Exam Project

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Exploring Tensor Decomposition Strategies for Entanglement Analysis in Quantum Systems

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SUMMARY

Evaluate the presence of the entanglement in a tensor network

Write an algorithm to carry out the decomposition of a rank-3 tensor T of dimensions (d_1, d_2, d_3):

- construct random unitary U of dimension (d_1, d_1)
- separate one leg d_1 space into two factors of dimension (d_4, d_5)
- contract T and U via d_1, obtaining T' of rank-4 of dimensions (d_4, d_5, d_2, d_3)
- decompose T' via SVD as (d_4, d_2) vs (d_5, d_3)

Evaluate entropy of entanglement using singular values distribution

RANDOM UNITARY MATRICES

Random matrix: large number of physical models

N x N Unitary Matrix U $\longrightarrow$ $UU^* = U^*U = 1_N,$
The set of unitary matrices U(N) forms a compact Lie Group whose dimension is N^2.
We can map this into a probability space, with distribution given by the measure invariant under group multiplication $\longrightarrow$ *HAAR MEASURE*

Our aim is to efficiently create random unitary matrices correctly and numerically stable (not biased).
Need an **algorithm** based on the invariant properties of the Haar measure.

The columns of a N x N unitary matrix are orthonormal vectors in $\mathbb{C}^n$.

By applying the Gram-Schmidt orthonormalization to the columns of an arbitrary complex matrix Z of full rank with normal random entries, we should get a matrix Q distributed as the Haar measure:
$\longrightarrow$ not stable algorithm

Gram-Schmidt realizes QR-decomposition $Z = QR$ with R an upper-triangular, invertible matrix.

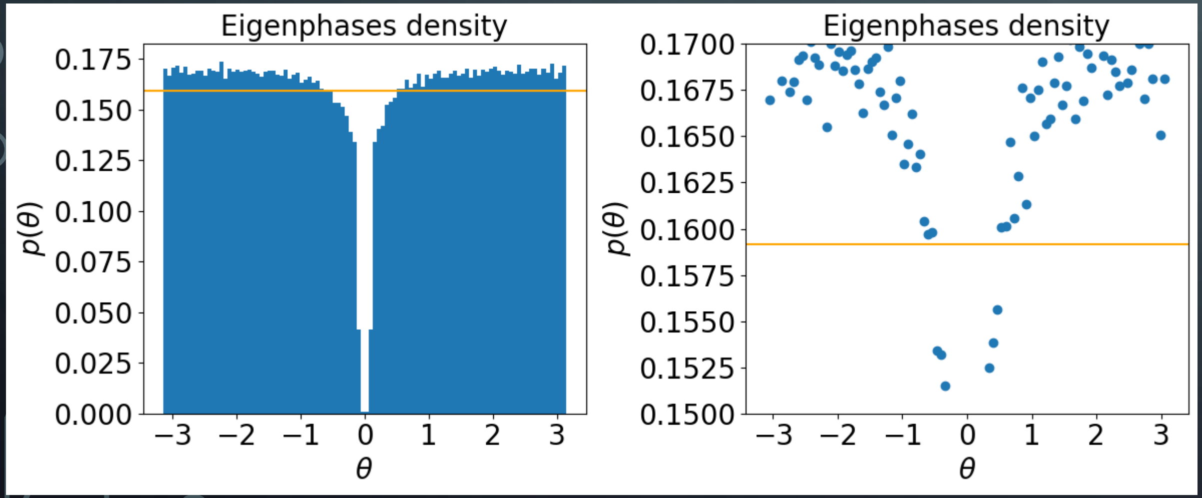
$\longrightarrow$ use Q as unitary

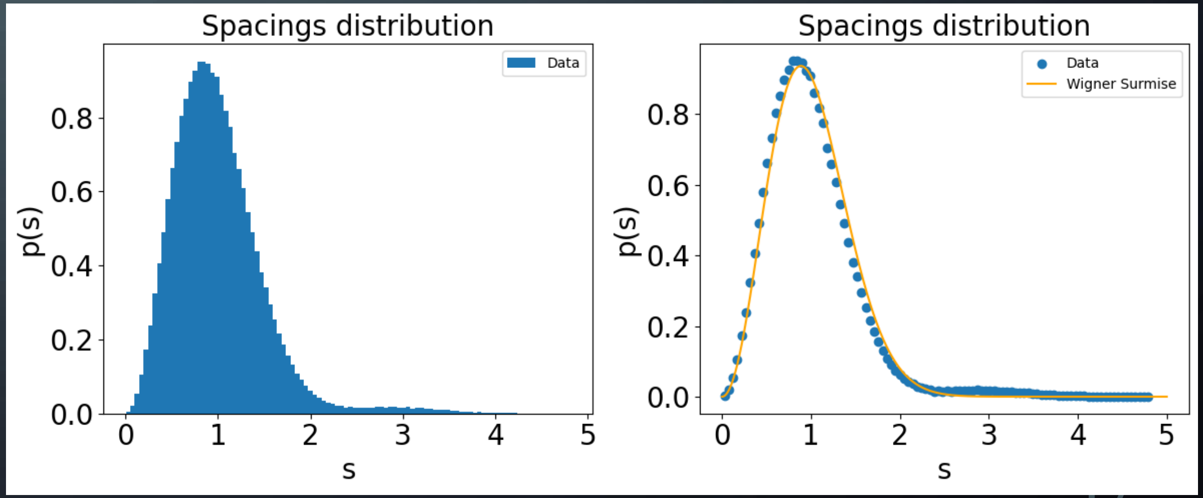
Unfortunately the output is not unitary.

The distribution of eigenvalues with the Haar measure should be uniform in the unit circle and the density should be constant $\rho(\Theta) = \frac{1}{2\pi}$.

