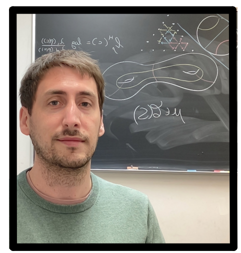
Shear-bend coordinates for pleated surfaces in $PSL_2\mathbb{C}$

Filippo Mazzoli (University of Virginia),
joint work with



Sara Maloni, Giuseppe Martone, & Tengren Zhang

AMS-SMF-EMS Meeting, Grenoble, July 20^{th}, 2022.

Motivation

Goal: Study $\mathrm{Hom}(\Gamma, PSL_d\mathbb{K})$, with
- $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,
- $\Gamma := \pi_1(S)$, S oriented closed surface, $\chi(S) < 0$.

Focus: Coord's on "large" open subsets of $\mathrm{Hom}(\Gamma, PSL_d\mathbb{K})$ generalizing techniques from 2 & 3-dimensional hyperbolic geometry

I. shear coord's in $PSL_d\mathbb{R}$

(d = 2 : Thurston, Bonahon '96
d > 2 : Fock-Goncharov '06, Bonahon-Dreyer '17)

II. shear-bend coord's in $PSL_d\mathbb{C}$

(d = 2 : Thurston, Bonahon '96
d > 2 : MMMZ)

When $d = 2$:

- $PSL_2\mathbb{R} = \mathrm{Isom}^+(\mathbb{H}^2) \rightsquigarrow \mathcal{X}_{d.f.}(\Gamma, PSL_2\mathbb{R}) = \mathrm{Teich}(S)$ $(\sqcup\ \mathrm{Teich}(\bar{S}))$
- $PSL_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3) \rightsquigarrow QF(S) \subsetneq \left\{ [\rho] \in \mathcal{X}(\Gamma, PSL_2\mathbb{C}) \;\middle|\; \begin{array}{c}\rho \text{ holonomy of} \\ \text{a pleated surface}\end{array} \right\} \not\subseteq \mathcal{X}_{d.f.}(\Gamma, PSL_2\mathbb{C})$

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Hitchin : $\mathrm{Hit}_d(S) :=$ conn. comp. of $\iota_{irr}(\mathcal{X}_{d.f.}(\Gamma, PSL_2\mathbb{R})) \subset \mathcal{X}(\Gamma, PSL_d\mathbb{R})$

$$\cong_{diff} \mathbb{R}^{(d^2-1)|\chi(S)|} \quad (\text{if } d=2,\ \mathrm{Hit}_d(S) = \mathrm{Teich}(S))$$

Gromov ∂ of Cayley(Γ)

Labourie : i) $[\rho] \in \mathrm{Hit}_d(S) \Rightarrow$ $\exists\, \xi : \partial\Gamma \longrightarrow \mathcal{F}lags(\mathbb{R}^d)$ ρ-equivariant

(Guichard: $\Leftarrow$) (Hölder) continuous, hyperconvex Frenet curve

ξ is transverse & generic, i.e.

(hyperconvex Frenet $\Rightarrow$ $\forall x,y,z \in \partial\Gamma$ $\forall i,j \geq 1 : i+j=d$ & $\forall k,h,\ell \geq 1 : k+h+\ell = d$

$$\xi(x)^i + \xi(y)^j = \xi(x)^k + \xi(y)^h + \xi(z)^\ell = \mathbb{R}^d \ )$$

ii) ρ Hitchin $\Rightarrow$ ρ Borel-Anosov, i.e. the (lift of) the geodesic flow on

$$T^1S \times_\rho \mathbb{R}^d := T^1\tilde{S} \times \mathbb{R}^d \big/ (v,X) \sim (\gamma v, \rho(\gamma)X)$$

uniformly expands/contracts in the directions determined by ξ.

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Geodesic laminations

Let $\tilde{S} \to S$ univ. cover.

