Pleated Surfaces for $SO_0(2,n)$-maximal representations

Filippo Mazzoli (University of Virginia),

joint work with Gabriele Viaggi (Heidelberg University).

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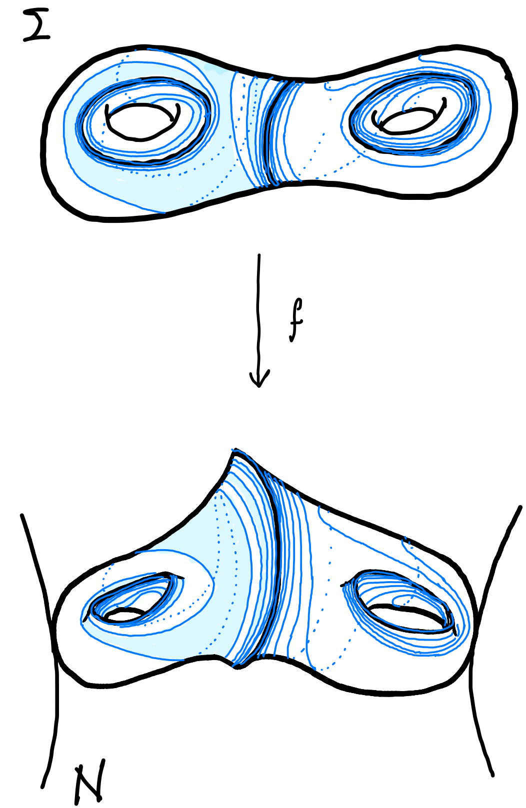
March 12, 2022.

Pleated surfaces (in hyp 3-mfld N)



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i) (Σ, h) hyp surface, N hyp 3-mfld

ii) $\exists$ max geodesic lamination λ (closed subset foliated by geodesics) s.t. $\forall g \subset \lambda$

$f(g)$ geodesic in N

and $\forall T \subset S \setminus \lambda$

$f|_T$ totally geodesic immersion in N

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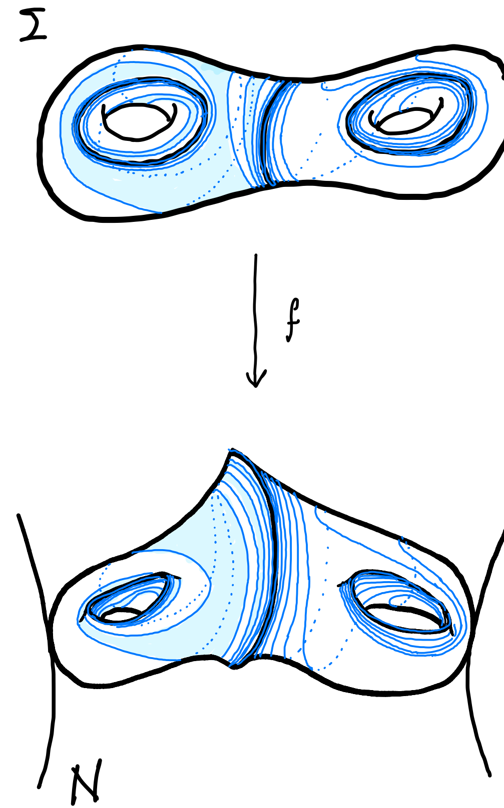

