Università Ca' Foscari Venezia

Theoretical Questions on Stochastic Calculus

Exam Preparation Booklet

Course: Stochastic Calculus for Finance

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Contents

0.1 Question 1

Explain what an ODE is, and what it means for a function $x(t)$ to be a (classical) solution. State the theorems that you know and that establish: (a) existence of a solution; (b) local existence and uniqueness of a solution; (c) global existence and uniqueness of a solution. Compare them and explain, give suitable examples.

Answer

Definition of ODE

An **ordinary differential equation (ODE)** is an equation where the unknown is a function $x(t)$ of one real variable (often time t), and the function appears together with its derivatives. A general first–order ODE can be written in normal form as

$$x'(t) = f(t, x(t)), \quad t \geq t_0.$$

Classical Solution

A function $x : [t_0, T] \to \mathbb{R}$ is a **classical solution** to the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq t_0, \\ x(t_0) = x_0, \end{cases}$$

if:

i) x is differentiable on $[t_0, T]$;

ii) it satisfies $x'(t) = f(t, x(t))$ for all $t \in [t_0, T]$;

iii) it satisfies the initial condition $x(t_0) = x_0$.

Theorems of Existence and Uniqueness

- **Peano's Existence Theorem:** If $f(t, x)$ is continuous in a neighborhood of (t_0, x_0), then there exists at least one solution $x(t)$ to the Cauchy problem on some interval around t_0. (Existence, but not uniqueness).
- **Picard–Lindelöf (Cauchy–Lipschitz) Theorem:** If $f(t, x)$ is continuous in t and *Lipschitz* continuous in x, then there exists a *unique* local solution to the Cauchy problem.
- **Global Existence and Uniqueness:** If the assumptions above hold on the entire domain (e.g. f is globally Lipschitz or satisfies suitable growth conditions preventing blow–up), then the solution can be extended uniquely to all $t \geq t_0$.

Comparison

- Peano theorem ensures existence but possibly many solutions.
- Picard–Lindelöf ensures both existence and uniqueness, but only locally in time.
- Global results require additional conditions to extend the solution to all times.

Examples

- **Non-uniqueness (Peano):**

$$x'(t) = \sqrt{x(t)}, \quad x(0) = 0.$$

  Both $x(t) \equiv 0$ and $x(t) = \frac{t^2}{4}$ are solutions, showing non-uniqueness since $f(x) = \sqrt{x}$ is not Lipschitz at 0.

- **Unique local (and global) solution (Picard–Lindelöf):**

$$x'(t) = t + x, \quad x(0) = 1.$$

  Here $f(t, x) = t + x$ is Lipschitz in x. The unique solution is

$$x(t) = Ce^t - t - 1.$$

- **Global existence:**

$$x'(t) = -x(t), \quad x(0) = x_0.$$

  Solution: $x(t) = x_0 e^{-t}$, which exists uniquely for all $t \geq 0$.

0.2 Question 2

Say what a linear ODE is. Show that, under suitable assumptions on the coefficients (exemplify), a Cauchy problem for a linear ODE has a unique solution. Prove that such solution is given by the known solution formula.

Answer

Definition

A **first–order linear ODE** is an equation of the form

$$x'(t) = a(t)x(t) + b(t), \quad t \geq t_0,$$

where $a, b : I \to \mathbb{R}$ are given real functions on an interval $I \ni t_0$. The associated **Cauchy problem** is

$$\begin{cases} x'(t) = a(t)x(t) + b(t), \\ x(t_0) = x_0. \end{cases}$$

Existence and Uniqueness (Picard–Lindelöf)

If $a(\cdot)$ and $b(\cdot)$ are continuous on I, then $f(t, x) = a(t)x + b(t)$ is continuous and Lipschitz in x on compatti:

$$|f(t, x) - f(t, y)| = |a(t)||x - y| \leq L|x - y| \quad (t \in K \Subset I).$$

Hence the Cauchy problem admits a **unique local** solution; if a, b are continuous on all of I and no blow-up occurs, the solution extends **uniquely** on I.

Derivation (Proof via integrating factor)

Consider

$$x'(t) - a(t)x(t) = b(t).$$

Let the integrating factor be

$$\mu(t) = \exp\Big(\int_{t_0}^{t} a(s)\, \mathrm{d}s \Big), \qquad \mu(t) > 0,\ \mu'(t) = a(t)\mu(t).$$

Multiply the ODE by $\mu(t)$:

$$\mu(t)x'(t) - \mu(t)a(t)x(t) = \mu(t)b(t).$$

By the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu(t)x(t)\big) = \mu(t)b(t).$$

Integrate from t_0 to t:

$$\mu(t)x(t) - \mu(t_0)x(t_0) = \int_{t_0}^{t} \mu(r)b(r)\, \mathrm{d}r.$$

Since $\mu(t_0) = 1$, $x(t_0) = x_0$, we obtain the **solution formula**

$$\boxed{x(t) = \exp\Big(\int_{t_0}^{t} a(s)\, \mathrm{d}s \Big) \Big[x_0 + \int_{t_0}^{t} \exp\Big(- \int_{t_0}^{r} a(u)\, \mathrm{d}u \Big) b(r)\, \mathrm{d}r \Big].}$$

