Notes of Stochastic Calculus

A brief introduction to stochastic calculus and its applications in finance

Filippo Vicidomini

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Chapter 1

Deterministic Models and ODEs

1.1 Ordinary Differential Equations (ODEs)

An **ordinary differential equation (ODE)** is an equation involving an unknown function $x(t)$ and its derivatives with respect to an independent variable, typically time t. The general form of a first-order ODE is:

$$\frac{dx}{dt} = f(t, x(t)),$$

where f is a given function and $x(t)$ is the unknown function.

A function $x(t)$ is said to be a **classical solution** if it is differentiable and satisfies the equation for every t in an open interval, that is:

$$\frac{dx(t)}{dt} = f(t, x(t)) \quad \text{for all } t \in I,$$

where I is an open interval containing the initial point t_0, and $x(t_0) = x_0$.

Fundamental Theorems

1. **Existence (general)** — *Peano's Theorem*: If $f(t, x)$ is continuous in a neighborhood of the initial point (t_0, x_0), then there exists at least one local solution to the ODE:

   $$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0.$$

   However, uniqueness is not guaranteed.

2. **Local existence and uniqueness** — *Picard–Lindelöf Theorem (also known as Cauchy–Lipschitz)*: If $f(t,x)$ is continuous in a neighborhood of (t_0, x_0) and locally Lipschitz in x, then there exists an interval $I \ni t_0$ and a unique solution $x(t)$ defined on I that solves the initial value problem:
$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0.$$
3. **Global existence and uniqueness**: If f is globally Lipschitz in x and continuous in t, and if the solution remains bounded for all t, then the solution can be extended to the entire maximal interval of definition — that is, it is *globally defined*.

Comparison and Examples

- *Peano's theorem* guarantees existence but not uniqueness. Example:
$$\frac{dx}{dt} = \sqrt{|x|}, \quad x(0) = 0.$$
This ODE admits multiple solutions, such as $x(t) \equiv 0$ or $x(t) = \frac{1}{4}(t-c)^2$ for $t \geq c$.
- *Picard–Lindelöf theorem* guarantees uniqueness and continuous dependence on initial data, but only locally. Example:
$$\frac{dx}{dt} = x, \quad x(0) = 1.$$
The unique solution is $x(t) = e^t$, which is also globally defined in this case.
- *Global existence* requires stronger conditions (e.g., bounded growth of f). If f grows too fast, solutions may "blow up" in finite time. Example:
$$\frac{dx}{dt} = x^2, \quad x(0) = 1.$$
The solution is $x(t) = \frac{1}{1-t}$, which diverges as $t \to 1^-$: it is not globally defined.

1.2 Linear Ordinary Differential Equations

A **linear ordinary differential equation (ODE)** of first order has the form:
$$\frac{dx}{dt} + a(t)x(t) = b(t),$$

where $a(t)$ and $b(t)$ are given functions (typically continuous), and $x(t)$ is the unknown function.

A **Cauchy problem** for a linear ODE consists in solving:

$$\frac{dx}{dt} + a(t)x(t) = b(t), \quad x(t_0) = x_0.$$

Theorem (Existence and Uniqueness): If $a(t)$ and $b(t)$ are continuous on an open interval I containing t_0, then the Cauchy problem has a unique solution defined on all of I.

Proof (Sketch): We rewrite the equation in standard form:

$$\frac{dx}{dt} = -a(t)x(t) + b(t),$$

and observe that the right-hand side is continuous and Lipschitz in x (since it is linear in x), satisfying the hypotheses of the Picard–Lindelöf theorem. Hence, the solution exists and is unique.

General Solution Formula: We can solve the linear ODE using an integrating factor. Define:

$$\mu(t) := e^{\int_{t_0}^{t} a(s)\, ds},$$

which is always positive since the exponential of a continuous function is continuous and never zero.

Multiplying both sides of the ODE by $\mu(t)$, we get:

$$\mu(t)\frac{dx}{dt} + \mu(t)a(t)x(t) = \mu(t)b(t).$$

Using the product rule:

$$\frac{d}{dt}[\mu(t)x(t)] = \mu(t)b(t).$$

Integrating both sides from t_0 to t:

$$\mu(t)x(t) - \mu(t_0)x_0 = \int_{t_0}^{t} \mu(s)b(s)\, ds.$$

Solving for $x(t)$:

$$x(t) = \frac{1}{\mu(t)}\left(\mu(t_0)x_0 + \int_{t_0}^{t} \mu(s)b(s)\, ds\right).$$

Example: Solve

$$\frac{dx}{dt} + 2x = \sin(t), \quad x(0) = 0.$$

Here, $a(t) = 2$, $b(t) = \sin(t)$, so:

$$\mu(t) = e^{\int_0^t 2\,ds} = e^{2t}.$$

Then:

$$\frac{d}{dt}[e^{2t}x(t)] = e^{2t}\sin(t),$$

$$e^{2t}x(t) = \int_0^t e^{2s}\sin(s)\,ds.$$

Finally:

$$x(t) = e^{-2t}\int_0^t e^{2s}\sin(s)\,ds.$$

This expresses the unique solution.

1.3 Autonomous Ordinary Differential Equations

An **autonomous ODE** is a differential equation of the form:

$$\frac{dx}{dt} = f(x),$$

where the right-hand side $f(x)$ depends only on the dependent variable x, not explicitly on time t.

We assume $f(x)$ is continuous and locally Lipschitz, ensuring the existence and uniqueness of solutions to the Cauchy problem:

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0.$$

(a) Every solution is monotonic

Let $x(t)$ be a solution of the autonomous ODE. Then:

$$\frac{dx}{dt} = f(x(t)).$$

Since the sign of $\frac{dx}{dt}$ depends only on $x(t)$, and the function f does not explicitly depend on t, we have:

- If $f(x(t)) > 0$ on an interval, then $x(t)$ is strictly increasing on that interval. - If $f(x(t)) < 0$, then $x(t)$ is strictly decreasing. - If $f(x(t)) = 0$, then $x(t)$ is constant.

Thus, as long as $f(x(t)) \neq 0$, the sign of the derivative does not change, and $x(t)$ is strictly monotonic (either increasing or decreasing) on its interval of definition.

Example: Consider $\frac{dx}{dt} = x^2$, with initial condition $x(0) = 1$. Then $x(t) = \frac{1}{1-t}$, which is increasing on $(-\infty, 1)$.

(b) Convergence implies stationary point

Assume that $\lim_{t\to\infty} x(t) = C$ for some real number C. We want to prove that C must be a stationary point of the ODE, i.e., a solution of:

$$\frac{dx}{dt} = f(x) = 0.$$

Proof: Suppose, for contradiction, that $f(C) \neq 0$. Then by continuity of f, there exists a neighborhood of C, say $(C-\delta, C+\delta)$, where $f(x)$ keeps the same sign and is bounded away from 0.

Since $x(t) \to C$, there exists $T > 0$ such that $x(t) \in (C-\delta, C+\delta)$ for all $t > T$. But then $\frac{dx}{dt} = f(x(t))$ has constant sign and magnitude bounded away from 0, which implies that $x(t)$ continues to grow or decrease indefinitely — contradicting the assumption that $x(t) \to C$.

