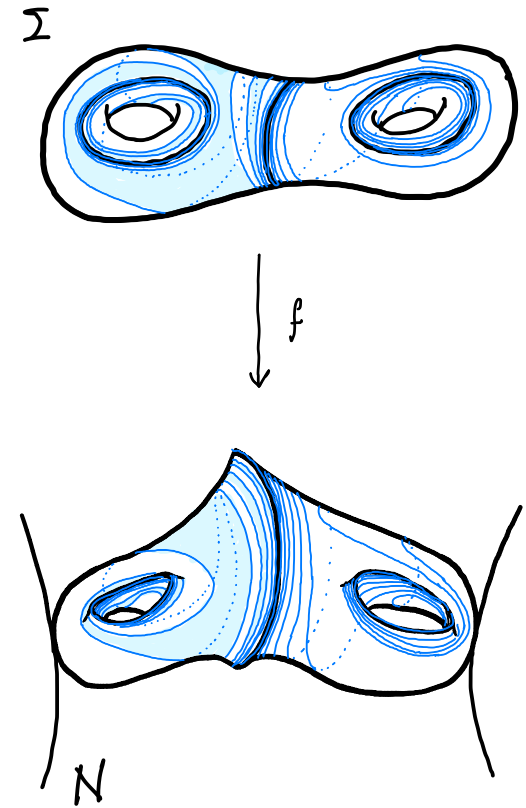

Thm (Thurston) Let $u: \Sigma \to M$ map s.t. $\Gamma \simeq u_*(\Gamma)$ has no parabolics, then for (almost every) max geod lam λ $\exists$ pleated surface homotopic to u with pleating locus $\subseteq \lambda$.

Many applications to Thurston's geometrization, Ending Lam thm, quasi-Fuchsian mflds & their convex core, ...

Q: Is there a good notion of pleated surfaces for $(\mathbb{H}^{2,n}, SO_0(2,n+1))$-manifolds with maximal holonomy ?

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Geometry in $\mathbb{H}^{2,n}$

$$\langle x, x \rangle_{(2,n+1)} := x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+3}^2$$

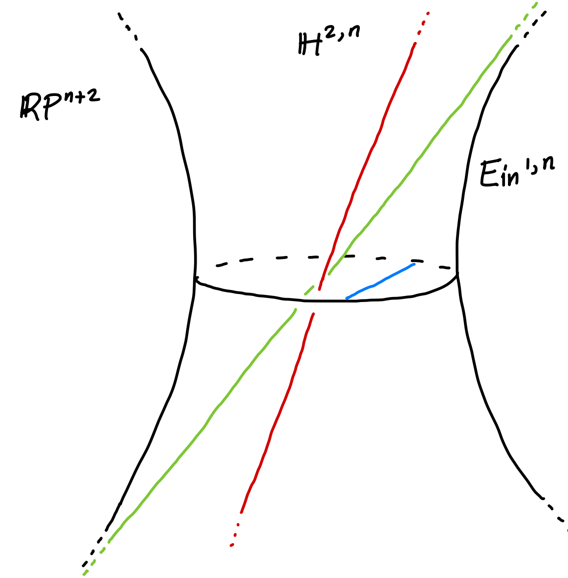
$$\mathbb{H}^{2,n} := P\left(\{ x \in \mathbb{R}^{2,n+1} \mid \langle x, x \rangle_{(2,n+1)} < 0 \} \right) \subset \mathbb{RP}^{n+2}$$

Pseudo-Riemannian mfld of signature $(2,n)$

$$\mathrm{Ein}^{1,n} := \partial \mathbb{H}^{2,n} = P\left(\{ x \in \mathbb{R}^{2,n+1} \mid \langle x, x \rangle_{(2,n+1)} = 0 \} \right)$$

Ideal boundary

$\mathrm{Isom}_0(\mathbb{H}^{2,n}) = SO_0(2,n+1) \curvearrowright \mathbb{H}^{2,n}$ & $\mathrm{Ein}^{1,n}$ (homogeneous spaces)