Verification (Plug-in)

Set $\mu(t) = \exp\Big(\int_{t_0}^t a\Big)$ and write

$$x(t) = \mu(t)\Big(x_0 + \int_{t_0}^t \mu(r)^{-1} b(r)\,\mathrm{d}r\Big).$$

Differentiate:

$$x'(t) = \mu'(t)\Big(x_0 + \int_{t_0}^t \mu(r)^{-1} b(r)\,\mathrm{d}r\Big) + \mu(t)\cdot\mu(t)^{-1} b(t).$$

Since $\mu'(t) = a(t)\mu(t)$, we get

$$x'(t) = a(t)\mu(t)\Big(\cdots\Big) + b(t) = a(t)x(t) + b(t),$$

quindi x soddisfa l'ODE. Inoltre $x(t_0) = \mu(t_0)\Big(x_0 + 0\Big) = x_0$. Per unicità (Picard–Lindelöf), questa è *la* soluzione del problema di Cauchy.

Example

Let

$$x'(t) = 2x(t) + t, \qquad x(0) = 1.$$

Here $a(t) = 2$, $b(t) = t$, $\mu(t) = e^{2t}$. Thus

$$x(t) = e^{2t}\left(1 + \int_0^t e^{-2r}\,r\,\mathrm{d}r\right) = e^{2t}\left(1 - \tfrac{1}{2}te^{-2t} - \tfrac{1}{4}e^{-2t} + \tfrac{1}{4}\right).$$

This (unique) solution is defined for all $t \in \mathbb{R}$.

0.3 Question 6

State and interpret the definition of: a) sigma algebra; b) sigma algebra generated by a random variable; c) filtration; d) stochastic process; e) stochastic process adapted to a filtration.

Answer

a) Sigma algebra

Let Ω be a sample space. A **σ-algebra** $\mathcal{F}$ on Ω is a collection of subsets of Ω such that:

i) $\Omega \in \mathcal{F}$;

ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closed under complementation);

iii) If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable unions).

By De Morgan's laws, $\mathcal{F}$ is also closed under countable intersections.

Interpretation: a σ-algebra represents the collection of events that can be "observed" or "measured" in a probabilistic experiment.

Example: On $\mathbb{R}$, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by all open intervals (a, b).

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b) Sigma algebra generated by a random variable

Given a random variable $X : \Omega \to \mathbb{R}$, the **$\sigma$-algebra generated by** X, denoted $\sigma(X)$, is the smallest σ-algebra such that X is measurable. Formally,

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}.$$

Interpretation: $\sigma(X)$ contains exactly the events that can be described in terms of the knowledge of X.

Example: If $X(\omega) = 1$ when a coin toss is Head and 0 otherwise, then $\sigma(X) = \{\emptyset, \Omega, \{X = 1\}, \{X = 0\}\}$.

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c) Filtration

A **filtration** $\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing family of σ-algebras:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \text{for all } 0 \leq s \leq t.$$

Interpretation: $\mathcal{F}_t$ represents the information available up to time t. As time progresses, information increases.

Example: For a Brownian motion $W(t)$, the *natural filtration* is $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$, i.e. all events determined by the past trajectory of W up to time t.

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d) Stochastic process

A **stochastic process** is a family $\{X(t)\}_{t \geq 0}$ of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Interpretation: $X(t)$ describes the random evolution of a system in time. - Fixing t, $X(t)$ is a random variable on Ω. - Fixing $\omega \in \Omega$, $t \mapsto X(t, \omega)$ is a trajectory (sample path).

Example: A random walk $M_n = \sum_{j=1}^{n} X_j$ with i.i.d. ± 1 steps is a stochastic process indexed by $n \in \mathbb{N}$.

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e) Adapted stochastic process

A stochastic process $\{X(t)\}_{t \geq 0}$ is **adapted** to a filtration $\{\mathcal{F}_t\}$ if, for each t, $X(t)$ is $\mathcal{F}_t$-measurable.

Interpretation: At time t, the value $X(t)$ depends only on the information available up to t, not on the future.

Example: The Brownian motion $W(t)$ is adapted to its natural filtration $\{\mathcal{F}_t\}$, since $W(t)$ is $\mathcal{F}_t$-measurable by construction.

0.4 Question 7

Give the definition of the martingale property for a stochastic process and interpret it. Give suitable examples of stochastic processes with this property.

Answer

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}$ a filtration. A stochastic process $X(t)$ adapted to $\{\mathcal{F}_t\}$ is called a **martingale** if:

i) $\mathbb{E}[|X(t)|] < \infty$ for all t;

ii) For all $s < t$,
$$\mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s).$$

Interpretation

A martingale represents a **fair game**: given the information available up to time s, the best prediction of the value at time t is exactly the current value $X(s)$. This means the process has no drift: it does not systematically increase or decrease.