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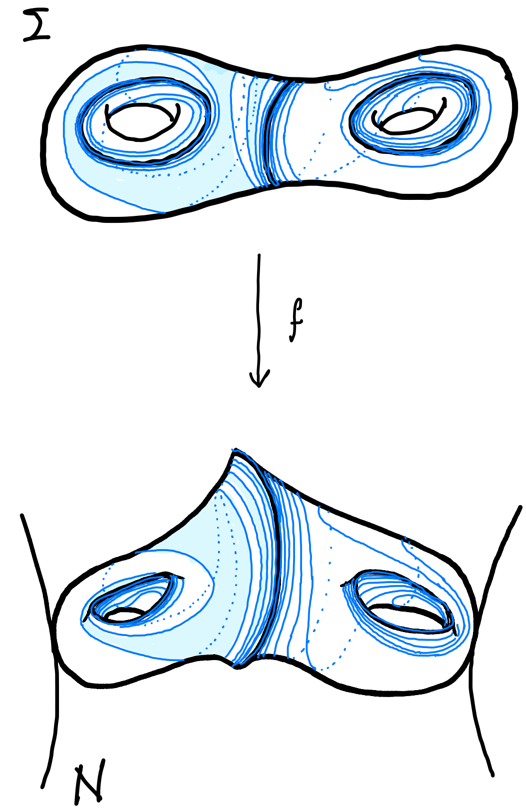

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Many applications to Thurston's geometrization, Ending Lam thm, quasi-Fuchsian mflds & their convex core, ...

Q: Is there a good notion of pleated surfaces for $(\mathbb{H}^{2,n}, SO_0(2,n+1))$-manifolds with maximal holonomy ?

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Geometry in $\mathbb{H}^{2,n}$

$$\langle x,x\rangle_{(2,n+1)} := x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+3}^2$$

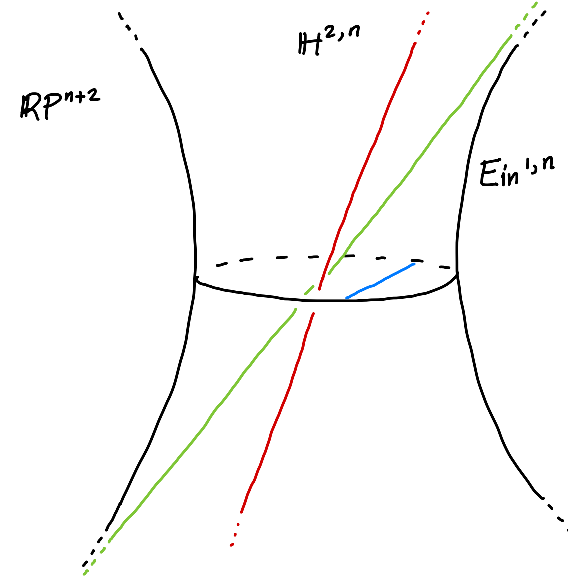
$$\mathbb{H}^{2,n} := P\left(\{x \in \mathbb{R}^{2,n+1} \mid \langle x,x\rangle_{(2,n+1)} < 0\}\right) \subset \mathbb{RP}^{n+2}$$

Pseudo-Riemannian mfld of signature $(2,n)$

$$\mathrm{Ein}^{1,n} := \partial \mathbb{H}^{2,n} = P\left(\{x \in \mathbb{R}^{2,n+1} \mid \langle x,x\rangle_{(2,n+1)} = 0\}\right)$$

Ideal boundary

$\mathrm{Isom}_0(\mathbb{H}^{2,n}) = SO_0(2,n+1) \curvearrowright \mathbb{H}^{2,n}$ & $\mathrm{Ein}^{1,n}$ (homogeneous spaces)



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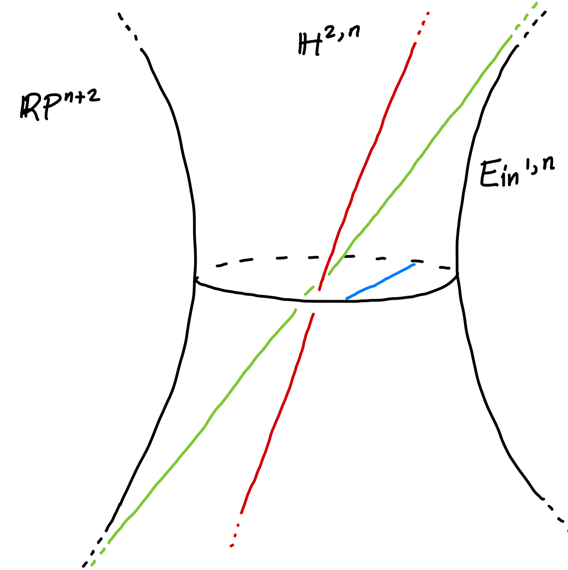