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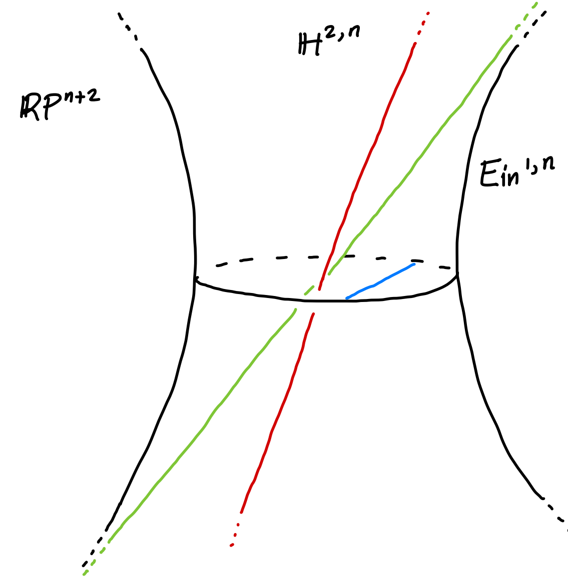

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Subspaces

1-dim: geodesics $\ell = PV$, for some $V \subset \mathbb{R}^{2,n+1}$ with $\dim = 2$

- ℓ is spacelike if $\langle \ell'(t), \ell'(t) \rangle > 0$
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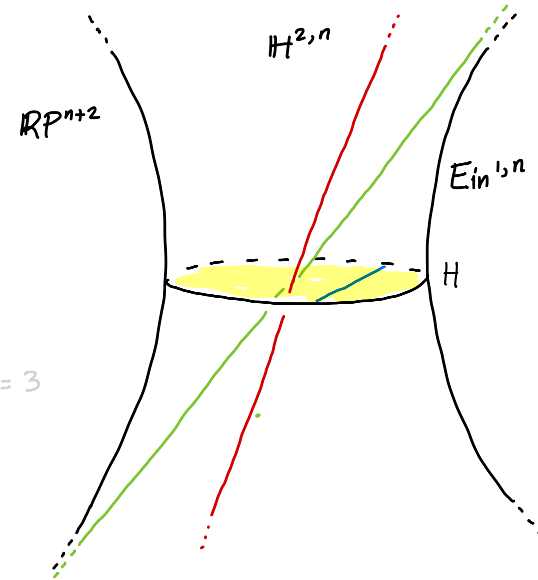

2-dim: totally geodesic spacelike planes $H = PW$, $\dim W = 3$ and $\langle \cdot, \cdot \rangle|_H > 0$

$$\boxed{\Rightarrow H \simeq_{isom} \mathbb{H}^2}$$

If $x, y \in \mathbb{H}^{2,n}$ joined by a spacelike geodesic $[x,y]$, then

$$d_{\mathbb{H}^{2,n}}(x,y) := L([x,y]) \quad \longleftarrow \quad \text{Alert! No triang ineq !!}$$

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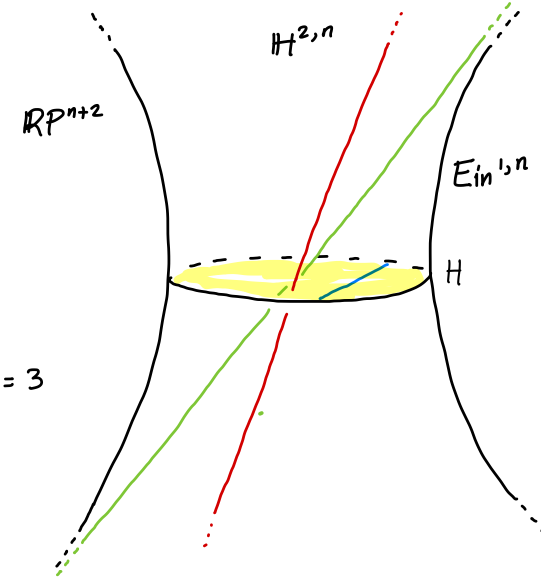