(oriented) geodesic g of $\tilde{S}$ $:=$ pair $(g^+, g^-) \in \partial\Gamma \times \partial\Gamma \setminus \{\text{diagonal}\}$

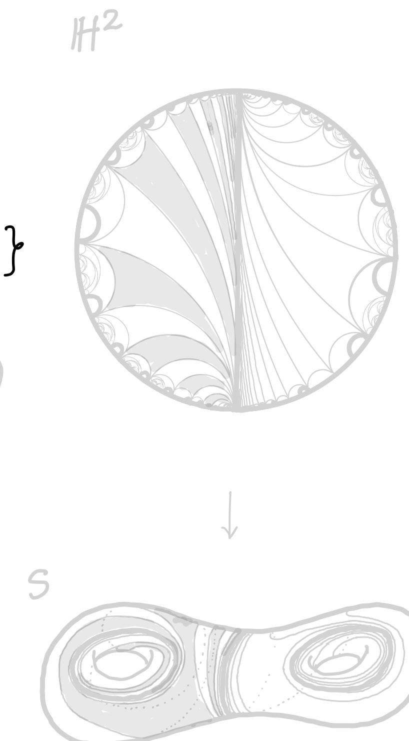
(oriented) ideal triangle T of $\tilde{S}$ $:=$ triple $(x_T, y_T, z_T) \in (\partial\Gamma)^3 \setminus \{\text{diagonals}\}$

(Every $r \in \mathrm{Hom}_{d.f}(\Gamma, PSL_2\mathbb{R})$ induces $\phi_0 : \partial\Gamma \to \partial\mathbb{H}^2$ r-equiv homeo)

(geodesic) lamination of S: $\tilde{\lambda} \subset \{\text{geodesics of } \tilde{S}\}$ Γ-invariant closed subset

s.t. $\forall g \neq h \in \tilde{\lambda}$ g, h disjoint

$\lambda := \tilde{\lambda}/\Gamma$ is maximal if $S \setminus \lambda = \bigsqcup$ interior of ideal triangles



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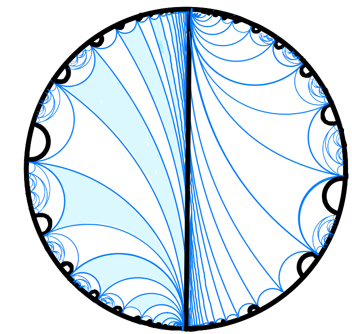

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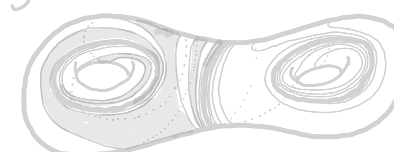

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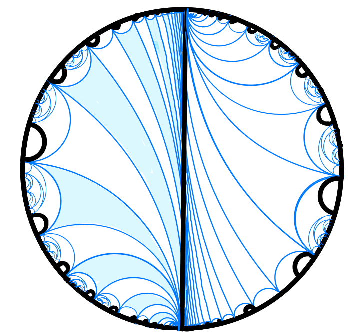

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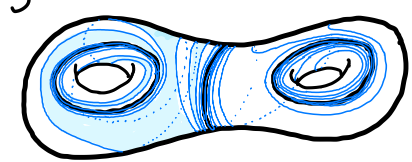

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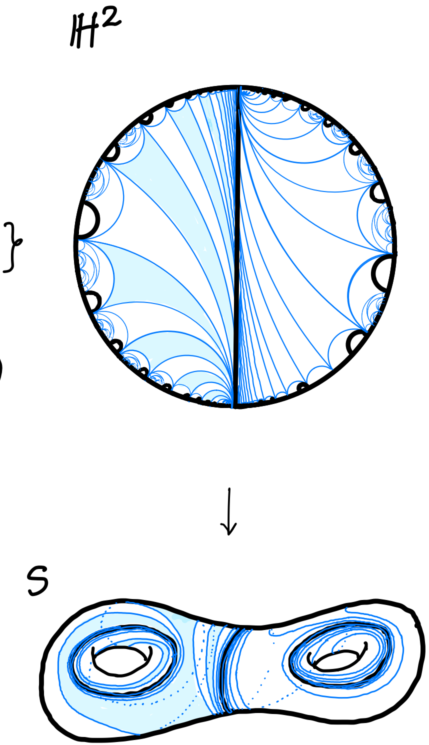

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Fix λ max lam. $\forall T$ triangle of $\tilde{\lambda}$, $\forall k,h,\ell \geq 1 : k+h+\ell = d$,

define triangle invariants

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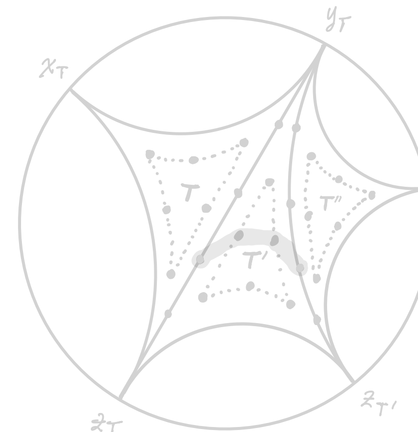

