1 Viability of Raman coupling for realistic quantum computation

Raman coupling allows to dynamically couple two long-lived atomic levels $|0\rangle$ and $|1\rangle$, that do not have a direct dipole transition, by means of an intermediate level $|e\rangle$ strongly optically coupled (electric dipole) to both of them. By carefully choosing detunings and amplitudes of two lasers, it is possible to obtain full transition $|0\rangle \to |1\rangle$ without ever populating substantially the intermediate, short-lived level, albeit at second perturbative order. Consequently, the fast decay of the intermediate level does not really have an impact.

But how much true is all of that? Let us find out in this exercise.

1. To begin with consider only the spontaneous emissions from level $|e\rangle$ into EACH of the two lower-energy levels. How do the spectral lines broaden? Is there interference between the two emission processes?

2. Convert the spontaneous emission line-broadening into a set of Lindblad (jump) operators. How many do you need?

3. Write now the full master equation of a coupled three-level lambda / Raman system, plus the spontaneous emission terms. Is it reasonable to work in Rotating-Wave approximation? Highlight the RWA validity regime.

4. Integrate the master equation dynamics in time, and find the appropriate regimes where the emission has negligible impact. Compare with your intuition and interpret the result.

5. Let us now go beyond simple spontaneous emission. Repeat the calculation of line broadening for a two-level atom (energy separation ω_{eg}), but this time, instead of using a photon vacuum (zero temperature) we use a laser state (coherent state α at a given mode $k = \omega_L/c, \lambda$, vacuum elsewhere). Interpret your result in terms of stimulated emission/absorption.

6. (Open ended) Try once again to address the open-system 3-level problem. But this time attempt to add all the contributions (spontaneous emission, stimulated emission and absorption) to the Master Equation.

2 The Mølmer-Sørensen gate

The Cirac-Zoller gate performs poorly in realistic platforms because it requires perfect cooling of the phonon mode (say the center-of-mass COM mode), and its fidelity drops at higher temperatures. A better solution came out in '99, and it does not even need tightly-focused lasers!

Precisely, let us consider two lasers. Each of the two lasers shines on BOTH ions inside a harmonic trap (trap frequency ω_T) in a perfectly symmetric way (same coupling of each laser with ion A and ion B). Each ion can be modeled as a two-level system $|g\rangle$ and $|e\rangle$, with transition frequency ω_{eg}. The first laser is detuned by $+\delta$ from the COM red sideband, the second laser is detuned by $-\delta$ from the COM blue sideband (δ can be positive or negative but it is small compared to ω_T).

1. Assume for now that the system starts at the level $|ggn\rangle$ where n is the number of phonons in the COM mode. Can you identify all the quantum states (of system ions + phonons) that are dynamically coupled to it (in RWA) ?

2. In this 6-dimensional space, two states are actually decoupled from the other 4 by the exchange symmetry (ion 1 $\leftrightarrow$ ion 2). Analyze the remaining four states, and write the effective Hamiltonian upon them. Highlight the validity range of the RWA and Lamb-Dicke regime.

3. Show that the Hamiltonian can be considered as a perturbation capable of lifting the (rotating frame) degeneracy between $|ggn\rangle$ and $|een\rangle$ at second perturbative order. Highlight the validity of the perturbative regime.

4. Generate a maximally-entangled state on two ion qubits using this mechanism. Highlight the required time duration of the laser pulses to achieve maximal entanglement. Does it depend on the phonon number n?

5. Now assume that the phonons are in a thermal state at finite temperature. Can you still generate the maximally entangled state? Argument the technological implications.

6. (Open ended) Argue that MS-mechanism plus single qubit resources are sufficient for universal quantum computation (at least for two qubits). How many additional lasers would you need?

3 From Adiabatic Passage to STIRAP

The "Stimulated Raman Adiabatic Passage" (STIRAP) is a trick to *adiabatically* (= slowly) transfer the population/probability from state $|0\rangle$ to state $|1\rangle$ of a three-level lambda system $|0\rangle \leftrightarrow |e\rangle \leftrightarrow |1\rangle$. Since it is an adiabatic process, let us refresh our mind about adiabatic driven dynamics by solving the Landau-Zener problem.

Consider first a two-level driven system (in the rotating frame) described by the Hamiltonian $\hat{H} = \Delta\hat{\sigma}_z + (\Omega/2)(\hat{\sigma}_+ + \hat{\sigma}_-)$, where $\Delta = \alpha t$, for $-\tau < t < \tau$.