Furthermore, the spacings distribution should follow the Wigner Surmise $p(s) = \frac{32}{\pi^2}\, s^2\, e^{-\frac{4}{\pi} s^2}$

Compute the eigenvalue density and spacing distribution:





Algorithm:

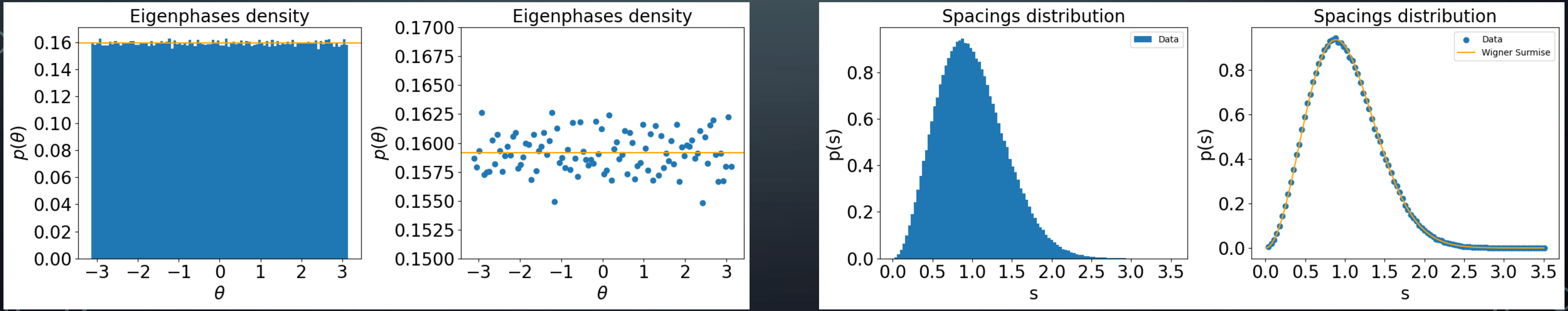
a) Take a N x N complex matrix Z whose entries are complex standard normal random variables

b) Feed Z into any QR decomposition routine, $Z = QR$

c) We create a diagonal matrix $\Lambda = \begin{pmatrix} \frac{r_{11}}{|r_{11}|} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{r_{NN}}{|r_{NN}|} \end{pmatrix}$, where the r_{ii} are diagonal elements of R

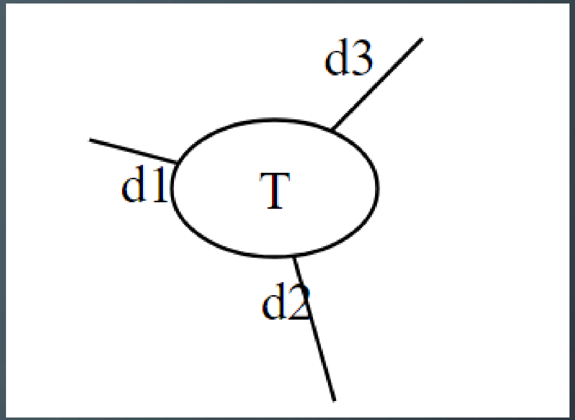
d) Now the diagonal elements of $R' = \Lambda^{-1}R$ are always real and strictly positive, therefore the matrix $Q' = Q\Lambda$ is distributed with Haar measure

Compute again the eigenvalue density and spacing distribution:



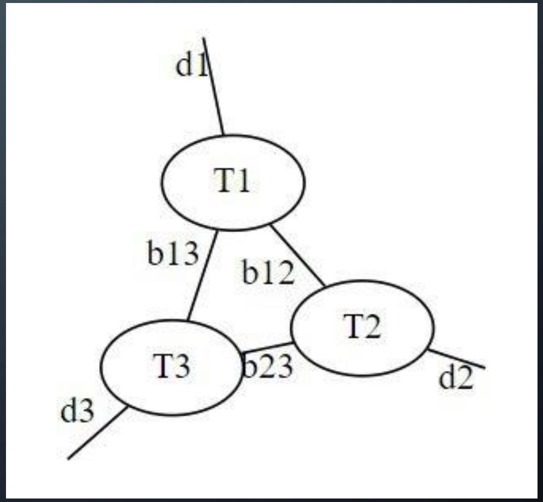
DEFINE TENSOR T

We have used the MPS approach:



Three tensors of rank-3, with one physical dimension each. Bond dimensions are used for defining entanglement between subsystems.

Each tensor is initialized with random normal variables draw from a Normal distribution with 0 mean and 1 variance. Normalization is ensured.

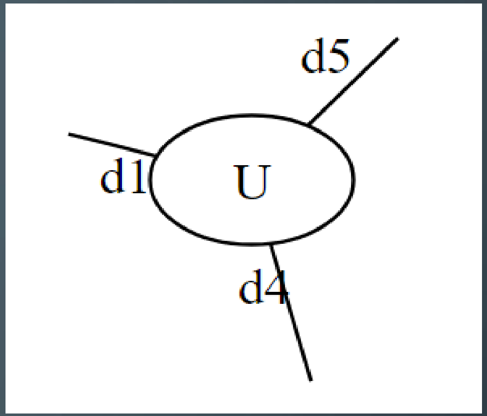


Contract along bond dimensions to get the final tensor T of rank-3, like the one above.
Normalization is again ensured.