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Subspaces

1-dim: geodesics $\ell = PV$, for some $V \subset \mathbb{R}^{2,n+1}$ with $\dim = 2$

- ℓ is spacelike if $\langle \ell'(t), \ell'(t) \rangle > 0$
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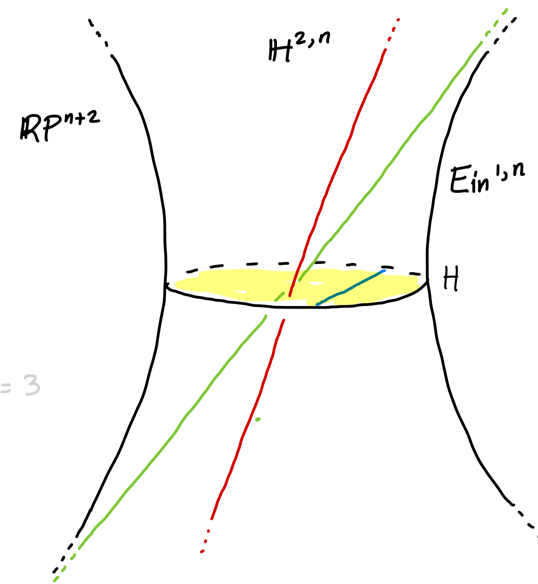

2-dim: totally geodesic spacelike planes $H = PW$, $\dim W = 3$ and $\langle \cdot, \cdot \rangle|_H > 0$

$$\boxed{\Rightarrow H \simeq_{isom} \mathbb{H}^2}$$

If $x, y \in \mathbb{H}^{2,n}$ joined by a spacelike geodesic $[x,y]$, then

$$d_{\mathbb{H}^{2,n}}(x,y) := L([x,y]) \quad \longleftarrow \quad \text{Alert! No triang ineq !!}$$

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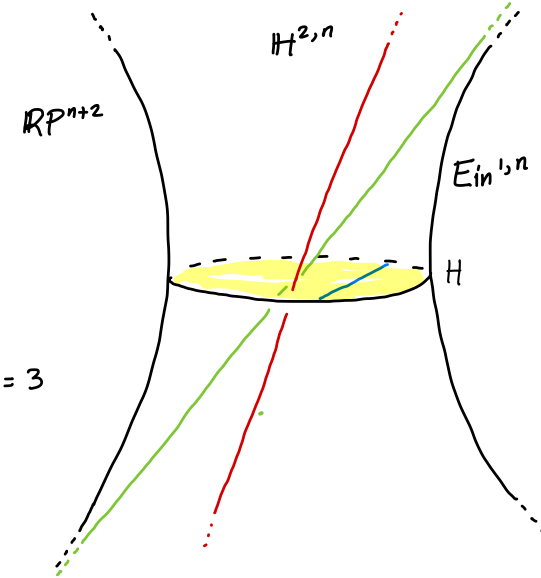

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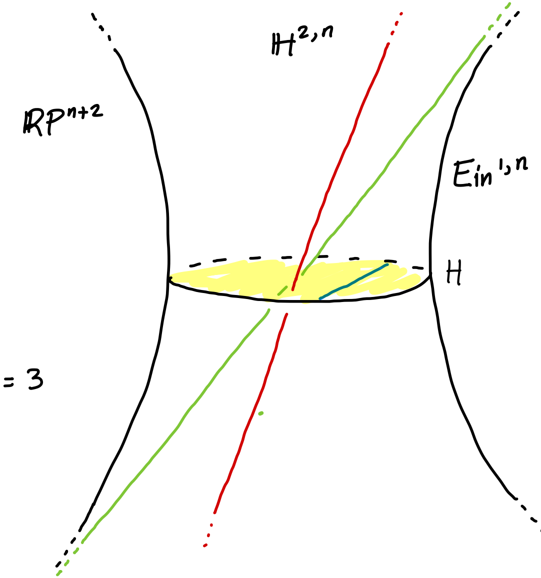