1. Plot the spectrum as a function of time and consider the ground state at $t = -\tau$. Numerically compute the probability to end in the excited state at $t = \tau$ for different values of the parameters (Ω, α, τ). Compare this result with the analytical prediction (Landau-Zener) $P_e \approx 1 - \exp(-\pi\Omega^2/(2\alpha))$, valid for sufficiently large τ. The transfer is successful if the process is adiabatic, namely small α.

And now back to the STIRAP. The protocol uses two lasers, and works like this. Turn ON laser B (the one that couples $|e\rangle$ with $|1\rangle$, and resonant with it). Then slowly turn ON laser A (the one that couples $|0\rangle$ with $|e\rangle$, and resonant with it). Then slowly turn OFF laser B. Then turn OFF laser A.

Ok, that's somehow counterintuitive, right? Wrong! Let's makes some considerations.

2. At every intermediate stage of this protocol, write down what are the instantaneous eigenstates of the Hamiltonian (for the atom). Of those three, identify which one is the 'dark state', i.e. the state that is not optically shifted due to the lasers. Show that the dark state goes indeed from $|0\rangle$ to $|1\rangle$. Track down the energy gaps from this level to the other two.

3. Now apply the adiabatic theorem from above to this problem. Observe that, when the adiabatic condition holds, the population is (almost) fully transfered from $|0\rangle$ to $|1\rangle$.

4. Estimate the imperfection of the transfer by numerical integration and decide if you can apply Landau-Zener theory to approximately predict the result. Think if this depends on the pulses shape.

4 Collapse and Revival beyond the ideal case

The features of collapse and revival do not perfectly appear in experimental setups as they are affected by the lack of ideal conditions. In particular, we will address deviation from the ideal limit of a coherent state with average number of particles $\bar{n} \equiv \langle \hat{n} \rangle \gg 1$. We will explore this situation here below in the cavity and for the Bose-Hubbard model. Consider the coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \, . \tag{1}$$

1. Demonstrate that $|\alpha\rangle$ satisfies $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, it is normalized to 1 and also that $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where the displacement operator is $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$.

2. Show that $\hat{D}^{-1}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$ and use this result to prove that the average number of particles in a coherent state is $\bar{n} = |\alpha|^2$.

Assume now that the system is a cavity in the photon vacuum and consider an atom with an oscillating dipole moment $d = d_0 \cos\omega t$, which we use to populate the cavity with photons. The Hamiltonian (acting only on the photons space) is $\hat{H} = -d\hat{E}$. Set perfect resonance $\omega = \omega_c$.

3. Show that the vacuum $|0\rangle$ evolves into a coherent state under time-evolution (*hint*: show that the time-evolution operator has the form of a the displacement operator in the appropriate frame) and verify the result numerically by performing the time-evolution and comparing with the result (1). Note: you have to choose a finite Hilbert space for the photons. Take a sufficiently large cutoff $0 \le n \le n_{\max}$, with $n_{\max} = 100$.

4. The photon number grows with time. Identify the time at which the generated coherent state has $\bar{n} = 8$ photons and verify it numerically.

5. We now turn off the oscillating dipole and consider a coherent state with $\bar{n} = 8$ photons coupled to an atom in its excited state, as done in class. Assume exact resonance between the cavity frequency and the two-level atom. Numerically compute the time evolution of the system under the Jaynes-Cummings Hamiltonian and extract the excited state population, $P_e(t)$. Compare the curve with the exact result $P_e(t) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \cos(2gt\sqrt{n+1})$ showing the agreement.

6. Verify the presence of three relevant time-scales: τ_{Rabi}, τ_{collapse} and τ_{revival} and quantitatively compare them with the results obtained analytically [*hint*: for the collapse part, fit the curve with an appropriate ansatz and compare the fitting parameters with the expected ones calculated in class (Rabi frequency and collapse time)].

7. Given the low photon occupation of the coherent state ($\bar{n} = 8$), you should have found a small discrepancy with the expected values which depends on $\bar{n}$. Repeat the calculation for the collapse time more accurately to find the improved result $\tau_{\text{collapse}} = \sqrt{2(\bar{n}+1)/(\bar{n}g^2)}$ and verify it agrees with the numerical output for different values of $\bar{n}$. If needed, decrease the photon occupation enough to see a sizeable difference.

8. Let us now consider a different physical system displaying similar features: a small number of bosonic atoms on a single site of an optical lattice. Atoms are prepared in a coherent state and let evolve under the Hamiltonian $\hat{H}_{\text{BH}} = \frac{U}{2}\hat{n}(\hat{n}-1)$. Show that this generates a collapse and revival of the superfluid parameter $\psi(t) = \langle \alpha(t)|\hat{a}|\alpha(t)\rangle$: derive the corresponding equation for $\psi(t)$, the time-scales (Rabi, collapse and revival) and their relation to U and $\bar{n}$. Finally check your results numerically.