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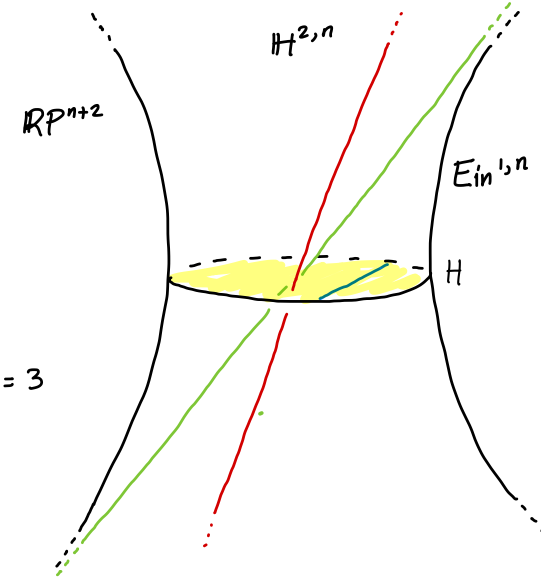

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Maximal representations

Wide class of surface groups repr's in G Lie group of Hermitian type

Here $G := SO_0(2,n+1)$, $\Gamma := \pi_1(\Sigma_g)$, $g \geq 2$

Thm (Burger-Iozzi-Labourie-Wienhard) $\rho : \Gamma \longrightarrow SO_0(2,n+1)$ max

iff $\exists\, \xi : \partial\Gamma \longrightarrow \mathrm{Ein}^{1,n}$ ρ-equiv homeo + spacelike, i.e.

$\forall x, y, z \in \partial\Gamma$ distinct $\exists\, H$ spacelike plane in $\mathbb{H}^{2,n}$ s.t. $\xi(x), \xi(y), \xi(z) \in \partial H$

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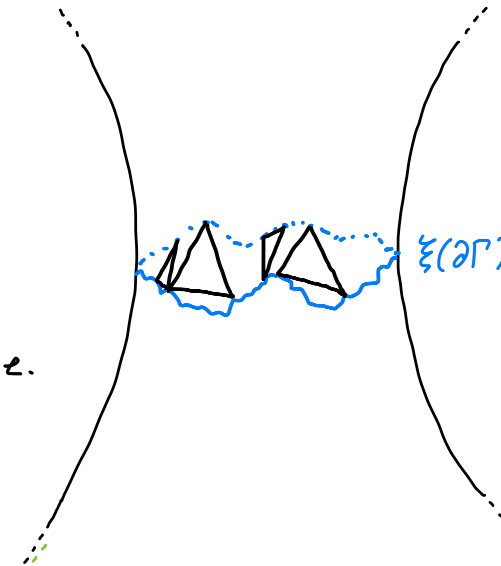

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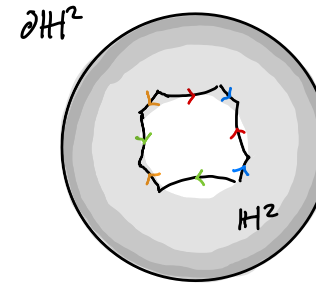
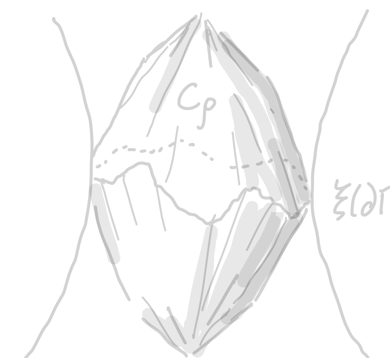


$\underline{n=0}$: $(\mathbb{H}^2, SO_0(2,1))$ $\rho : \Gamma \longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$ maximal iff

ρ Fuchsian, i.e. $\rho(\Gamma)\backslash\mathbb{H}^2$ hyperbolic surface



$\underline{n=1}$: $(\mathbb{H}^{2,1}, SO_0(2,2)) = (AdS^3, \mathrm{Isom}_0(AdS^3) \cong PSL_2\mathbb{R} \times PSL_2\mathbb{R})$