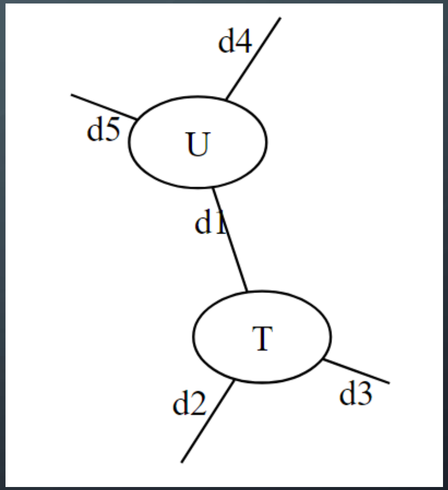
DEFINE U AND T'

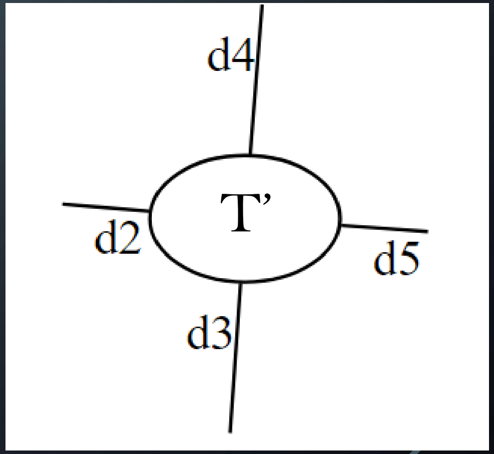
Create a random unitary matrix as seen before with dimension (d_1, d_1), and separate one space d_1 into two factors of dimension (d_4, d_5):

$d_1 \prec d_4, d_5$ inverse of a *d-nary coding* of index d_1



Now we have a tensor T of rank 3 (d_1, d_2, d_3) and a tensor U of rank 3 (d_1, d_4, d_5)





Contract T and U via d_1 thus obtaining T' of rank-4 of dimensions (d_2, d_3, d_4, d_5)

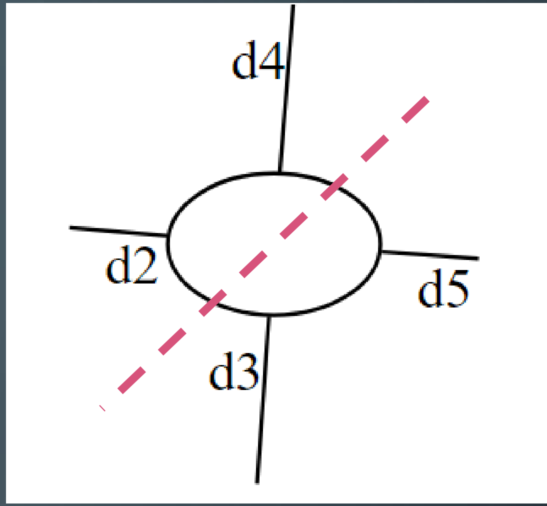
Contraction formulas:

$$\sum_{d_1} T_{d_2 d_3, d_1} U_{d_1, d_4, d_5} \equiv T_{d_2, d_3 d_1} U_{d_1, d_4, d_5} = T'_{d_2, d_3, d_4, d_5}$$

SVD

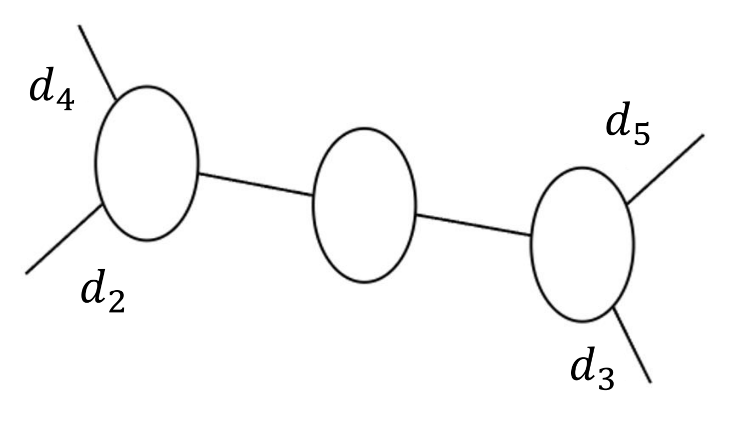
Decompose T' via SVD as (d_2 , d_4) against (d_3 , d_5) finding the three matrices: S, V, D

Select the singular values λ, diagonal of V, and from these calculate the entropy.



The SVD is analogous to the Schmidt decomposition, where the singular values are the Schmidt coefficients.

⟶ a quantum state is said to be entangled, if the number of singular values different from zero is strictly greater than 1



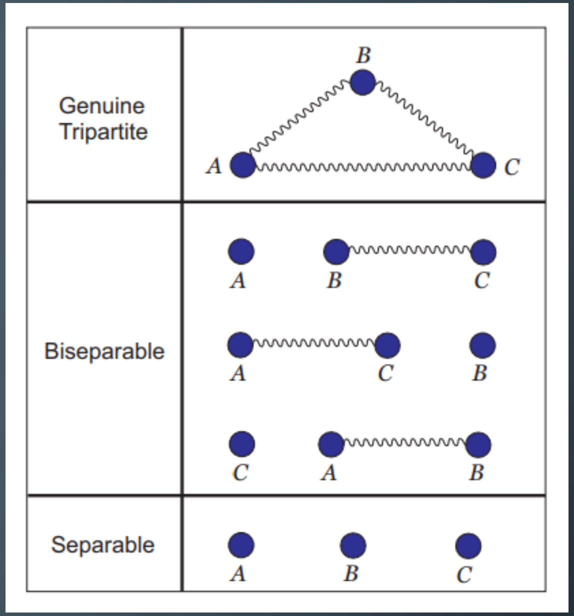
SVD formulas:

$$T'_{(d_4,d_2)(d_5,d_3)} =_{fuse} T'_{a,b} =_{SVD} \sum_{k=1}^{\min\{d_a,d_b\}} S_{a,k}V_{k,k}D_{k,b} =_{split} \sum_{k=1}^{\min\{d_a,d_b\}} S_{(d_4,d_2),k}\, V_{k,k}\, D_{k,(d_5,d_3)}$$

ENTANGLEMENT

We measure the entanglement of 3 distinct cases:

- separable
- biseparable systems
- tripartite systems



Through SVD and the Von Neumann entropy, evaluate the entanglement entropy using the singular values of the Schmidt decomposition of the bipartition (d_2 , d_4) against (d_3 , d_5) and the bipartition (d_2 , d_3) against (d_4 , d_5) :

$$S = -\sum_{k=1}^{N} \lambda_k^2 \log(\lambda_k^2)$$

In this way, it is possible to study how the entanglement is distributed on the tensor network.