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Maximal representations

Wide class of surface groups repr's in G Lie group of Hermitian type

Here $G := SO_0(2,n+1)$, $\Gamma := \pi_1(\Sigma_g)$, $g \geq 2$

Thm (Burger-Iozzi-Labourie-Wienhard) $\rho : \Gamma \longrightarrow SO_0(2,n+1)$ max

iff $\exists\, \xi : \partial\Gamma \longrightarrow \mathrm{Ein}^{1,n}$ ρ-equiv homeo + spacelike, i.e.

$\forall x, y, z \in \partial\Gamma$ distinct $\exists\, H$ spacelike plane in $\mathbb{H}^{2,n}$
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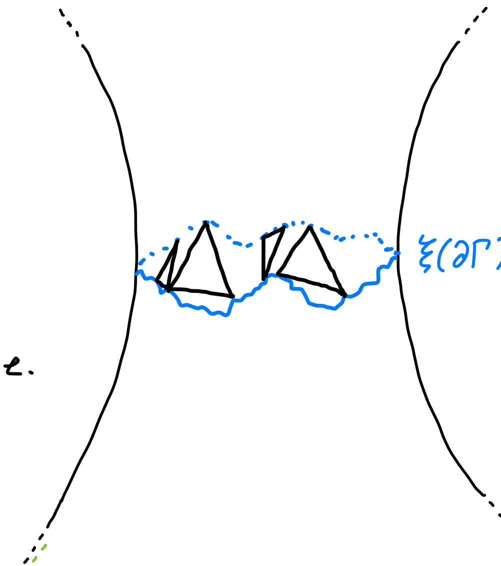

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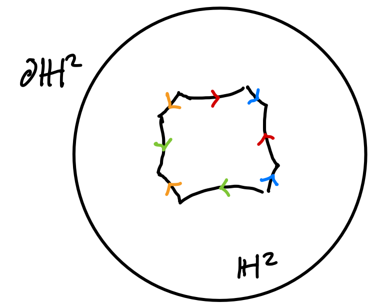
Examples:

n=0: $(\mathbb{H}^2, SO_0(2,1))$ $\quad \rho : \Gamma \longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$ maximal iff

ρ Fuchsian, i.e. $\rho(\Gamma)\backslash\mathbb{H}^2$ hyperbolic surface



n=1: $(\mathbb{H}^{2,1}, SO_0(2,2)) = (AdS^3, \mathrm{Isom}_0(AdS^3) \cong \mathbb{P}SL_2\mathbb{R} \times \mathbb{P}SL_2\mathbb{R})$

$\rho = (\rho_\ell, \rho_r) : \Gamma \longrightarrow \mathbb{P}SL_2\mathbb{R} \times \mathbb{P}SL_2\mathbb{R}$ maximal iff

both ρ_ℓ & ρ_r are Fuchsian

$\forall n$: $\exists\, C_\rho$ closed ρ-equiv convex s.t. $\Gamma \overset{\rho}{\curvearrowright} C_\rho$ cocompact, $M_\rho := C_\rho/\rho(\Gamma)$

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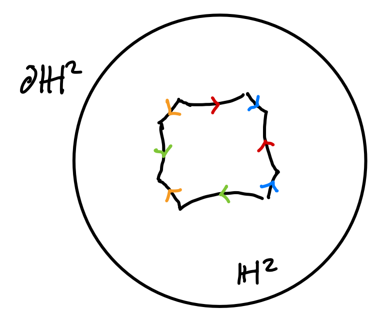