$\rho = (\rho_l, \rho_r) : \Gamma \longrightarrow PSL_2\mathbb{R} \times PSL_2\mathbb{R}$ maximal iff

both ρ_l & ρ_r are Fuchsian

$\underline{\forall n}$: $\exists\, C_\rho$ closed ρ-equiv convex s.t. $\Gamma \overset{\rho}{\curvearrowright} C_\rho$ cocompact, $M_\rho := C_\rho/\rho(\Gamma)$

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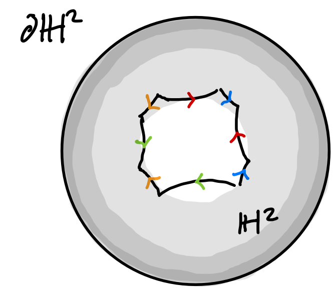

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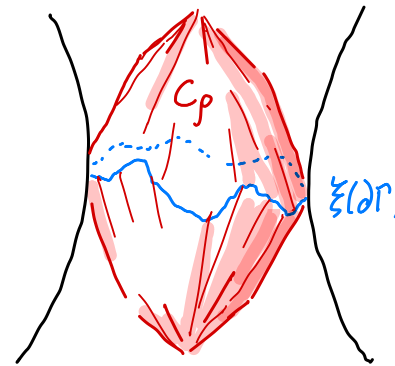

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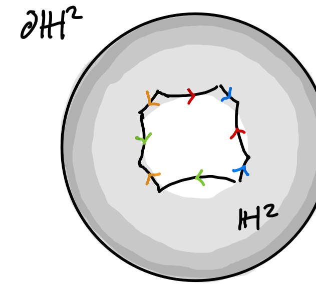

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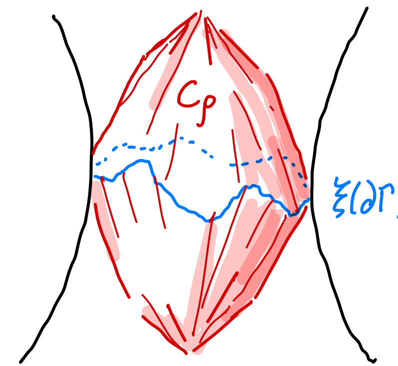

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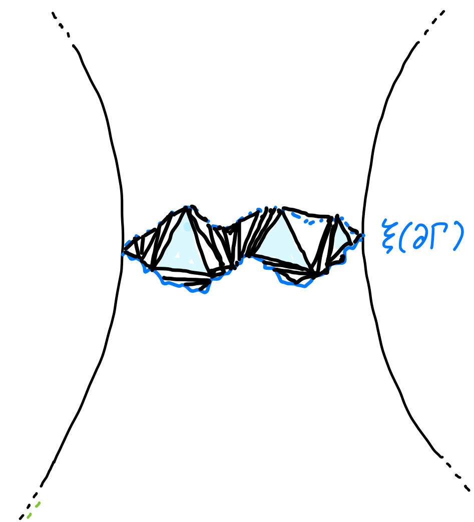

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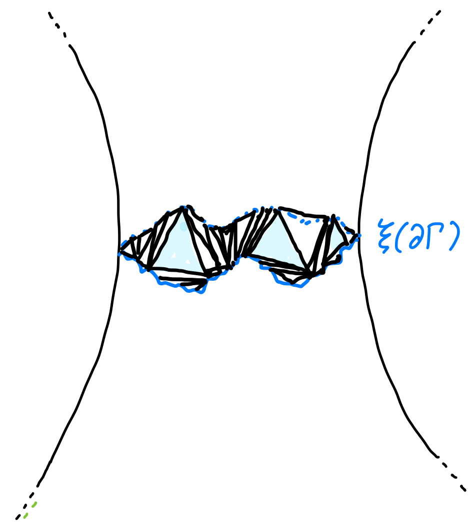