Consequently, $\mathbb{E}[X(t)] = \mathbb{E}[X(0)]$ for all t.

Examples

- **Symmetric random walk**: $M_n = \sum_{j=1}^{n} X_j$ with $X_j = \pm 1$ with equal probability, is a martingale with respect to the natural filtration.
- **Brownian motion** $W(t)$: is a martingale with respect to its natural filtration.
- **Itô integrals**: if $\Delta(t)$ is adapted and square-integrable, then
$$I(t) = \int_0^t \Delta(s)\,\mathrm{d}W(s)$$
is a martingale with zero mean.

Counterexample

A Geometric Brownian Motion
$$S(t) = S(0)e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$
is not a martingale if $\alpha \neq 0$, since it has exponential drift. However, under the risk-neutral measure $\mathbb{Q}$, the discounted price $e^{-rt}S(t)$ is a martingale. This property is fundamental in financial mathematics (e.g., Black–Scholes model).

0.5 Question 8

Describe the construction of a Brownian motion.

Answer

A Brownian motion, also known as a Wiener process, is a stochastic process $W(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the following properties:

i) $W(0) = 0$ almost surely;

ii) $W(t)$ has independent increments: for $0 \leq t_0 < t_1 < \cdots < t_n$, the increments

$$W(t_1) - W(t_0),\ W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$$

are independent random variables;

iii) $W(t)$ has Gaussian increments: for $s < t$, the increment $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$;

iv) $W(t)$ has continuous trajectories almost surely.

Construction via Random Walks

One can construct a Brownian motion as the limit of suitably rescaled symmetric random walks:

- Consider a sequence $(X_j)_{j\geq 1}$ of i.i.d. random variables with

$$\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = \tfrac{1}{2}.$$

- Define the partial sums (a symmetric random walk):

$$M_k = \sum_{j=1}^{k} X_j, \quad M_0 = 0.$$

Then $\mathbb{E}[M_k] = 0$, $\mathrm{Var}(M_k) = k$.

- Define the scaled random walk:

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor}, \quad t \geq 0.$$

- As $n \to \infty$, the processes $W^{(n)}(t)$ converge in distribution to a process $W(t)$ that satisfies the above four properties.

The limit process $W(t)$ is called a **Brownian motion**.

Properties

From this construction, Brownian motion inherits:

- Mean zero: $\mathbb{E}[W(t)] = 0$;
- Variance linear in time: $\mathrm{Var}(W(t)) = t$;
- Independent, Gaussian increments;
- Quadratic variation: $[W, W]_t = t$;
- Martingale property: $\mathbb{E}[W(t) \mid \mathcal{F}_s] = W(s)$ for $s < t$.

Interpretation

Brownian motion models continuous-time randomness:

- In physics, it describes the irregular motion of particles suspended in a fluid.
- In finance, it underlies models of asset price fluctuations (e.g., geometric Brownian motion in the Black–Scholes framework).

0.6 Question 9

Describe the construction of a random walk and of a scaled random walk. Show that a Brownian motion can be obtained as a limit of scaled random walks.

Answer

Random Walk

Let $\{X_j\}_{j\geq 1}$ be a sequence of i.i.d. random variables with

$$\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = \tfrac{1}{2}.$$

Define the partial sums

$$M_k = \sum_{j=1}^{k} X_j, \quad M_0 = 0.$$

The process $\{M_k\}_{k\in\mathbb{N}}$ is called a **symmetric random walk**. Properties:

- $\mathbb{E}[M_k] = 0$, $\mathrm{Var}(M_k) = k$.
- Increments are independent and stationary: $M_{n+m} - M_n \sim \mathcal{N}(0, m)$.
- M_k is a martingale with respect to the natural filtration.

Scaled Random Walk

To approach a continuous-time process, rescale both time and space:

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor}, \quad t \geq 0.$$

Interpretation:

- Time is accelerated by factor n (steps of size $1/n$).
- Space is scaled down by $1/\sqrt{n}$ (variance normalisation).

Thus $W^{(n)}(t)$ is a piecewise constant, right–continuous process with jumps $\pm 1/\sqrt{n}$ at times k/n.

Limit Process: Brownian Motion

By Donsker's invariance principle (or functional Central Limit Theorem),

$$W^{(n)}(t) \xrightarrow{d} W(t), \quad \text{as } n \to \infty,$$

where $W(t)$ is a **Brownian motion**.

Proof idea:

- Finite-dimensional distributions: by the Central Limit Theorem, for fixed t,

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor} \overset{d}{=} \mathcal{N}(0, t).$$

- Independence of increments: inherited from independence of X_j.
- Continuous trajectories: obtained in the limit (the $W^{(n)}$ are piecewise constant, but converge in distribution to a continuous process).

Conclusion

A Brownian motion $W(t)$ is obtained as the scaling limit of a symmetric random walk.

$$W(t) = \lim_{n\to\infty} W^{(n)}(t) \quad \text{in distribution.}$$

Example

Simulating many paths of a scaled random walk with large n, the trajectories approximate continuous Brownian paths with variance t and independent Gaussian increments.