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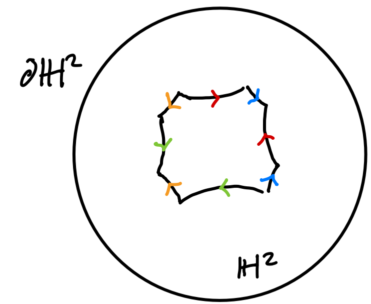

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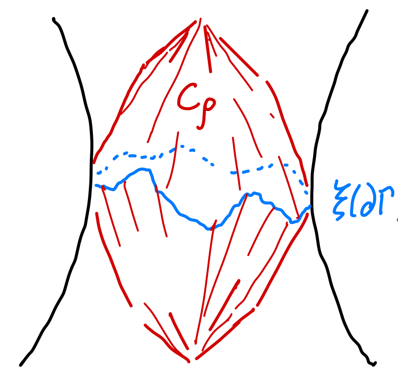

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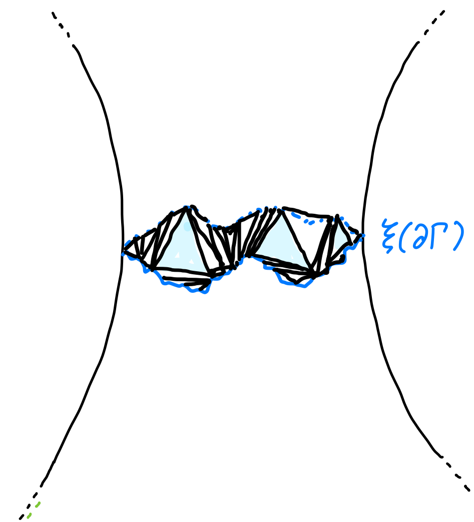

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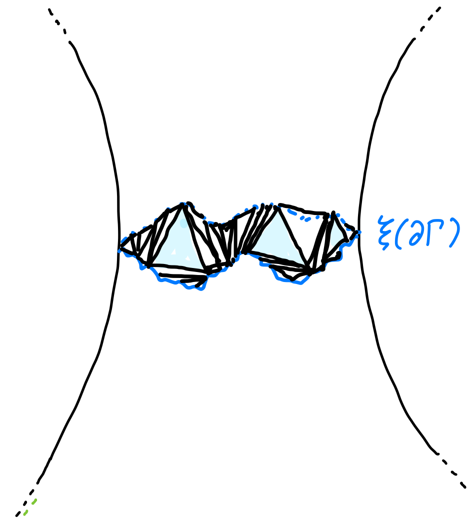

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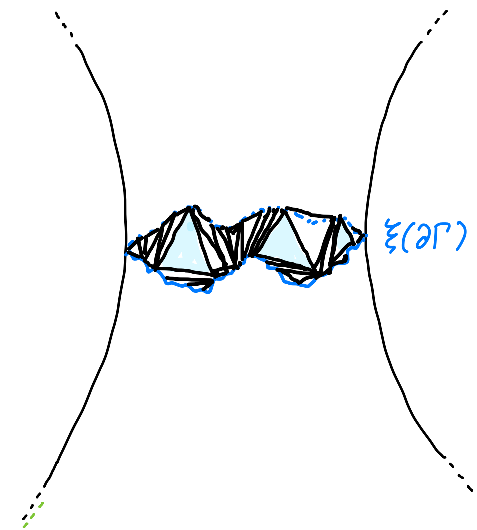
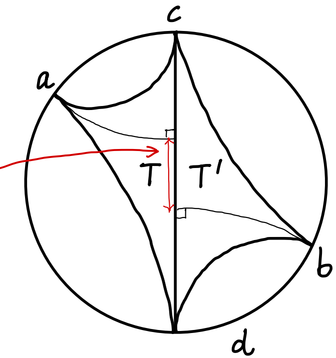


No path metric $\Rightarrow$ how to get a hyp structure on Σ ?