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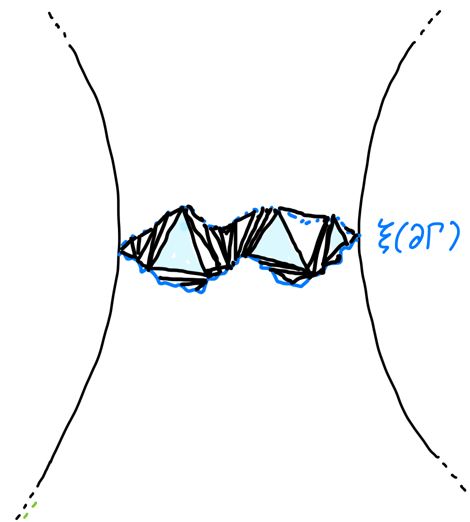

No path metric $\Rightarrow$ how to get a hyp structure on Σ ?

Main tool: ρ max repr, ξ limit map $\rightsquigarrow$ β^ρ cross ratio

$$\beta^\rho(u,v,w,z) = \left(\frac{\langle \xi u, \xi w \rangle_{(2,n+1)}}{\langle \xi u, \xi z \rangle_{(2,n+1)}} \frac{\langle \xi v, \xi z \rangle_{(2,n+1)}}{\langle \xi v, \xi w \rangle_{(2,n+1)}} \right)^{1/2}$$

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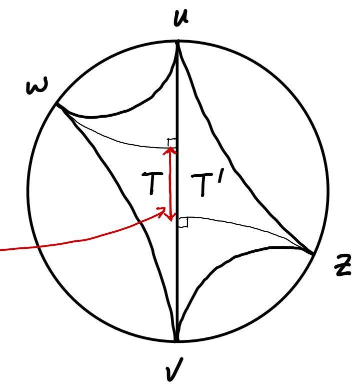
Why are cross ratios useful ?

If λ max lamination and β_o is the cross ratio on $\partial\tilde{\Sigma} \cong_h \partial\mathbb{H}^2$,

shear between ideal triangles

$$\sigma_\lambda(T,T') := \log|\beta_o(\xi u, \xi v, \xi w, \xi z)|$$



σ_λ extends to a function $(T,T') \mapsto \sigma_\lambda(T,T')$ on pairs of distinct triangles of λ satisfying :

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- $\sigma_\lambda(T,T') = \sigma_\lambda(\gamma T, \gamma T') \quad \forall \gamma \in \pi_1(\Sigma)$
- $\sigma_\lambda(T,T'') = \sigma_\lambda(T,T') + \sigma_\lambda(T',T'') \quad \forall T'$ separating T & T''

$\Bigg\}$ σ_λ is a Hölder cocycle ! $\sigma_\lambda \in \mathcal{H}(\lambda)$

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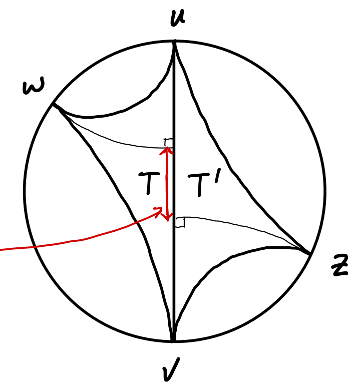

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Thm (Bonahon '96) $\forall \lambda$ the map

$$\{\text{hyp str's on } \Sigma\}/_{\text{isotopy}} \longrightarrow \mathcal{C}(\lambda) \subset \mathcal{H}(\lambda)$$

$$(\Sigma, h) \longmapsto \sigma_\lambda = \sigma_\lambda^h$$

is a diffeo onto its image.

We can generalize the shear cocycle machinery $\beta, \lambda \mapsto \sigma_\lambda^\beta \in \mathcal{H}(\lambda)$ obtaining:

Thm (M-Viaggi) Let β be a strictly positive and locally bounded cross ratio.

Then for all max lam's λ $\exists!$ X hyp surface with shear coord's σ_λ^β

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$$\{\text{hyp str's on } \Sigma\}/_{\text{isotopy}} \longrightarrow \mathcal{C}(\lambda) \subset \mathcal{H}(\lambda)$$

$$(\Sigma, h) \longmapsto \sigma_\lambda = \sigma_\lambda^h \qquad \text{shear coord's}$$

is a diffeo onto its image.