Hence, we must have:

$$f(C) = 0,$$

which means that $x(t) \equiv C$ is a stationary solution of the ODE.

Conclusion: If a solution of an autonomous ODE converges to a value C, then C must be an equilibrium point (also called a fixed point or stationary solution).

1.4 Bernoulli Cauchy Problem

Definition. The **Bernoulli differential equation** is a first-order nonlinear ordinary differential equation of the form:

$$\frac{dy}{dt} + P(t)y = Q(t)y^n,$$

where:

- $P(t)$ and $Q(t)$ are continuous functions on some interval $I \subset \mathbb{R}$;
- $n \in \mathbb{R} \setminus \{0, 1\}$ (so that the equation is nonlinear).

With an initial condition $y(t_0) = y_0$, this efines a **Cauchy problem**.

Reduction to a Linear Equation. We perform the change of variable:

$$z(t) = y(t)^{1-n}.$$

Using the chain rule:

$$\frac{dz}{dt} = (1-n)y^{-n}\frac{dy}{dt}.$$

Substitute the original ODE:

$$\frac{dy}{dt} = -P(t)y + Q(t)y^n,$$

into the expression for $\frac{dz}{dt}$:

$$\frac{dz}{dt} = (1-n)\left[-P(t)y^{1-n} + Q(t)\right] = (1-n)(-P(t)z + Q(t)).$$

Thus we obtain a
textbflinear ODE in
(z(t)
):

$$\frac{dz}{dt} + (1-n)P(t)z = (1-n)Q(t).$$

Solution. Solve the linear ODE for $z(t)$ using the integrating factor method. Then recover $y(t)$ from:

$$y(t) = (z(t))^{\frac{1}{1-n}}.$$

Example. Solve the Cauchy problem:

$$\frac{dy}{dt} + 2y = 2y^3, quady(0) = 1.$$

Here, $P(t) = 2$, $Q(t) = 2$, and $n = 3$.

Change of variable:

$$z(t) = y(t)^{-2}.$$

Then:

$$\frac{dz}{dt} = -2y^{-3}\frac{dy}{dt} = -2(-2y + 2y^3) = 4y - 4y^3 = 4z^{-1/2} - 4z^{-3/2},$$

but better to proceed via the linear method:

$$\frac{dz}{dt} = -4z + 4.$$

Solve:

$$\frac{dz}{dt} + 4z = 4 \Rightarrow z(t) = 1 + Ce^{-4t},$$

with $z(0) = y(0)^{-2} = 1 \Rightarrow C = 0$. So $z(t) = 1$, and:

$$y(t) = z(t)^{-1/2} = 1.$$

Conclusion. Bernoulli equations can be systematically reduced to linear equations and solved using standard methods. The nonlinear nature is effectively eliminated through a power substitution.

1.5 The Logistic Equation

The **logistic equation** is a classical model for population growth that includes the effect of limited resources. It is given by:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right),$$

where: - $x(t)$ is the population at time t, - $r > 0$ is the intrinsic growth rate, - $K > 0$ is the carrying capacity of the environment.

Interpretation: - For small x, the term $\left(1 - \frac{x}{K}\right) \approx 1$, so the equation behaves like exponential growth: $\frac{dx}{dt} \approx rx$. - As $x \to K$, $\frac{dx}{dt} \to 0$: growth slows down and the population stabilizes at the carrying capacity. - If $x > K$, the growth rate becomes negative: the population decreases back toward K.

The logistic model captures the idea that growth is self-limiting due to competition for resources.

Qualitative analysis: The equilibria are found by solving $\frac{dx}{dt} = 0$, which yields:

$$x = 0 \quad \text{and} \quad x = K.$$

- $x = 0$ is an unstable equilibrium (any small positive population grows away from zero). - $x = K$ is a stable equilibrium (the population stabilizes at K).

We can also analyze the sign of the derivative: - If $0 < x < K$, then $\frac{dx}{dt} > 0$: population increases. - If $x > K$, then $\frac{dx}{dt} < 0$: population decreases.

Explicit solution:

To solve the logistic equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right),$$

we separate variables:

$$\frac{dx}{x(1 - x/K)} = r\,dt.$$

We use partial fractions:

$$\frac{1}{x(1 - x/K)} = \frac{1}{x} + \frac{1}{K - x} \cdot \frac{1}{K}.$$

Actually, it's better to write:

$$\frac{1}{x(1 - x/K)} = \frac{1}{K}\left(\frac{1}{x} + \frac{1}{K - x}\right).$$

Then integrate both sides:

$$\int \left(\frac{1}{x} + \frac{1}{K-x}\right) dx = \int r\, dt.$$

$$\ln|x| - \ln|K-x| = rKt + C.$$

This simplifies to:

$$\ln\left(\frac{x}{K-x}\right) = rKt + C.$$

Exponentiating both sides:

$$\frac{x}{K-x} = Ae^{rKt}, \quad \text{where } A = e^C.$$

Solving for $x(t)$:

$$x(t) = \frac{KAe^{rKt}}{1+Ae^{rKt}} = \frac{K}{1+Be^{-rKt}}, \quad \text{where } B = \frac{1}{A}.$$

Determine B using the initial condition $x(0) = x_0$:

$$x_0 = \frac{K}{1+B}, \quad \Rightarrow B = \frac{K-x_0}{x_0}.$$

Final explicit solution:

$$x(t) = \frac{K}{1+\left(\frac{K-x_0}{x_0}\right)e^{-rKt}}.$$

Conclusion: - The population grows toward the carrying capacity K. - The growth is initially exponential, then slows down due to competition. - The logistic equation models self-limited growth more realistically than exponential models.

1.6 The Solow Model for Capital Accumulation

The **Solow model** describes the dynamics of capital accumulation in an economy over time. In its simplest form, without population or technological growth, the model assumes that the capital per worker $k(t)$ evolves according to the ODE:

$$\frac{dk}{dt} = sf(k) - \delta k,$$

where:

- $k(t)$ is the capital per worker at time t,
- $f(k)$ is the production function (output per worker as a function of capital),
- $s \in (0, 1)$ is the constant savings rate,
- $\delta > 0$ is the depreciation rate of capital.

Interpretation: - The term $sf(k)$ represents the portion of output that is saved and reinvested as capital. - The term δk represents the loss of capital due to depreciation. - The difference $sf(k) - \delta k$ gives the net rate of capital accumulation.

Example: Cobb–Douglas production function

A common assumption is the Cobb–Douglas production function:

$$f(k) = k^{\alpha}, \quad \text{with } \alpha \in (0, 1).$$

Then the Solow equation becomes:

$$\frac{dk}{dt} = sk^{\alpha} - \delta k.$$

Qualitative analysis:

We can analyze the behavior of solutions by studying the sign of $\frac{dk}{dt}$: - If $k = 0$, then $\frac{dk}{dt} = 0$ (absorbing state). - If $0 < k < k^*$, then $\frac{dk}{dt} > 0$: capital increases. - If $k > k^*$, then $\frac{dk}{dt} < 0$: capital decreases.