<u>Main tool</u>: ρ max repr, ξ limit map $\rightsquigarrow$ β^ρ cross ratio

shear between ideal triangles

$$\sigma_\lambda^\rho(T,T') := \log(-\beta^\rho(\xi a, \xi b, \xi c, \xi d))$$



<u>Thm 1</u> (M-Viaggi) Let β be a positive and locally bounded cross ratio.

Then for all (but $<\infty$ many) max lam λ $\exists X$ hyp surface with shear coord's σ_λ^β (Hölder cocycle transverse to λ)

When $\beta = \beta^\rho$, $\rho : \Gamma \to SO_0(2, n+1)$ then true for ALL max geod lam!

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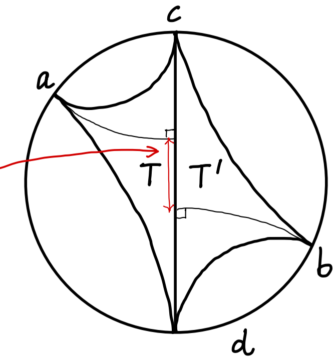

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Thm 2 (M-Viaggi '22) Let $\rho : \Gamma \to SO_0(2,n+1)$ be a max repr. Then

$\forall \lambda$ max geod lam there exist unique

- $S_\lambda \hookrightarrow C_\rho / \rho(\Gamma)$ embedded spacelike surface
- X_λ hyp structure, ← intrinsic description using cross ratio
- homeo $u : S_\lambda \to X_\lambda$ such that

  i) $u^{-1}(g)$ spacelike geo $\forall g \subset \lambda$ &

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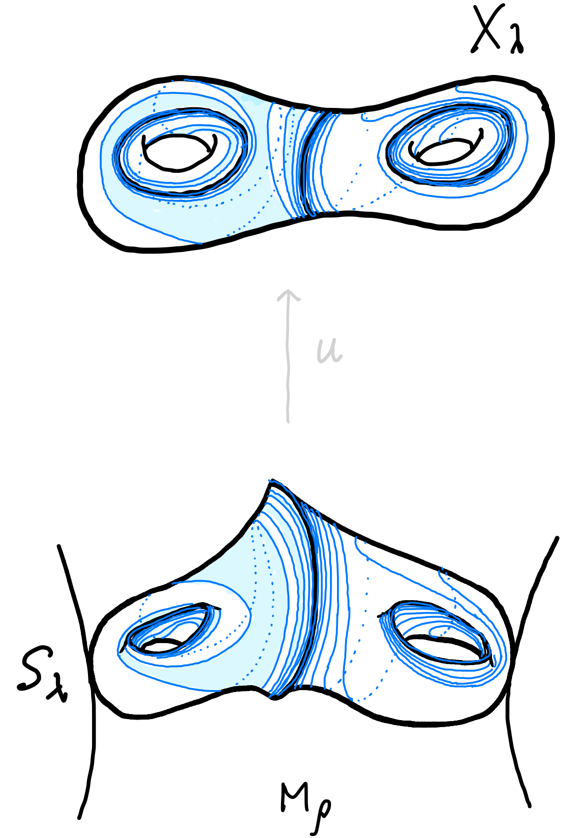

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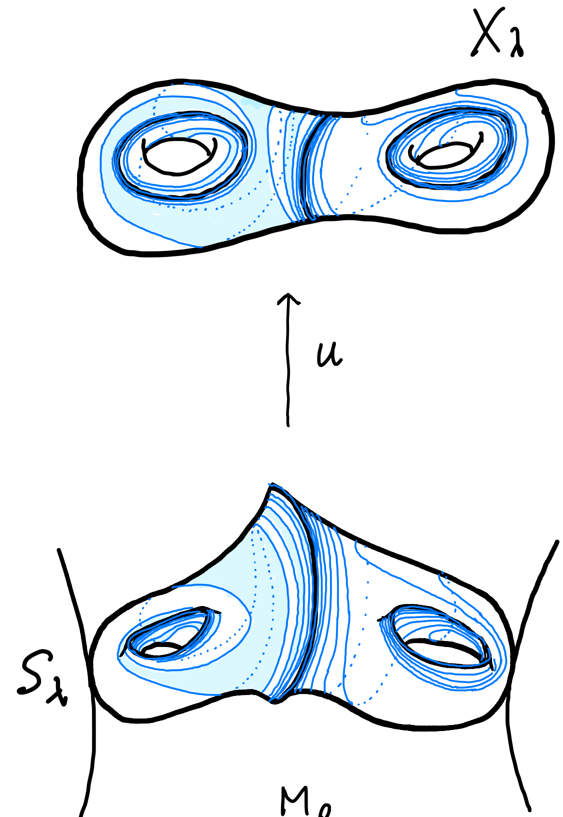