We can generalize the shear cocycle machinery $\beta, \lambda \mapsto \sigma_\lambda^\beta \in \mathcal{H}(\lambda)$ obtaining:

Thm 1 (M-Viaggi) Let β be a strictly positive and locally bounded cross ratio.

Then for all max lam's λ $\exists! X = X_\lambda^\beta$ hyp surface with shear coord's σ_λ^β.

Thm 2 (M-Viaggi) Let $\rho : \Gamma \to SO_0(2,n+1)$ be a max repr. Then

$\forall \lambda$ max geod lam there exist unique

- $S_\lambda \hookrightarrow C_\rho / \rho(\Gamma)$ embedded spacelike surface
- X_λ hyp structure, ← intrinsic description using cross ratio
- homeo $u : S_\lambda \to X_\lambda$ such that

  i) $u^{-1}(g)$ spacelike geo $\forall g \subset \lambda$ &

  $u^{-1}(T)$ spacelike triangle $\forall T \subset X_\lambda \setminus \lambda$

  ii) $d_{\mathbb{H}^2}(\tilde{u}(x), \tilde{u}(y)) \leq d_{\mathbb{H}^{2,n}}(x,y)$

Moreover $L_{X_\lambda}(\gamma) \leq L_\rho(\gamma)$ $\forall \gamma \in \Gamma \setminus \{e\}$, with $=$ iff $\gamma \subset \lambda$.

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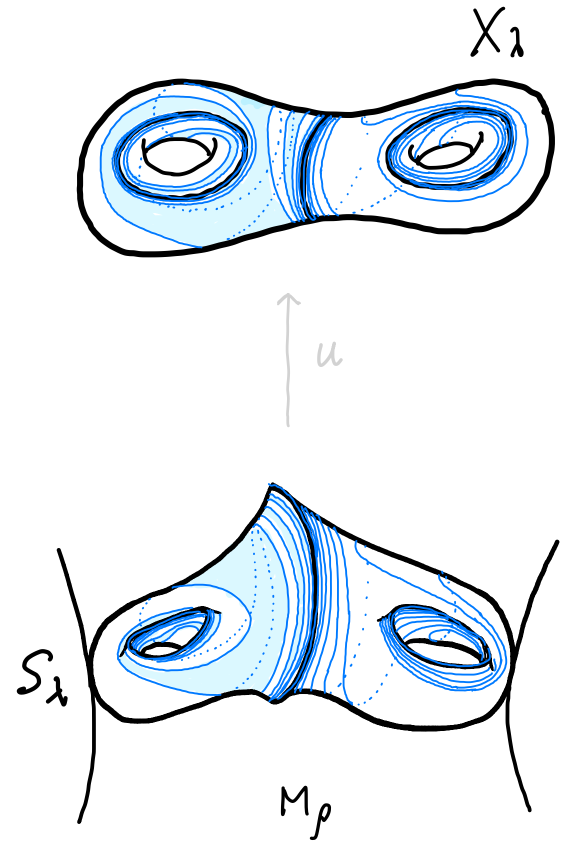

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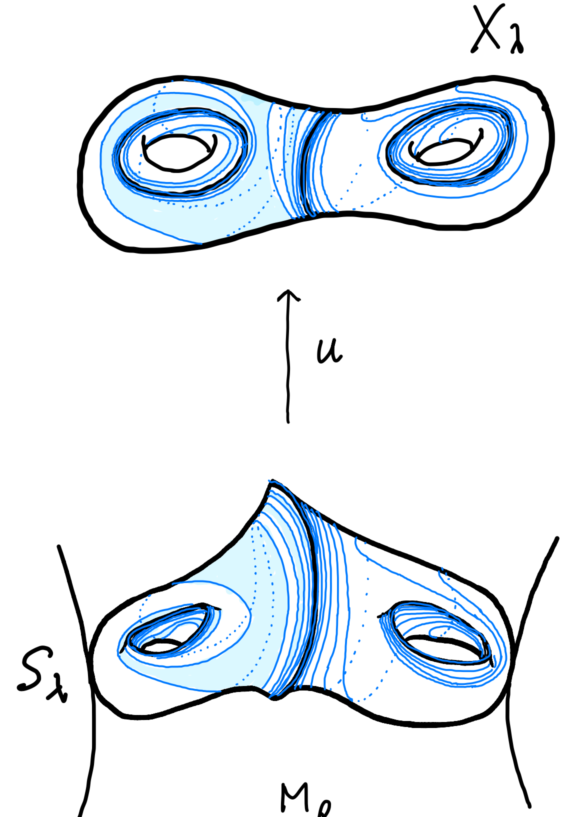

