Università Ca' Foscari Venezia

Theoretical Questions on Stochastic Calculus

Exam Preparation Booklet

Course: Stochastic Calculus for Finance

Student: Filippo Vicidomini
Master's Degree in Engineering Physics

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0.1 Question 7

Give the definition of the martingale property for a stochastic process and interpret it. Give suitable examples of stochastic processes with this property.

Answer

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}$ a filtration. A stochastic process $X(t)$ adapted to $\{\mathcal{F}_t\}$ is called a **martingale** if:

i) $\mathbb{E}[|X(t)|] < \infty$ for all t;

ii) For all $s < t$,
$$\mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s).$$

Interpretation

A martingale represents a **fair game**: given the information available up to time s, the best prediction of the value at time t is exactly the current value $X(s)$. This means the process has no drift: it does not systematically increase or decrease.

Consequently, $\mathbb{E}[X(t)] = \mathbb{E}[X(0)]$ for all t.

Examples

- **Symmetric random walk**: $M_n = \sum_{j=1}^{n} X_j$ with $X_j = \pm 1$ with equal probability, is a martingale with respect to the natural filtration.
- **Brownian motion** $W(t)$: is a martingale with respect to its natural filtration.
- **Itô integrals**: if $\Delta(t)$ is adapted and square-integrable, then
$$I(t) = \int_0^t \Delta(s)\, \mathrm{d}W(s)$$
is a martingale with zero mean.

Counterexample

A Geometric Brownian Motion
$$S(t) = S(0) e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$
is not a martingale if $\alpha \neq 0$, since it has exponential drift. However, under the risk-neutral measure $\mathbb{Q}$, the discounted price $e^{-rt} S(t)$ is a martingale. This property is fundamental in financial mathematics (e.g., Black–Scholes model).

0.2 Question 8

Describe the construction of a Brownian motion.

Answer

A Brownian motion, also known as a Wiener process, is a stochastic process $W(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the following properties:

i) $W(0) = 0$ almost surely;

ii) $W(t)$ has independent increments: for $0 \le t_0 < t_1 < \cdots < t_n$, the increments

$$W(t_1) - W(t_0),\ W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$$

are independent random variables;

iii) $W(t)$ has Gaussian increments: for $s < t$, the increment $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$;

iv) $W(t)$ has continuous trajectories almost surely.

Construction via Random Walks

One can construct a Brownian motion as the limit of suitably rescaled symmetric random walks:

- Consider a sequence $(X_j)_{j\ge1}$ of i.i.d. random variables with

$$\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = \tfrac{1}{2}.$$

- Define the partial sums (a symmetric random walk):

$$M_k = \sum_{j=1}^{k} X_j, \quad M_0 = 0.$$

  Then $\mathbb{E}[M_k] = 0$, $\mathrm{Var}(M_k) = k$.

- Define the scaled random walk:

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor}, \quad t \ge 0.$$

- As $n \to \infty$, the processes $W^{(n)}(t)$ converge in distribution to a process $W(t)$ that satisfies the above four properties.

The limit process $W(t)$ is called a **Brownian motion**.

Properties

From this construction, Brownian motion inherits:

- Mean zero: $\mathbb{E}[W(t)] = 0$;
- Variance linear in time: $\mathrm{Var}(W(t)) = t$;
- Independent, Gaussian increments;
- Quadratic variation: $[W, W]_t = t$;
- Martingale property: $\mathbb{E}[W(t) \mid \mathcal{F}_s] = W(s)$ for $s < t$.

Interpretation

Brownian motion models continuous-time randomness:

- In physics, it describes the irregular motion of particles suspended in a fluid.
- In finance, it underlies models of asset price fluctuations (e.g., geometric Brownian motion in the Black–Scholes framework).