(d = 5)

holds only if adjacent

$\Longrightarrow (\sigma, \tau)$ are a λ-cocyclic pair, i.e.

i) symmetries: $\sigma_i(T,T') = \sigma_{d-i}(T',T)$, $\tau_{hk\ell}(x_T, y_T, z_T) = \tau_{k\ell h}(y_T, z_T, x_T)$,

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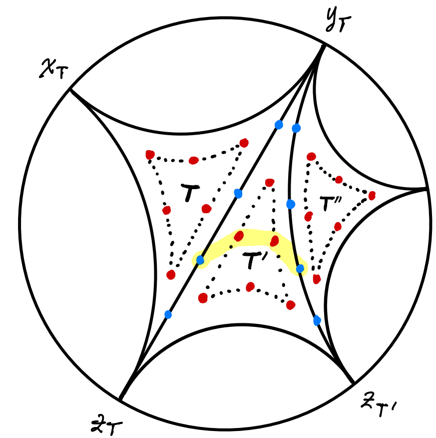

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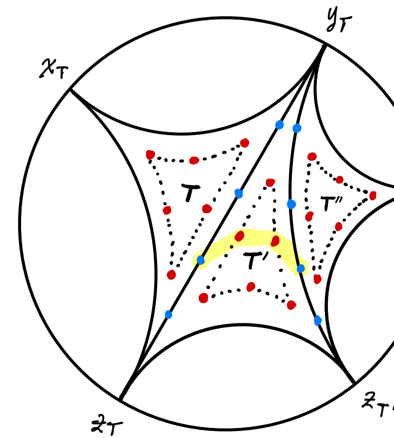

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Fix λ max lam. $\forall T$ triangle of $\tilde{\lambda}$, $\forall k,h,\ell \geq 1 : k+h+\ell = d$,
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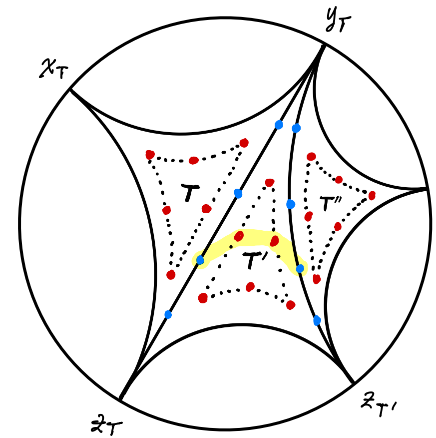
$$\tau_{kh\ell}(x_T, y_T, z_T) := \log\left(\frac{\xi(x_T)^{h+1} \wedge \xi(y_T)^{k} \wedge \xi(z_T)^{\ell-1}}{\xi(x_T)^{h-1} \wedge \xi(y_T)^{k} \wedge \xi(z_T)^{\ell+1}} \frac{\cdots}{\cdots} \frac{\cdots}{\cdots} \right) \in \mathbb{R}$$



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and shears between triangles T, T' $\forall i \in \{1,\dots,d-1\}$

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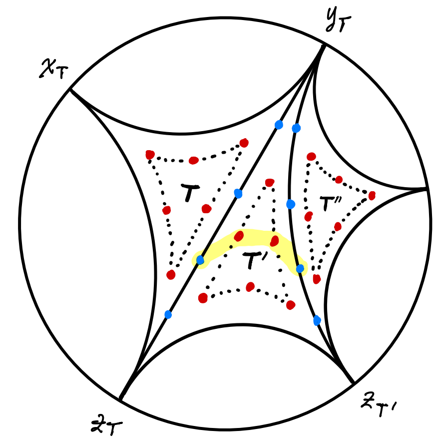

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Shear coord's for $\mathrm{Hit}_d(S)$

Thm (Bonahon-Dreyer '17, Bonahon '96 if $d=2$) $\forall d \geq 2$ $\forall \lambda$ max lam the map

$$\varphi_d^\lambda : \mathrm{Hit}_d(S) \longrightarrow \mathcal{C}(\lambda, d, \mathbb{R}) \subset \left\{ \begin{array}{c} \mathbb{R}\text{-valued} \\ \lambda\text{-cocyclic pairs} \end{array} \right\}$$
$$[\rho] \longmapsto (\sigma^\rho, \tau^\rho)$$

is a real analytic diffeomorphism, whose Image $= \mathcal{C}(\lambda, d, \mathbb{R}) = \bigcap_{\#<\infty}$ half-spaces is determined by suitable length functions.