The steady state (equilibrium) is given by:

$$sk^{*\alpha} = \delta k^* \quad \Rightarrow \quad k^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

At the steady state k^*, the capital per worker remains constant over time: the economy stabilizes.

Explicit solution:

We solve the equation:

$$\frac{dk}{dt} = sk^{\alpha} - \delta k.$$

This is a Bernoulli differential equation. Define:

$$\frac{dk}{dt} + \delta k = sk^{\alpha}.$$

Divide both sides by k^α:

$$k^{-\alpha}\frac{dk}{dt} + \delta k^{1-\alpha} = s.$$

Let $u = k^{1-\alpha}$. Then:

$$\frac{du}{dt} = (1-\alpha)k^{-\alpha}\frac{dk}{dt}.$$

Multiply both sides by $\frac{1}{1-\alpha}$:

$$\frac{1}{1-\alpha}\frac{du}{dt} = \frac{dk}{dt}k^{-\alpha}.$$

From the original equation:

$$\frac{1}{1-\alpha}\frac{du}{dt} = s - \delta k^{1-\alpha} = s - \delta u.$$

So we obtain the linear ODE:

$$\frac{du}{dt} + (1-\alpha)\delta u = (1-\alpha)s.$$

This is a linear first-order ODE. The integrating factor is:

$$\mu(t) = e^{(1-\alpha)\delta t}.$$

Multiplying both sides:

$$\frac{d}{dt}\left(e^{(1-\alpha)\delta t}u(t)\right) = (1-\alpha)se^{(1-\alpha)\delta t}.$$

Integrating both sides:

$$e^{(1-\alpha)\delta t}u(t) = \frac{s}{\delta}\left(e^{(1-\alpha)\delta t}\right) + C.$$

Solve for $u(t)$:

$$u(t) = \frac{s}{\delta} + Ce^{-(1-\alpha)\delta t}.$$

Recall $u(t) = k(t)^{1-\alpha}$, so:

$$k(t) = \left(\frac{s}{\delta} + Ce^{-(1-\alpha)\delta t}\right)^{\frac{1}{1-\alpha}}.$$

Using the initial condition $k(0) = k_0$, we find:

$$C = k_0^{1-\alpha} - \frac{s}{\delta}.$$

Final explicit solution:

$$k(t) = \left(\frac{s}{\delta} + \left(k_0^{1-\alpha} - \frac{s}{\delta}\right) e^{-(1-\alpha)\delta t}\right)^{\frac{1}{1-\alpha}}.$$

Interpretation: - If $k_0 < k^*$, capital grows toward k^*. - If $k_0 > k^*$, capital decreases toward k^*. - The economy converges to the steady state $k^* = \left(\frac{s}{\delta}\right)^{1/(1-\alpha)}$, regardless of the initial condition $k_0 > 0$.

Chapter 2

Foundations of Probability and Stochastic Processes

2.1 Basic Definitions in Probability Theory

a) σ-algebra

A **σ-algebra** $\mathcal{F}$ on a set Ω is a collection of subsets of Ω such that:

- $\Omega \in \mathcal{F}$,
- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
- If $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Interpretation: A σ-algebra represents a collection of events for which we can define probabilities in a consistent way, and it is closed under the basic operations of probability theory (complementation, countable unions).

b) σ-algebra generated by a random variable

Given a random variable $X : \Omega \to \mathbb{R}$, the **$\sigma$-algebra generated by** X is:

$$\sigma(X) = \{X^{-1}(B) \subseteq \Omega \mid B \in \mathcal{B}(\mathbb{R})\},$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. In mathematics, a Borel set is any subset of a topological space that can be formed from its open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement. Borel sets are named after Émile Borel.

Interpretation: $\sigma(X)$ is the collection of all events that can be determined by observing the value of X. It represents the "information" contained in the variable X.

c) Filtration

A **filtration** is a family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras such that:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \quad \text{for all } 0 \leq s \leq t.$$

Interpretation: A filtration represents the evolution of information over time. $\mathcal{F}_t$ contains all the events that can be observed up to time t.

d) Stochastic process

A **stochastic process** is a family of random variables $(X_t)_{t\in T}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where each $X_t : \Omega \to \mathbb{R}$ (or more generally to some measurable space) for $t \in T$.

Interpretation: A stochastic process describes the evolution of a random quantity over time. Each X_t gives the state of the system at time t.

e) Adapted stochastic process

Let $(\mathcal{F}_t)_{t\in T}$ be a filtration. A stochastic process $(X_t)_{t\in T}$ is said to be **adapted** to the filtration if, for each $t \in T$, the random variable X_t is $\mathcal{F}_t$-measurable.

Interpretation: An adapted process is one where the value of X_t at time t only depends on the information available up to time t, and not on future events.

2.2 Martingales

In probability theory, a **martingale** is a stochastic process that models a "fair game" with no predictable trend. Formally, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space. A process $(M_t)_{t\in T}$ (discrete or continuous time) is a martingale if it is adapted, integrable, and satisfies:

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s \quad \text{a.s.}, \quad \forall\, s \leq t.$$

This means the expected future value of the process, given the present, equals the current value. In other words, a martingale has *no drift*—knowing the past gives no advantage in predicting the future.

Definition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space. A process $(M_t)_{t\in T}$ is a **martingale** with respect to $(\mathcal{F}_t, \mathbb{P})$ if:

- **Adaptedness:** M_t is $\mathcal{F}_t$-measurable for each t,
- **Integrability:** $\mathbb{E}[|M_t|] < \infty$ for all t,
- **Martingale property:** For all $s \leq t$,

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s \quad \text{(almost surely)}.$$

Intuition

A martingale is like a fair betting game: your expected future fortune is always equal to your current wealth. For example, in a fair coin-flipping game, a gambler's fortune after each round (win $+1$, lose -1) has zero expected change, hence forms a martingale. The key idea is that the process has no predictable trend: its expected increment is zero given the past.

2.2.1 Examples

- **Symmetric random walk:** Let $S_n = \sum_{i=1}^{n} X_i$ where $X_i \in \{-1, +1\}$ are i.i.d. with equal probability and $S_0 = 0$. Then (S_n) is a discrete-time martingale since:
$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n.$$
- **Standard Brownian motion:** The process $(B_t)_{t \geq 0}$ with $B_0 = 0$ and independent, mean-zero increments satisfies:
$$\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s \quad \text{for } s \leq t.$$
Thus, (B_t) is a martingale.
- **Discounted asset price under risk-neutral measure:** In mathematical finance, if S_t is the price of a risky asset and r the risk-free rate, then under the risk-neutral measure $\mathbb{Q}$, the discounted price $\tilde{S}_t = e^{-rt} S_t$ is a martingale:
$$\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t \mid \mathcal{F}_s] = \tilde{S}_s.$$
This reflects the "no arbitrage" condition: the expected discounted gain is zero.

Generalizations

- A process (X_t) is a **submartingale** if:
$$\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s \quad \text{for all } s \leq t.$$
It tends to increase in expectation over time.

- A process (X_t) is a **supermartingale** if:

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s \quad \text{for all } s \leq t.$$

It tends to decrease in expectation.