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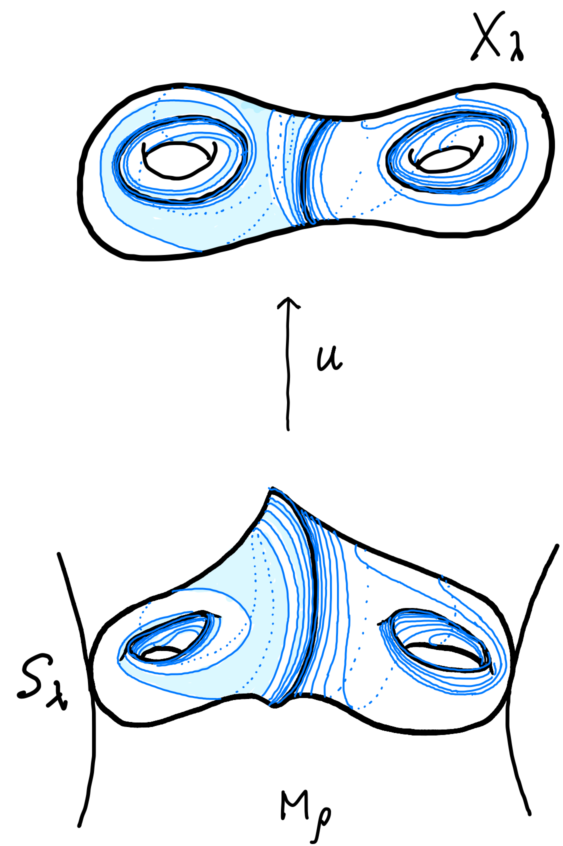

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Applications:

- In AdS^3, $\forall \lambda$ max lam we find

$$\varphi_\lambda : \mathcal{X}_{max}(\Gamma, SO_0(2,2)) \longrightarrow \operatorname{Im} \varphi_\lambda \subset \mathcal{U}(\lambda; \mathbb{R} + \tau\mathbb{R})$$
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← shear-bend cocycle

para-holomorphic & symplectic parametrization (analogue of the work of Bonahon for repr's in $\mathrm{Isom}^+(\mathbb{H}^3) = \mathbb{P}SL_2\mathbb{C}$

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- $\forall \lambda$ max $\quad \ell_\rho(\gamma) \geq \ell_{X_\lambda}(\gamma) \quad \forall \gamma \in \Gamma \Leftarrow$ Pseudo-Riemannian geometry of $\mathbb{H}^{2,n}$

When $n=1$, $\rho = (\rho_X, \rho_Y)$ satisfies $L_\gamma(\rho) = \frac{1}{2}\big(L_\gamma(X) + L_\gamma(Y)\big)$, and $\sigma_\lambda^\rho = \frac{1}{2}(\sigma_\lambda^X + \sigma_\lambda^Y)$

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Questions

i) CROSS RATIOS : Thm 1 applies to a wide class of cross ratios

(other Anosov repr's come with good cross ratios, e.g. Hitchin repr's, or max repr's in $Sp(2n,\mathbb{R})$, see Martone-Zhang, Labourie)

Q: Does σ_X^β have nice geom meaning for other types of ρ ?

ii) CONVEX CORE GEOMETRY : every $\rho : \Gamma \to SO_0(2,n+1)$ max has a cpt convex core $CC(\rho) \supset$ all pleated surfaces

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Thank you for your attention !