SEPARABLE AND BI-SEPARABLE CASES

In this case we can implement a simple algorithm to distinguish these 4 cases:

1. d_1, d_2, d_3 are not entangled
2. d_1 is entangled with d_2
3. d_1 is entangled with d_3
4. d_2 is entangled with d_3

Given d_1 we distinguish two cases $d_4 = 1$, $d_5 = d_1$ and $d_4 = d_1$, $d_5 = 1$ because different splitting introduces positive entropy when performing an SVD (d_2, d_4) vs (d_3, d_5)

For these two split we calculate the entropy in two cases S_{vert}, S_{horiz}, with two different SVD decomposition, a vertical one and an horizontal one.

We consider the entropy $S = 0$, when it is smaller than a certain threshold $\varepsilon = 10^{-12}$

So at the end we find 4 entropy values, calculated starting from singular values:

$S_{vert}(d_4 = 1, d_5 = d_1)$
$S_{vert}(d_4 = d_1, d_5 = 1)$

$S_{horiz}(d_4 = 1, d_5 = d_1)$
$S_{horiz}(d_4 = d_1, d_5 = 1)$

<u>The method is:</u>

- If all values $S \approx 0$, states are completely determined, SEPARABLE CASE
- If both vertical $S_{vert} > 0$ and $S_{horiz} \approx 0$, we have entanglement between d_2 and d_3
- If both horizontal $S_{horiz} > 0$ but $S_{vert}(d_4 = 1\ ) > 0$ and $S_{vert}(d_4 = d_1) \approx 0$, there is entanglement between d_1 and d_2
- If both horizontal $S_{horiz} > 0$ but $S_{vert}(d_4 = 1\ ) \approx 0$ and $S_{vert}(d_4 = d_1) > 0$, there is entanglement between d_1 and d_3

GENUINE TRIPARTITE CASE

In this case, all the entropies are always greater than zero, since there is no splitting for which the entropy is null.

It is possible to detect the genuine tripartite entanglement in this way.

Qualitatively approach:

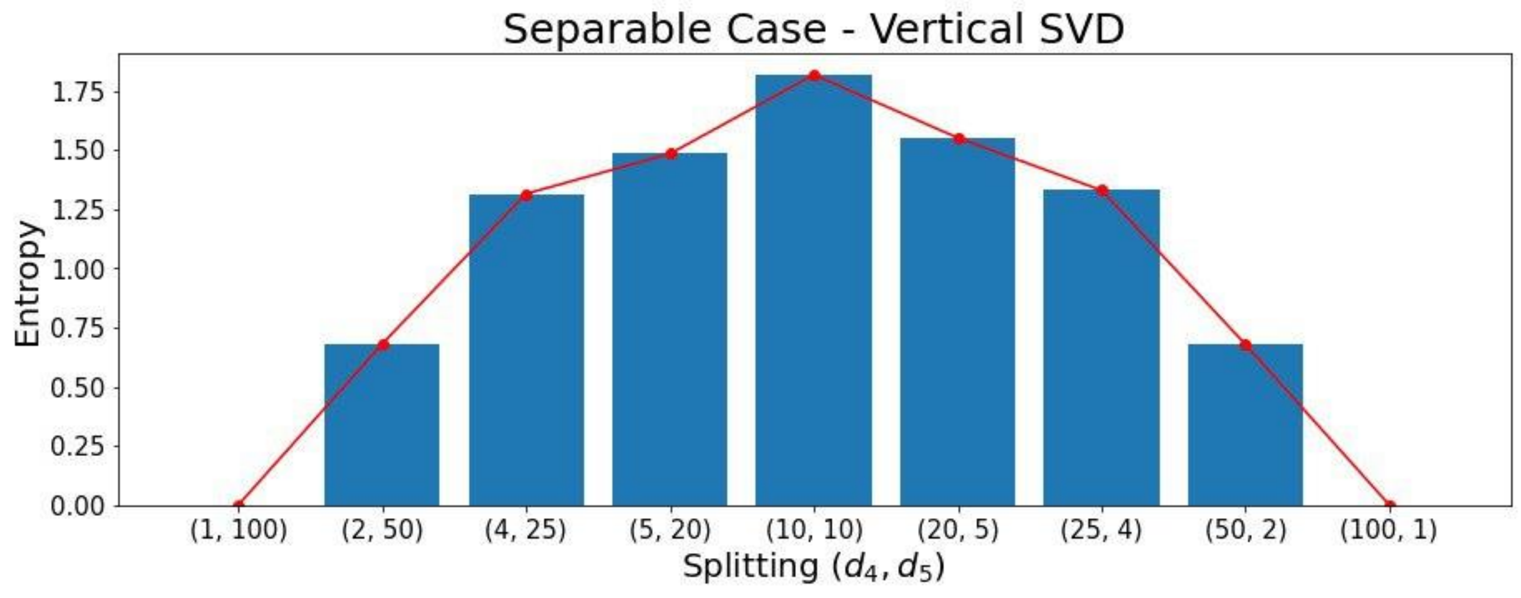
by inspecting the values of the entropies for a vertical SVD, (d_2 , d_4) against (d_3 , d_5), it is possible to infer how the entanglement is distributed in the system.