Note: Every martingale is both a submartingale and a supermartingale.

Chapter 3

Brownian Motion and Random Walks

3.1 Construction of Brownian Motion

Definition. A stochastic process $(B_t)_{t\geq 0}$ is called a **standard Brownian motion** (or **Wiener process**) if it satisfies the following properties:

1. $B_0 = 0$ almost surely;
2. (B_t) has **independent increments**: for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent;
3. The increments are **stationary and Gaussian**: for all $0 \leq s < t$,
$$B_t - B_s \sim \mathcal{N}(0, t - s);$$
4. The sample paths $t \mapsto B_t(\omega)$ are almost surely continuous.

Interpretation: Brownian motion models the limit of a symmetric random walk in continuous time. It is a fundamental object in probability theory and plays a central role in stochastic calculus, statistical physics, and financial mathematics.

Construction via Scaled Random Walks (Donsker's Invariance Principle):

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. real-valued random variables with zero mean and unit variance:

$$\mathbb{E}[X_i] = 0, \qquad \mathrm{Var}(X_i) = 1.$$

Define the partial sums:

$$S_k = \sum_{i=1}^{k} X_i.$$

For each $n \in \mathbb{N}$, define the rescaled and interpolated process:

$$W_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} + \frac{1}{\sqrt{n}} (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}, \quad t \in [0, 1].$$

Then, by the **Donsker invariance principle** (functional central limit theorem), the sequence of processes $(W_n(t))_{t\in[0,1]}$ converges in distribution in $C[0, 1]$ (with the uniform topology) to standard Brownian motion $(B_t)_{t\in[0,1]}$ as $n \to \infty$.

Alternative Construction via Karhunen–Loève Expansion:
Brownian motion on the interval $[0, 1]$ can also be constructed as an infinite series:

$$B_t = \sum_{n=1}^{\infty} Z_n \sqrt{\lambda_n} e_n(t),$$

where:

- (Z_n) is a sequence of i.i.d. standard normal random variables;
- $(\lambda_n, e_n(t))$ are the eigenvalues and eigenfunctions of the covariance operator of Brownian motion, typically:

  $$\lambda_n = \frac{1}{((n - 1/2)\pi)^2}, \quad e_n(t) = \sqrt{2} \sin\left((n - 1/2)\pi t\right),$$

  for Dirichlet boundary conditions.

This representation highlights the Gaussian structure and regularity properties of Brownian paths.

Remarks:

- Brownian motion is a continuous-time martingale with respect to its natural filtration.
- Sample paths of B_t are almost surely continuous everywhere but nowhere differentiable.
- Brownian motion is the unique (in law) continuous process with independent and stationary Gaussian increments.
- It satisfies the scaling property: for any $c > 0$, the process $(B_{ct})_{t\geq 0}$ has the same distribution as $(\sqrt{c}B_t)_{t\geq 0}$.

3.2 Increments and Properties of the Symmetric and Scaled Random Walk

Symmetric Random Walk. Let $(X_i)_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables defined by:

$$\mathbb{P}(X_i = +1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Define the symmetric random walk as the sequence of partial sums:

$$S_n := \sum_{i=1}^{n} X_i, \quad \text{for } n \in \mathbb{N}.$$

Increments: The process has stationary and independent increments:

$$S_n - S_{n-1} = X_n.$$

Expectation and Variance: Since X_i is symmetric,

$$\mathbb{E}[X_i] = 0, \quad \mathbb{V}ar(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = 1.$$

Then:

$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 0, \quad \mathbb{V}ar(S_n) = \sum_{i=1}^{n} \mathbb{V}ar(X_i) = n.$$

Quadratic Variation: Define $\Delta S_i := S_i - S_{i-1} = X_i$. Then the discrete quadratic variation up to time n is:

$$[S]_n := \sum_{i=1}^{n} (\Delta S_i)^2 = \sum_{i=1}^{n} X_i^2 = n \quad \text{almost surely.}$$

Scaled Random Walk. To approximate Brownian motion, we define the scaled random walk:

$$W_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}, \quad t \in [0, 1].$$

This process resamples the symmetric walk over a finer and finer time grid and rescales the amplitude to ensure convergence.

Increments: For $0 \leq s < t \leq 1$,

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}} \left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} X_i.$$

This is a normalized sum of $k := \lfloor nt \rfloor - \lfloor ns \rfloor$ i.i.d. terms.

Mean and Variance of Increments:

$$\mathbb{E}[W_n(t) - W_n(s)] = 0, \quad \mathbb{V}ar(W_n(t) - W_n(s)) = \frac{k}{n} \approx t - s.$$

This shows that the variance of the scaled process mimics that of Brownian motion.

Quadratic Variation: Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$ be a partition of $[0, 1]$. The quadratic variation of W_n over this partition is defined as:

$$[W_n]_1 := \sum_{j=1}^{m} \left(W_n(t_j) - W_n(t_{j-1})\right)^2.$$

Each squared increment has expected value:

$$\mathbb{E}\left[\left(W_n(t_j) - W_n(t_{j-1})\right)^2\right] \approx t_j - t_{j-1},$$

so summing over all intervals:

$$\mathbb{E}[[W_n]_1] \approx \sum_{j=1}^{m}(t_j - t_{j-1}) = 1.$$

Thus, the expected quadratic variation over $[0, 1]$ converges to 1, consistent with the pathwise properties of Brownian motion.

Conclusion: The symmetric random walk has discrete time and linearly growing variance, with deterministic quadratic variation. When rescaled appropriately in time and space, the scaled random walk $W_n(t)$ converges in distribution (by Donsker's theorem) to a standard Brownian motion. The convergence of mean, variance, and quadratic variation all reinforce this result and provide a rigorous bridge between discrete and continuous stochastic processes.

3.3 Properties of Brownian Motion

Definition Recap. A stochastic process $(B_t)_{t \geq 0}$ is a **standard Brownian motion** if:

1. $B_0 = 0$ almost surely;
2. B_t has **independent increments**;
3. $B_t - B_s \sim \mathcal{N}(0, t - s)$ for all $0 \leq s < t$;

4. Paths $t \mapsto B_t$ are continuous almost surely.

Main Properties of Brownian Motion:

- **Gaussian increments:** All finite-dimensional distributions are multivariate normal.
- **Stationary increments:** The law of $B_{t+h} - B_t$ depends only on h.
- **Independent increments:** For $0 \le t_1 < t_2 < \cdots < t_n$, the increments $B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.
- **Martingale property:** $\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s$ for $s \le t$, where $\mathcal{F}_t$ is the natural filtration.
- **Scaling invariance:** For any $c > 0$, $(B_{ct}) \overset{d}{=} (\sqrt{c} B_t)$.
- **Time-homogeneity:** The process $(B_{t+s} - B_s)$ is again a Brownian motion independent of $\mathcal{F}_s$.
- **Nowhere differentiable paths:** With probability 1, the sample paths $t \mapsto B_t$ are continuous but nowhere differentiable.
- **Quadratic variation:** $[B]_t = t$, i.e., the quadratic variation grows linearly with time.