5 Quantum simulation of a magnetic field

Atoms in optical lattices are neutral objects and this makes the analogue quantum simulation of certain electromagnetic phenomena of condensed matter physics (e.g. the Quantum Hall effect) a priori unrealistic with this platform. There are however strategies to make a neutral atom behave as a charged particle in a fictitious magnetic field. One such strategy exploits the Aharonov-Bohm effect by making the tunneling amplitudes complex. In this project, we will show an example of how such effects can be designed.

Consider first only two sites of an optical lattices $|1\rangle$ and $|2\rangle$ with a real tunneling coefficient $J > 0$ and an energy offset Δ, such that the Hamiltonian reads $\hat{H}_{12} = -J(|1\rangle\langle 2| + \text{H.c.}) + \Delta|2\rangle\langle 2|$. If $\Delta \gg J$, an atom initially located on the site $|1\rangle$ will not tunnel. We now restore tunneling by adding a time dependent drive $\hat{V}_2(t) = \Delta_\text{d}\cos(\omega t + \phi)|2\rangle\langle 2|$.

1. Show that when $\hbar\omega = \Delta$, the dynamics is governed by the effective Hamiltonian (for large ω and in the appropriate frame): $\hat{H}_\text{eff} = -J_\text{eff}|1\rangle\langle 2| + \text{H.c.}$, with $J_\text{eff} = J\mathcal{J}_1(\Delta_\text{d}/\hbar\omega)e^{i\phi}$ and $\mathcal{J}_m(x)$ being a Bessel function of the first kind. After deriving the result analytically, check the corresponding validity numerically.

2. Show that the phase ϕ can be a priori removed from the effective model by a gauge (unitary) transformation and is therefore an unphysical quantity. Will this phase affect the micromotion? Provide an example.

We now proceed to create a closed path for the atom by adding a third site. You can picture the system as three sites laying at the vertex of an equilateral triangle. We add to the previous two-site Hamiltonian the term $\hat{H}_3 = -J(|2\rangle\langle 3| + |3\rangle\langle 1| + \text{H.c.}) + \Delta|3\rangle\langle 3|$ and a drive $\hat{V}_3(t) = \Delta_\text{d}\cos(\omega t - \phi)|3\rangle\langle 3|$. The full Hamiltonian is $\hat{H}_{12} + \hat{H}_3 + \hat{V}_2 + \hat{V}_3$. Notice the minus sign in the phase of $\hat{V}_3(t)$ with respect to $\hat{V}_2(t)$.

3. Derive the effective Hamiltonian of this 3-level system and check the result numerically. How is the coupling between site $|2\rangle$ and $|3\rangle$ renormalized?

4. Determine the value(s) of parameters at which all the effective tunnelings can be tuned to be equal in absolute values (*Hint*: you can control the argument of the Bessel functions).

Let us indicate the obtained tunneling amplitude of each pair of sites (in absolute value) as $|\tilde{J}|$. The final effective Hamiltonian has the form $\hat{H} = -\tilde{J}e^{i\phi}|1\rangle\langle 2| - \tilde{J}|2\rangle\langle 3| - \tilde{J}e^{i\phi}|3\rangle\langle 1| + \text{H.c.}$.

5. Provide analytical results for this model and numerically show that an atom initially prepared in $|1\rangle$ performs on average a closed orbit with a given chirality (clockwise or counterclockwise). Determine the rotation frequency. Can you invert the chirality of motion?

6. Differently from step 2, there is now a gauge invariant quantity related to the complex tunnelings. Consider the hopping Hamiltonian $\hat{H} = -Je^{i\phi_{12}}|1\rangle\langle 2| - Je^{i\phi_{23}}|2\rangle\langle 3| - Je^{i\phi_{31}}|3\rangle\langle 1| + \text{H.c.}$. Show that the total phase accumulated in a loop $\Phi = \phi_{12} + \phi_{23} + \phi_{31}$ is invariant upon a U(1) gauge transformation $|i\rangle \to e^{i\theta_i}|i\rangle$, for $i = 1, 2, 3$, where θ_i are arbitrary local phases.

7. A U(1) gauge symmetry corresponds to the invariance of electrodynamics and one can identify the total phase Φ as related to a fictitious magnetic flux $\Phi = (e/\hbar)\oint \mathbf{A}\cdot d\mathbf{l} \approx (e/\hbar)BS$, where S is the area of the triangular plaquette. The atom thus behaves as if it had a charge e in the presence of a magnetic field B. Estimate the numerical value of B for $\phi = \pi/4$ (take the lattice spacing to be the typical one of optical lattices and chose e to be the elementary charge).