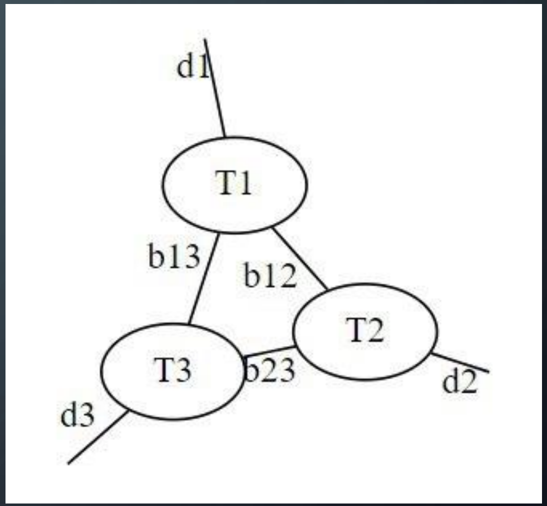
If the entropies tend to lower values for certain splittings, i.e. giving more dimension to T_2 rather than T_3, then it is probable that T_1 is more entangled with T_2 rather than T_3 and viceversa.

RESULTS

Generate a tensor in the possible combinations: completely separable and biseparable

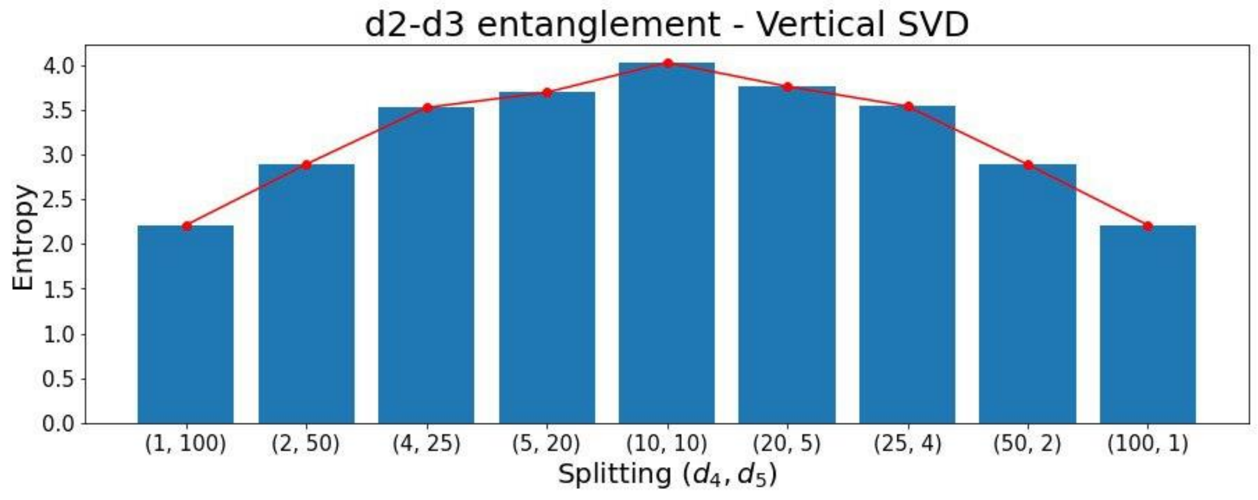
- Completely separable

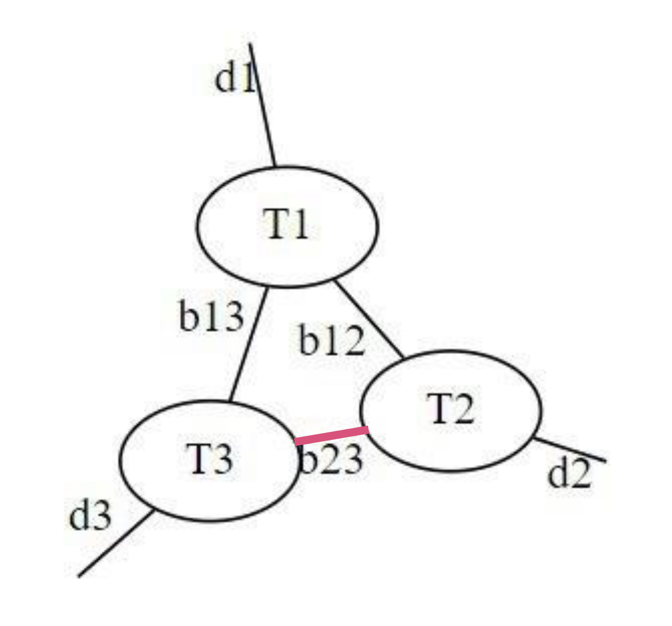




RESULTS