Comments and Applications:

1. Martingale Property. This property implies that Brownian motion is a fair game: its expected future value, given the past, equals the current value. This underpins the concept of *no arbitrage* in financial mathematics. *Example:* In the Black-Scholes model, discounted asset prices (under the risk-neutral measure) are modeled as martingales.

2. Nowhere Differentiability. Although Brownian paths are continuous, they are highly irregular and not smooth. This motivates the use of **Itô calculus**, which handles integration with respect to such rough paths. *Example:* In stochastic differential equations (SDEs), we use Itô integrals instead of classical integrals.

3. Independent Increments. This property ensures that future movements of the process are unaffected by the past, making Brownian motion a natural model for unpredictable phenomena. *Example:* In modeling noise in physical systems or stock price fluctuations.

4. Scaling Property. Brownian motion is self-similar. This is useful in studying fractals and in multi-scale modeling in physics and finance.

Chapter 4

Itô Calculus

4.1 Diffusion Processes and the Need for Itô Calculus

Definition (Diffusion Process). A **diffusion process** $(X_t)_{t\geq 0}$ is a continuous-time stochastic process that satisfies a **stochastic differential equation** (SDE) of the form:

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t,$$

where:

- B_t is a standard Brownian motion;
- $\mu(t, x)$ is the **drift coefficient**, governing the deterministic trend of the process;
- $\sigma(t, x)$ is the **diffusion coefficient**, governing the stochastic fluctuations;
- The function μ and σ are assumed to be sufficiently regular (e.g., measurable and satisfying growth and Lipschitz conditions).

Examples.

- The Brownian motion itself: $dX_t = dB_t \Rightarrow \mu = 0, \sigma = 1$;
- The Ornstein–Uhlenbeck process: $dX_t = -\theta X_t\,dt + \sigma\,dB_t$;
- The Geometric Brownian motion (used in finance): $dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$.

Why Classical Calculus Fails. In deterministic analysis, we use classical tools such as differential calculus and chain rule (i.e., $df(x(t)) = f'(x(t))dx(t)$). However, Brownian motion paths are almost surely nowhere differentiable and have infinite total variation over any interval. Therefore, classical calculus cannot be applied directly.

Why We Need Itô Calculus.

- **Irregularity of paths:** Since B_t has infinite variation, we must define a new type of integral: the **Itô integral**, which is well-defined for integrands adapted to the filtration generated by B_t.
- **Modified chain rule:** The classical chain rule is replaced by **Itô's formula**, which accounts for the stochastic nature of the process. For a twice differentiable function $f(t, X_t)$:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} \, dB_t.$$

- **Foundation for stochastic models:** Itô calculus provides the mathematical foundation to define, solve, and analyze SDEs, which are essential in modeling random dynamics in physics, biology, economics, and especially in financial mathematics.

Conclusion. The study of stochastic processes—especially those driven by Brownian motion—requires a new calculus that respects their randomness and irregularity. Itô calculus fills this gap, making it possible to define integrals and derive dynamics of functions of stochastic processes.

4.2 Construction of the Itô Integral

Goal. Define the stochastic integral

$$\int_0^t H_s \, dB_s$$

for a process (H_s) adapted to the natural filtration $(\mathcal{F}_s)$ generated by a standard Brownian motion (B_s), and satisfying suitable integrability conditions. This integral plays a fundamental role in stochastic calculus and in solving stochastic differential equations (SDEs).

Step 1: Itô Integral for Simple (Elementary) Processes

A *simple process* is a left-continuous, adapted process that is piecewise constant:

$$H_s = \sum_{i=0}^{n-1} H_i \, \mathbf{1}_{(t_i, t_{i+1}]}(s),$$

where:

- $0 = t_0 < t_1 < \cdots < t_n = t$ is a finite partition of $[0, t]$;
- Each H_i is $\mathcal{F}_{t_i}$-measurable and square-integrable.

The Itô integral of such a process is defined as:

$$\int_0^t H_s \, dB_s := \sum_{i=0}^{n-1} H_i \left(B_{t_{i+1}} - B_{t_i} \right).$$

Example. Let B_t be Brownian motion and define

$$H_s = \begin{cases} 1 & \text{if } s \in (0, 1], \\ 2 & \text{if } s \in (1, 2]. \end{cases}$$

Then,

$$\int_0^2 H_s \, dB_s = (B_1 - B_0) + 2(B_2 - B_1).$$

This is a linear combination of independent normal variables with mean zero.

Step 2: Extension to General Square-Integrable Processes
Let H be an $\mathcal{F}_t$-adapted process such that:

$$\mathbb{E}\left[\int_0^t H_s^2 \, ds \right] < \infty.$$

Then, there exists a sequence of simple processes (H_s^n) such that:

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^t |H_s^n - H_s|^2 \, ds \right] = 0.$$

We define the Itô integral of H as the limit (in L^2):

$$\int_0^t H_s \, dB_s := \lim_{n \to \infty} \int_0^t H_s^n \, dB_s.$$

Key Properties of the Itô Integral

- **Linearity:**

$$\int_0^t (aH_s + bG_s)\, dB_s = a\int_0^t H_s\, dB_s + b\int_0^t G_s\, dB_s.$$

- **Itô Isometry:**

$$\mathbb{E}\left[\left(\int_0^t H_s\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2\, ds\right].$$

- **Martingale Property:** If H is adapted and square-integrable, then the process

$$M_t = \int_0^t H_s\, dB_s$$

is a martingale with respect to $(\mathcal{F}_t)$.

Conceptual Insight

- The Itô integral captures how H_s interacts with the random fluctuations of Brownian motion.
- Classical Riemann–Stieltjes integration fails for Brownian motion because of its infinite variation.
- Unlike the Stratonovich integral, the Itô integral evaluates the integrand at the *left endpoint* of each interval in the partition. This asymmetry leads to Itô's lemma including a correction term.

Example: Itô Integral of a Deterministic Function
Let $H_s = s$, a deterministic and continuous function. Then:

$$\int_0^t s\, dB_s$$

is a Gaussian random variable with mean zero and variance:

$$\mathbb{E}\left[\left(\int_0^t s\, dB_s\right)^2\right] = \int_0^t s^2\, ds = \frac{t^3}{3}.$$

Hence:

$$\int_0^t s\, dB_s \sim \mathcal{N}\left(0, \frac{t^3}{3}\right).$$

Conclusion

The Itô integral extends classical integration into the stochastic realm. It is first defined for simple adapted processes and then extended by density in L^2. Its key properties—linearity, isometry, and martingale behavior—make it an indispensable tool in the theory and application of stochastic differential equations.

4.3 Properties of the Itô Integral

Let $H = (H_t)_{t \geq 0}$ be an $\mathcal{F}_t$-adapted process satisfying:

$$\mathbb{E}\left[\int_0^T H_s^2 \, ds\right] < \infty \quad \text{for some fixed } T > 0.$$

Then the Itô integral $\int_0^t H_s \, dB_s$ is well-defined for all $t \in [0, T]$, and satisfies the following fundamental properties:

1. Linearity. For any two integrable adapted processes H_t and G_t, and constants $a, b \in \mathbb{R}$,

$$\int_0^t (aH_s + bG_s) \, dB_s = a \int_0^t H_s \, dB_s + b \int_0^t G_s \, dB_s.$$

2. Itô Isometry.

$$\mathbb{E}\left[\left(\int_0^t H_s \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, ds\right].$$

Interpretation: The Itô integral preserves the L^2 norm up to a change of domain: the "variance" of the integral equals the expected "energy" of the integrand.