Applications:

- In AdS^3, $\forall \lambda$ max lam we find

$$\begin{aligned} \varphi_\lambda : \mathcal{X}_{max}(\Gamma, SO_0(2,2)) &\longrightarrow Im\, \varphi_\lambda \subset \mathcal{U}(\lambda; \mathbb{R} + \tau \mathbb{R}) \\ \rho &\longmapsto \hat{\sigma}^{\rho}_{\lambda} \end{aligned}$$

$\leftarrow$ shear bend cocycle

para-holomorphic & symplectic parametrization

(if λ finite, then there is a similar result using $\mathbb{P}ho^{2,n}$-structures, compare with Collier-Tholozan-Toulisse)

- $\forall \lambda$ max $\quad \ell_\rho(\gamma) \geq \ell_{X_\lambda}(\gamma) \quad \forall \gamma \in \Gamma \Leftarrow$ Pseudo-Riemannian geometry of $\mathbb{H}^{2,n}$

When $n=1$, $\rho = (\rho_X, \rho_Y)$ satisfies $L_\gamma(\rho) = \frac{1}{2}\left(L_\gamma(X) + L_\gamma(Y)\right)$, and $\sigma_\lambda^\rho = \frac{1}{2}(\sigma_\lambda^X + \sigma_\lambda^Y)$

$$\Longrightarrow \boxed{L_\gamma\left(\tfrac{1}{2}(\sigma_\lambda^X + \sigma_\lambda^Y)\right) \leq \tfrac{1}{2}\left(L_\gamma(X) + L_\gamma(Y)\right)}$$ (Bestvina-Bromberg-Fujiwara-Souto, Théret)

lenghs are convex along shear paths

- recover Collier-Tholozan-Toulisse: $\forall \rho : \Gamma \to SO_0(2,n+1)$ max

$$\boxed{\exists X \text{ hyp str } \& \ \exists k \leq 1 \text{ s.t. } \ell_\rho(\gamma) \geq k\, \ell_X(\gamma) \quad \forall \gamma \in \Gamma}$$

and $k=1 \Leftrightarrow \rho$ Fuchsian (but w/o Higgs bundles!)

$\left(\Longrightarrow \text{entropy}(\rho) \leq 1 \ \& \ =1 \Leftrightarrow \rho \text{ Fuchsian}\right)$

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($\Rightarrow$ entropy$(\rho) \leq 1$ & $=1 \Leftrightarrow \rho$ Fuchsian)

Questions

i) CROSS RATIOS: Thm 1 applies to a wide class of cross ratios (other Anosov repr's come with good cross ratios)

Q: Does σ_λ^β have nice geom meaning for other types of ρ?

ii) CONVEX CORE GEOMETRY: every $\rho : \Gamma \longrightarrow SO_0(2,n+1)$ max has a cpt convex core $CC(\rho) \supset$ all pleated surfaces

Q: Can we use pleated surfaces to understand geometry of convex core? Already for $d=1$ not much is known about the interior of the convex core

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has a cpt convex core $CC(\rho) \supset$ all pleated surfaces

Q: Can we use pleated surfaces to understand geometry

of convex core ? Already for $d=1$ not much is known about

the <u>interior</u> of the convex core

Thank you for your attention!