Q: What about $PSL_d\,\mathbb{C}$?

$G = PSL_2\mathbb{C}$: Bonahon '96 constructed shear-bend coord's on $\mathcal{R}_\lambda$, with $QF(S) \subsetneq \mathcal{R}_\lambda \not\subset \mathcal{X}_{d.f.}(\Gamma, PSL_2\mathbb{C})$, via equivariant pleated surfaces

Credits : Jeffrey Brock, Mickey Mouse example

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Pleated surfaces in higher rank

Dfn (MMMZ) Let $d \geq 2$, λ max lam. A d-pleated surface with pleating locus λ is a pair $(\rho : \Gamma \to PSL_d\mathbb{C},\ \xi : \partial\tilde{\lambda} \subset \partial\Gamma \to \mathcal{F}lags(\mathbb{C}^d))$ satisfying:

- ξ ρ-equivariant, $\xi_{\tilde{\lambda}} : \tilde{\Lambda}^{or} \to \mathcal{F}lags(\mathbb{C}^d)^2$ (locally Hölder) continuous,
- ξ λ-transverse and λ-generic,
- ρ is λ-Borel Anosov, i.e. the lift of the geodesic flow $(\varphi_t)_t$ over

$$T^1\lambda \times_\rho \mathbb{C}^d := T^1\tilde{\lambda} \times \mathbb{C}^d \big/ (v,X) \sim (\gamma v, \rho(\gamma)X) \longrightarrow T^1\lambda$$

uniformly contracts/expands in the directions prescribed by ξ.

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Main result

Given (ρ, ξ) d-pleated surface, let

$$\sigma_i(T,T') = \log\left(\overset{\mathbb{C}^*}{\text{same expression as Bonahon-Dreyer}}\right), \quad \tau_{hk\ell}(x_T, y_T, z_T) = \log\left(\overset{\mathbb{C}^*}{\text{same expression as Bonahon-Dreyer}}\right) \in \mathbb{C}/2\pi i\mathbb{Z}$$

Rmk (σ,τ) well defined since ξ is λ-transverse & λ-generic

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Thm (MMMZ, Bonahon '96 for $d=2$) $\forall d \geq 2$, $\forall \lambda$ max lam, the map

$$\begin{aligned} \Psi_d^\lambda : \tilde{\mathcal{R}}_d^\lambda &\longrightarrow C(\lambda, d, \mathbb{C}/2\pi i\mathbb{Z}) \times \mathcal{X}_d \quad \longleftarrow \text{affine space over } PSL_d\mathbb{C} \\ \rho &\longmapsto (\sigma^\rho, \tau^\rho; \xi(x_{T_0}), \xi(y_{T_0}), \xi(z_{T_0})^{1}) \end{aligned}$$

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Steps of the proof:

i) $\forall (\xi, \rho)$ d-pleated surface, its shear-bend coord's $(\sigma, \tau) \in C(\lambda, d, \mathbb{C}/2\pi i\mathbb{Z})$

(extend length functions $\ell_i^{\mathbb{R}}$ from Dreyer, Bonahon-Dreyer to $\ell_i^{\mathbb{C}}$ + homological interpretation of $C(\lambda, d, \mathbb{C}/2\pi i \mathbb{Z})$)

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(Generalizes the case $d=2$, requires notion of eruption from Wienhard-Zhang-Zhu)

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Thank you for your attention!

Remarks/questions

- $\forall d \geq 2$ and $\forall \lambda$ (finite) max lam, $\exists\, [\rho] \in \mathcal{R}_d^\lambda$ non-discrete.

- if $d > 2$ and λ finite max lam, then $\widetilde{\mathcal{R}}_d^\lambda \subset \mathrm{Hom}(\Gamma, PSL_d\mathbb{C})$ contains reducible representations, and

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contains non-Hausdorff points of $\mathcal{X}(\Gamma, PSL_d\mathbb{C})$. Can we characterize them in terms of shear-bend coord's ?

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