3. Martingale Property. If H_t is adapted and square-integrable, then:

$$M_t := \int_0^t H_s \, dB_s \quad \text{is a martingale.}$$

Moreover,

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s, \quad \text{for } 0 \leq s \leq t.$$

4. Non-anticipativity. The value of the integral up to time t only depends on the values of H_s for $s \leq t$. This reflects the causal nature of the integral: future information is not used.

5. Continuity. The process $\left(\int_0^t H_s\,dB_s\right)_{t\in[0,T]}$ has continuous paths almost surely.

6. Quadratic Variation. If $M_t = \int_0^t H_s\,dB_s$, then:

$$[M]_t = \int_0^t H_s^2\,ds.$$

Interpretation: The quadratic variation of a stochastic integral reflects the instantaneous variance of the integrand. This is essential in stochastic calculus.

Application: Solving an SDE. Consider the stochastic differential equation:

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t, \quad X_0 = x_0.$$

Using Itô's formula for $\log X_t$, we can compute the explicit solution:

$$X_t = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

This shows how the Itô integral provides a fundamental tool to express and solve stochastic differential equations.

Conclusion. The Itô integral is not only a mathematically rigorous generalization of classical integration, but also a cornerstone of stochastic calculus. Its properties enable the development of a rich theory for modeling randomness in time-dependent systems.

4.4 Expectation of an Itô Integral and Application

Let $I(t) = \int_0^t H_s\,dB_s$ be an Itô integral, where $(H_t)_{t\geq 0}$ is an adapted process such that

$$\mathbb{E}\left[\int_0^t H_s^2\,ds\right] < \infty.$$

Then:

$$\mathbb{E}[I(t)] = \mathbb{E}\left[\int_0^t H_s\,dB_s\right] = 0.$$

Reason: The Itô integral is a martingale with zero initial condition, and martingales have constant expectation over time. Since $I(0) = 0$, the expected value remains zero:

$$\mathbb{E}[I(t)] = \mathbb{E}[I(0)] = 0.$$

Application: No-Arbitrage in Finance
In the Black–Scholes model, the price of a financial asset S_t is modeled as:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t.$$

To evaluate the fair price of a contingent claim (e.g. an option), one works under the risk-neutral measure $\mathbb{Q}$, under which the discounted asset price:

$$\tilde{S}_t = e^{-rt} S_t$$

satisfies the SDE:

$$d\tilde{S}_t = \sigma \tilde{S}_t \, dW_t,$$

where W_t is a $\mathbb{Q}$-Brownian motion.
Then, the expected discounted price is:

$$\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t] = \tilde{S}_0.$$

This is because the stochastic integral $\int_0^t \sigma \tilde{S}_s \, dW_s$ has zero expectation.
Conclusion:
The fact that the Itô integral has zero expectation underlies the pricing of financial derivatives and the no-arbitrage principle in modern financial mathematics.

4.5 Expected Value of a Stochastic Process – Worked Example

Example: Let X_t follow a Geometric Brownian Motion defined by the SDE:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = x > 0,$$

with constants $\mu, \sigma \in \mathbb{R}$, and $(W_t)_{t \geq 0}$ a standard Brownian motion.
We want to compute:

$$\mathbb{E}[X_t].$$

Step 1 – Solve the SDE

Divide both sides by $X_t \neq 0$:

$$\frac{dX_t}{X_t} = \mu \, dt + \sigma \, dW_t.$$

This is the logarithmic derivative of X_t, so we consider:

$$Y_t = \ln X_t.$$

By Itô's formula:

$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t.$$

Now integrate both sides from 0 to t:

$$Y_t = \ln X_t = \ln x + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Exponentiating:

$$X_t = x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Step 2 – Compute the Expected Value

We now compute:

$$\mathbb{E}[X_t] = \mathbb{E}\left[x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)\right].$$

Factor out constants:

$$= x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t\right) \mathbb{E}\left[\exp(\sigma W_t)\right].$$

Since $W_t \sim \mathcal{N}(0, t)$, we know:

$$\mathbb{E}\left[\exp(\sigma W_t)\right] = \exp\left(\frac{1}{2}\sigma^2 t\right).$$

Therefore:

$$\mathbb{E}[X_t] = x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t\right) \exp\left(\frac{1}{2}\sigma^2 t\right) = x \exp(\mu t).$$

Final Answer

$$\boxed{\mathbb{E}[X_t] = x\, e^{\mu t}}$$

This shows that the expected value of a geometric Brownian motion grows exponentially at rate μ, independently of σ.

4.6 A Transformation of GBM and Its Expected Value

Let $X(t)$ be a geometric Brownian motion with drift $\mu = 1$, volatility $\sigma = 1$, and initial condition $X(0) = 1$. That is, $X(t)$ satisfies the SDE:

$$dX(t) = X(t)\,dt + X(t)\,dW(t).$$

Define a new process:

$$Y(t) = \sqrt{X(t)} = X(t)^{1/2}.$$

Step 1 – Apply Itô's Formula

We apply Itô's formula to the function $f(x) = x^{1/2}$. We compute:

$$f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}.$$

Then:

$$\begin{aligned}
dY(t) &= f'(X(t))\,dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2 \\
&= \frac{1}{2}X(t)^{-1/2}\left(X(t)\,dt + X(t)\,dW(t)\right) + \frac{1}{2}\left(-\frac{1}{4}X(t)^{-3/2}\right)X(t)^2\,dt \\
&= \frac{1}{2}\sqrt{X(t)}\,dt + \frac{1}{2}\sqrt{X(t)}\,dW(t) - \frac{1}{8}\sqrt{X(t)}\,dt \\
&= \left(\frac{3}{8}\right)Y(t)\,dt + \left(\frac{1}{2}\right)Y(t)\,dW(t).
\end{aligned}$$

Step 2 – Identify the Process

We conclude that $Y(t)$ satisfies the SDE:

$$dY(t) = \frac{3}{8}Y(t)\,dt + \frac{1}{2}Y(t)\,dW(t), \quad Y(0) = 1,$$

which is again a geometric Brownian motion, with:

$$\mu_Y = \frac{3}{8}, \quad \sigma_Y = \frac{1}{2}.$$

Step 3 – Compute the Expected Value

The expected value of a GBM is given by:

$$\mathbb{E}[Y(t)] = Y(0)\, e^{\mu_Y t} = e^{(3/8)t}.$$

Final Result:

$$\boxed{\mathbb{E}[Y(t)] = e^{\frac{3}{8}t}}$$

Why the Expected Value of $Y(t)$ is $e^{\frac{3}{8}t}$

We showed that $Y(t)$ satisfies the SDE:

$$dY(t) = \frac{3}{8}Y(t)\, dt + \frac{1}{2}Y(t)\, dW(t), \quad Y(0) = 1.$$

This is a geometric Brownian motion with drift $\mu = \frac{3}{8}$ and volatility $\sigma = \frac{1}{2}$. The explicit solution is:

$$\begin{aligned} Y(t) &= \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \\ &= \exp\left(\left(\frac{3}{8} - \frac{1}{8}\right)t + \frac{1}{2}W(t)\right) \\ &= \exp\left(\frac{1}{4}t + \frac{1}{2}W(t)\right). \end{aligned}$$