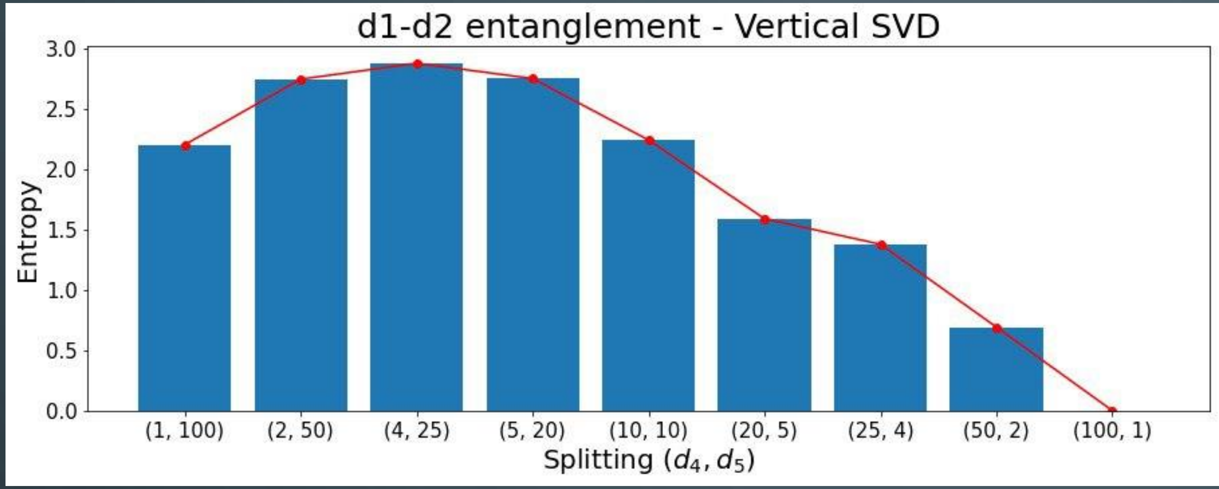
- Biseparable: T_2 and T_3

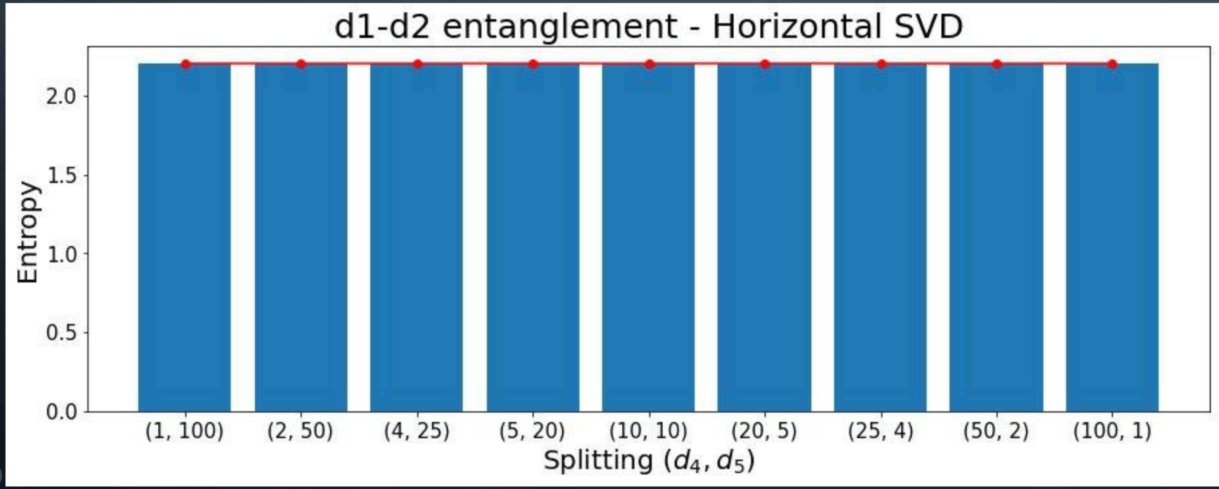


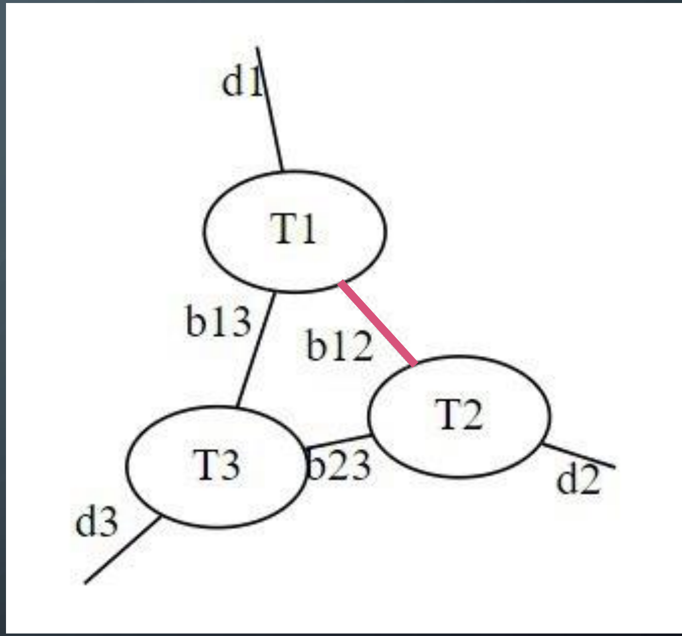


RESULTS

- Biseparable: T_1 and T_2

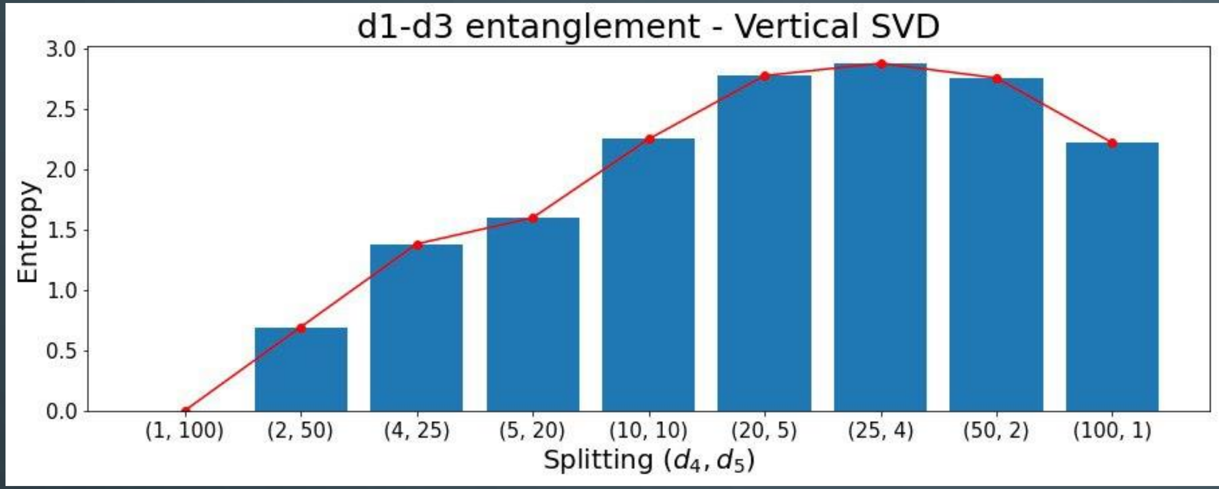


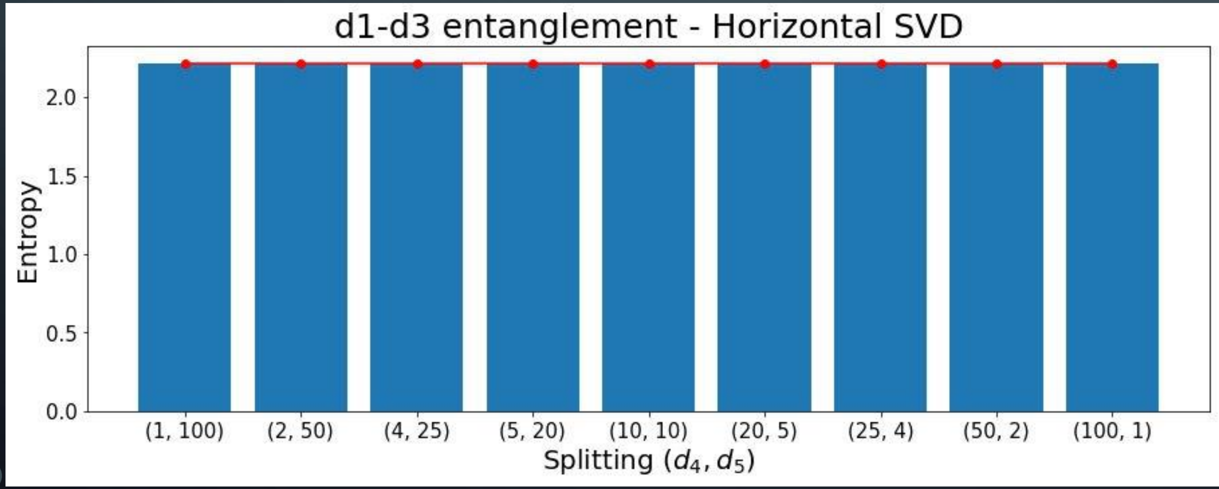


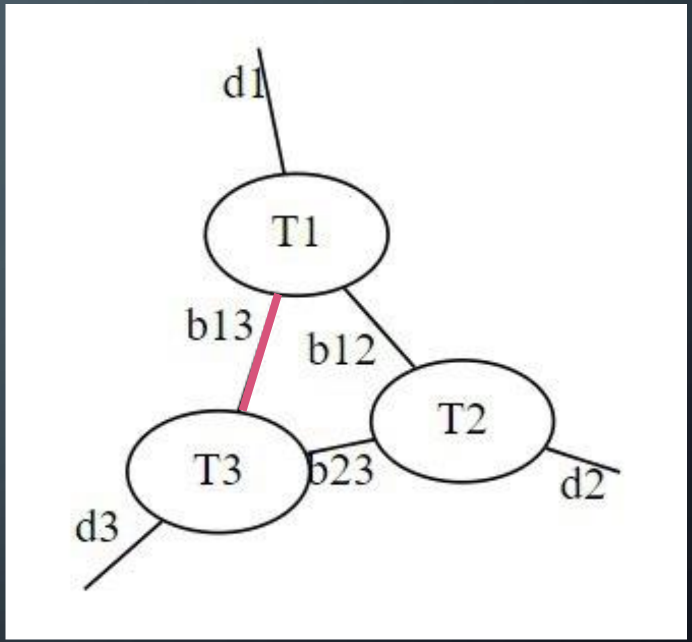


RESULTS

- Biseparable: T_1 and T_3

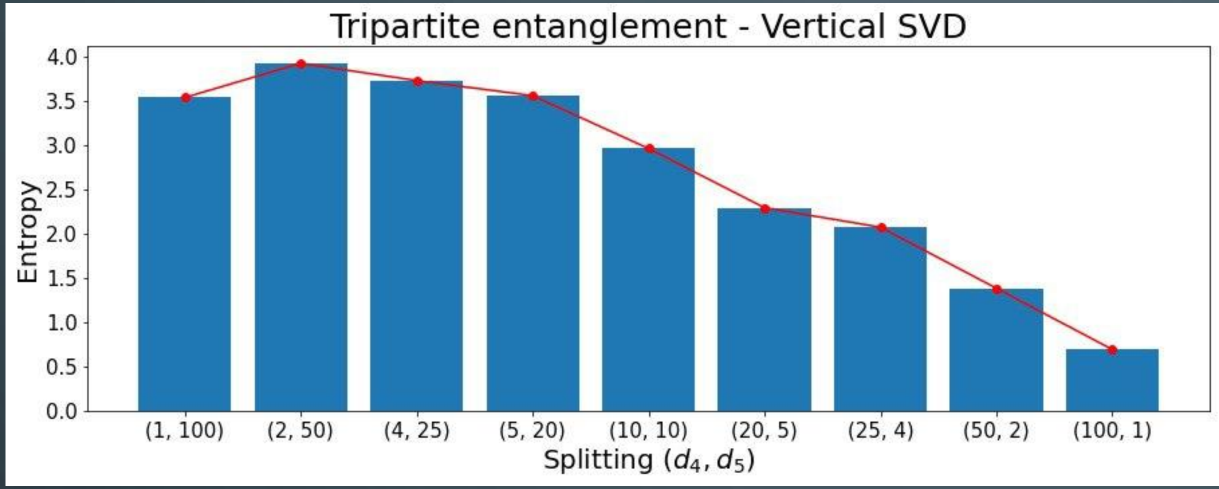


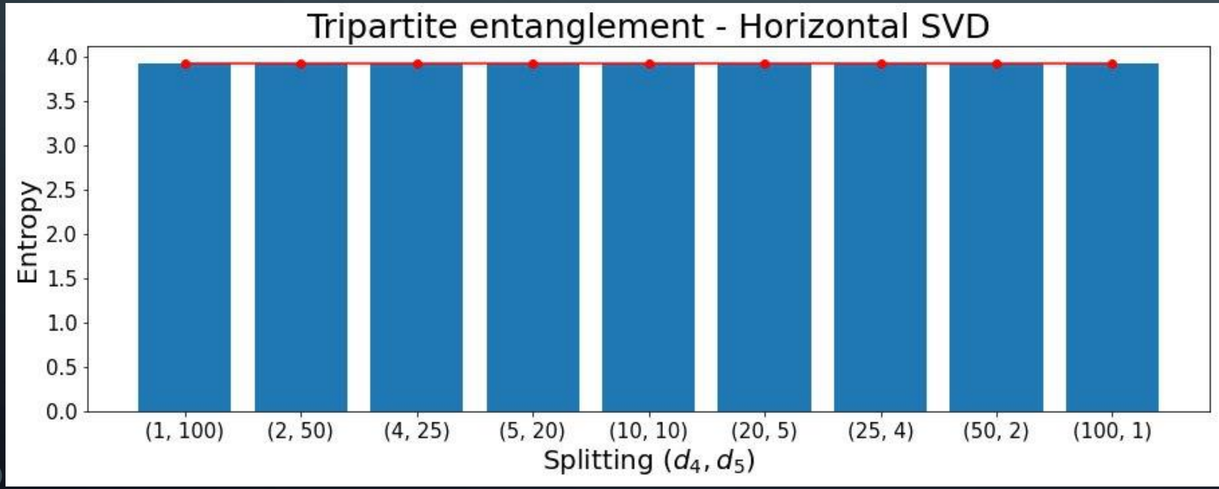


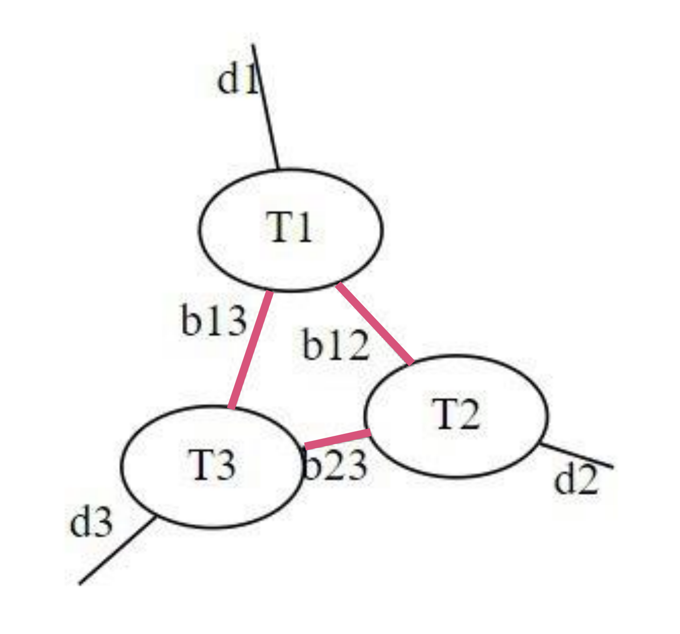