Now compute the expected value:

$$\mathbb{E}[Y(t)] = \mathbb{E}\left[\exp\left(\frac{1}{4}t + \frac{1}{2}W(t)\right)\right] = \exp\left(\frac{1}{4}t\right) \cdot \mathbb{E}\left[e^{\frac{1}{2}W(t)}\right].$$

Since $W(t) \sim \mathcal{N}(0, t)$, we know:

$$\mathbb{E}[e^{aW(t)}] = e^{\frac{1}{2}a^2 t}, \quad \text{so } \mathbb{E}[e^{\frac{1}{2}W(t)}] = e^{\frac{1}{8}t}.$$

Thus:

$$\mathbb{E}[Y(t)] = e^{\frac{1}{4}t} \cdot e^{\frac{1}{8}t} = \boxed{e^{\frac{3}{8}t}}.$$

4.7 Itô Isometry and Application to the Vasicek Model

Itô Isometry. Let $H = (H_t)_{t\geq 0}$ be an adapted process such that:

$$\mathbb{E}\left[\int_0^T H_s^2\, ds\right] < \infty.$$

Then the Itô integral $I(t) := \int_0^t H_s \, dB_s$ satisfies:

$$\mathbb{E}\left[\left(\int_0^t H_s \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, ds\right], \quad \forall t \in [0, T].$$

This is a fundamental property of the Itô integral. It means that the mapping $H \mapsto \int H \, dB$ is an isometry from $L^2([0,T] \times \Omega)$ into $L^2(\Omega)$.

4.7.1 Application: Variance in the Vasicek Interest Rate Model

The Vasicek model for the short rate r_t is given by the stochastic differential equation:

$$dr_t = a(b - r_t) \, dt + \sigma \, dB_t, \quad r_0 \in \mathbb{R}.$$

This is a linear SDE, and its solution is:

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} \, dB_s.$$

Let us compute the variance of r_t. Since the deterministic part has zero variance, it remains to compute:

$$\operatorname{Var}(r_t) = \operatorname{Var}\left(\sigma \int_0^t e^{-a(t-s)} \, dB_s\right) = \sigma^2 \mathbb{E}\left[\left(\int_0^t e^{-a(t-s)} \, dB_s\right)^2\right].$$

Applying Itô's isometry:

$$\operatorname{Var}(r_t) = \sigma^2 \int_0^t e^{-2a(t-s)} \, ds = \sigma^2 \int_0^t e^{-2au} \, du = \frac{\sigma^2}{2a} \left(1 - e^{-2at}\right).$$

Interpretation: As $t \to \infty$, the variance of r_t tends to $\frac{\sigma^2}{2a}$, which shows the long-term stability of the Vasicek model around the mean-reverting level b.

Conclusion: The Itô isometry allows one to compute second moments of stochastic integrals and is especially useful in finance, for instance, in understanding the volatility and risk of interest rate models.

4.8 Itô–Doeblin Formula

Let X_t be an Itô process with dynamics

$$dX_t = \mu_t \, dt + \sigma_t \, dB_t, \quad X_0 = x_0,$$

where (B_t) is Brownian motion. For a twice-differentiable function $f(t,x)$ (with $\partial_t f$ continuous and f of class C^2 in x), the *Itô–Doeblin formula* (also known as Itô's lemma) gives the stochastic chain rule. In differential form it reads

$$d\big[f(t,X_t)\big] = \Big(f_t(t,X_t) + \mu_t\, f_x(t,X_t) + \tfrac{1}{2}\sigma_t^2\, f_{xx}(t,X_t)\Big)\,dt \;+\; \sigma_t\, f_x(t,X_t)\,dB_t,$$

where subscripts denote partial derivatives. Equivalently, in integral form:

$$f(t,X_t) - f(0,X_0) = \int_0^t \big(f_s(s,X_s) + \mu_s f_x(s,X_s) + \tfrac{1}{2}\sigma_s^2 f_{xx}(s,X_s)\big)\,ds + \int_0^t \sigma_s f_x(s,X_s)\,dB_s.$$

Here the extra term $\frac{1}{2}\sigma_t^2 f_{xx}\,dt$ is the *Itô correction* arising from the quadratic variation of B_t. This formula is the stochastic analogue of the classical chain rule, with the additional $(dB_t)^2 = dt$ term. It implies that $Y_t = f(t,X_t)$ is itself an Itô process. In contrast to deterministic calculus (which would give $df = f_t dt + f_x dX$), Itô calculus requires f to be $C^{1,2}$ (once in t, twice in x) and yields the extra second-derivative term.

Example 1: $f(x) = x^2$. As a simple illustration, let $Y_t = X_t^2$ with $dX_t = \mu_t dt + \sigma_t dB_t$. We have $f'(x) = 2x$, $f''(x) = 2$. By Itô's formula,

$$d(X_t^2) = 2X_t\,dX_t + d\langle X\rangle_t,$$

and since $(dX_t)^2 = \sigma_t^2\,dt$, this becomes

$$d(X_t^2) = 2X_t(\mu_t dt + \sigma_t dB_t) + \sigma_t^2 dt = \big(2X_t\mu_t + \sigma_t^2\big)\,dt + 2X_t\sigma_t\,dB_t.$$

In other words, the *drift* part of X_t^2 is $2X_t\mu_t + \sigma_t^2$, not just $2X_t\mu_t$. Integrating from 0 to t gives

$$X_t^2 = X_0^2 + \int_0^t (2X_s\mu_s + \sigma_s^2)\,ds + \int_0^t 2X_s\sigma_s\,dB_s.$$

Example 2: $f(x) = \ln x$ for Geometric Brownian Motion. Consider a geometric Brownian motion X_t solving

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dB_t, \quad X_0 > 0.$$

Take $Y_t = \ln X_t$. Here $f'(x) = 1/x$, $f''(x) = -1/x^2$. Itô's formula gives

$$d(\ln X_t) = \frac{1}{X_t}\,dX_t - \frac{1}{2X_t^2}\,(dX_t)^2 = \frac{1}{X_t}(\mu X_t dt + \sigma X_t dB_t) - \frac{1}{2}\frac{1}{X_t^2}(\sigma^2 X_t^2 dt).$$

Simplifying,

$$d(\ln X_t) = \mu\,dt + \sigma\,dB_t \;-\; \tfrac{1}{2}\sigma^2\,dt = \Big(\mu - \tfrac{1}{2}\sigma^2\Big)dt + \sigma\,dB_t.$$

Thus

$$\ln X_t = \ln X_0 + \sigma B_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)t,$$

and exponentiating yields the familiar solution $X_t = X_0 \exp\big(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\big)$.