RESULTS

- Tripartite



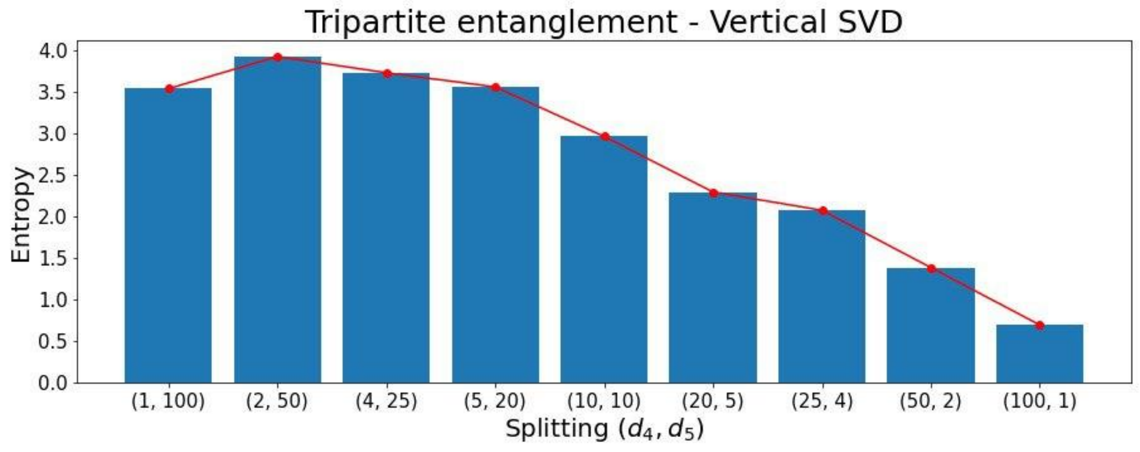




RESULTS

In the case of a genuine tripartite entanglement, it is possible to qualitatively infer the geometry of the systems.

- Case of T_1 more entangled with T_2, rather than T_3 and T_2, T_3 are not highly entangled:



Qualitatively, it is possible to say that the most probable geometry is T_1, T_2 highly entangled, respect to the other possibilities.

Unfortunately, nothing can be said for the other tensors.
Furthermore, if T_2, T_3 are highly entangled, also this approach fails.

CONCLUSIONS

Implementation of an algorithm for detecting the entanglement between sub-systems in a Tensor Network, through the use of a support unitary matrix and a SVD decomposition of the bipartition of the system.

It is possible to detect completely separable, biseparable cases and recognize if the system is highly entangled between its subsystems, i.e. genuine tripartite.

Thanks