Chain Rule Comparison

Classical:	$df = f_x\, dX$
Itô:	$df = f_x\, dX + \frac{1}{2} f_{xx}\, (dX)^2$

Figure 4.1: In the stochastic setting, the differential of $f(X_t)$ includes a second-order correction term due to quadratic variation.

4.9 Comparison of Classical and Itô Calculus

Table 4.1: Classico vs. Itô

Aspetto	**Calcolo classico**	**Calcolo di Itô**
Regola catena	$df = f_x\, dX$	$df = f_t dt + f_x dX + \frac{1}{2} f_{xx}(dX)^2$
Variazione quadr.	Trascurata	$(dB)^2 = dt$
Integrale	Riemann/Lebesgue	Integrale di Itô (media quadratica)
Reg. prodotto	$d(XY) = X\, dY + Y\, dX$	$d(XY) = X\, dY + Y\, dX + d\langle X, Y\rangle$
Regolarità f	C^1	$C^{1,2}$ (1 in t, 2 in x)
Contesto	Deterministico	Stocastico (con moto browniano)

4.10 Why the Itô Integral is Different

Brownian motion paths are almost nowhere differentiable and have infinite variation, so standard Riemann (or Lebesgue) integration theory does not apply to $\int B_s\, dB_s$. Instead, the Itô integral is defined via mean-square limits of adapted Riemann sums. This leads to martingale properties but also introduces extra terms. For example:

$$\int_0^t B_s\, dB_s \;=\; \tfrac{1}{2}B_t^2 - \tfrac{1}{2}t,$$

as can be derived from Itô's formula applied to $f(x) = x^2$. The $-\frac{1}{2}t$ term reflects the Itô correction.

4.11 Itô Process and Itô–Doeblin Formula

4.11.1 Definition of an Itô Process.

A stochastic process $(X_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)$ is said to be an **Itô process** if it can be written as:

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s,$$

where:

- μ_t is an $\mathcal{F}_t$-adapted process (called the drift),
- σ_t is an $\mathcal{F}_t$-adapted process (called the volatility),
- B_t is a standard Brownian motion.

Both integrals must be well-defined in the Itô sense, typically assuming that μ_t and σ_t are progressively measurable and satisfy:

$$\mathbb{E}\left[\int_0^T |\mu_s| \, ds\right] < \infty, \quad \mathbb{E}\left[\int_0^T |\sigma_s|^2 \, ds\right] < \infty.$$

Examples of Itô Processes:

- **Brownian motion:** $X_t = B_t$ with $\mu_t = 0$, $\sigma_t = 1$.
- **Ornstein–Uhlenbeck process:**

$$dX_t = \theta(\mu - X_t) \, dt + \sigma \, dB_t.$$

- **Geometric Brownian motion (Black–Scholes model):**

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t.$$

Itô–Doeblin Formula for Itô Processes. Let $f \in C^{1,2}([0,T] \times \mathbb{R})$, and let X_t be an Itô process as above. Then:

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) \, dt + \frac{\partial f}{\partial x}(t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \sigma_t^2 \, dt.$$

Application – Geometric Brownian Motion. Let X_t satisfy:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t,$$

and define $Y_t = \log X_t$. Then applying Itô's formula:

$$\frac{d}{dt} \log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} \sigma^2 X_t^2 \, dt = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \, dB_t.$$

Hence:

$$\log X_t = \log X_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t,$$

which implies the explicit solution:

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right).$$

This formula is fundamental in finance, as it models the evolution of asset prices under stochastic dynamics in the Black–Scholes framework.

Chapter 5

Stochastic Differential Equations

5.1 Stochastic Differential Equations (SDEs)

5.1.1 Definition.

A **Stochastic Differential Equation (SDE)** is an equation of the form

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t,$$

where:

- X_t is the unknown stochastic process,
- $\mu(t, X_t)$ is the drift coefficient (deterministic or stochastic),
- $\sigma(t, X_t)$ is the diffusion coefficient,
- B_t is a standard Brownian motion.

The solution to an SDE is defined as an Itô process that satisfies the integral version:

$$X_t = X_0 + \int_0^t \mu(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s.$$

5.1.2 Examples of Financially Relevant SDEs

- **Geometric Brownian Motion (Black–Scholes Model):**

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t.$$

It models the price of a risky asset under constant drift and volatility. The solution is:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

- **Ornstein–Uhlenbeck Process (Mean-Reverting Process):**

$$dX_t = \theta(\mu - X_t)\, dt + \sigma\, dB_t.$$

This process models quantities that revert to a long-term mean μ over time. Often used for interest rates or volatility.

- **Vasicek Interest Rate Model:**

$$dr_t = a(b - r_t)\, dt + \sigma\, dB_t.$$

This is an Ornstein–Uhlenbeck process applied to interest rates. It can become negative due to its Gaussian nature.

- **Cox–Ingersoll–Ross (CIR) Model:**

$$dr_t = a(b - r_t)\, dt + \sigma\sqrt{r_t}\, dB_t.$$

This is a mean-reverting model that ensures interest rates remain non-negative (as long as $2ab \geq \sigma^2$).

- **Heston Model for Stochastic Volatility:**

$$\begin{cases} dS_t = \mu S_t\, dt + \sqrt{v_t} S_t\, dB_t^S, \\ dv_t = \kappa(\theta - v_t)\, dt + \xi\sqrt{v_t}\, dB_t^v, \end{cases}$$

with $\mathrm{Cov}(dB_t^S, dB_t^v) = \rho\, dt$. This two-dimensional SDE models asset prices with stochastic volatility.

SDEs are fundamental in financial modeling for capturing the randomness inherent in asset prices, interest rates, and volatility, and are the basis for option pricing, portfolio optimization, and risk management.

5.2 Existence and Uniqueness Theorem for SDEs

5.2.1 Theorem (Existence and Uniqueness of Strong Solutions).

Let $\mu(t, x)$ and $\sigma(t, x)$ be measurable functions such that:

- (Local Lipschitz condition) For every $R > 0$, there exists a constant $L_R > 0$ such that for all $x, y \in \mathbb{R}$ with $|x|, |y| \leq R$,
$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L_R|x-y|;$$
- (Linear growth condition) There exists a constant $C > 0$ such that for all $x \in \mathbb{R}$,
$$|\mu(t,x)|^2 + |\sigma(t,x)|^2 \leq C(1+x^2).$$

Then, for every initial condition $X_0 = x_0$, the stochastic differential equation

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t$$

has a unique strong solution defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, adapted to the Brownian motion B_t.

5.2.2 Applications of the Theorem

(a) Geometric Brownian Motion (GBM):

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$$

Here, $\mu(t,x) = \mu x$ and $\sigma(t,x) = \sigma x$ are both globally Lipschitz and satisfy the linear growth condition:

$$|\mu x - \mu y| = |\mu||x-y|, \quad |\sigma x - \sigma y| = |\sigma||x-y|,$$

and

$$|\mu x|^2 + |\sigma x|^2 \leq C(1+x^2),$$

for some constant $C > 0$. Hence, the existence and uniqueness theorem applies and guarantees a unique strong solution. The explicit solution is:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

(b) Linear SDEs:

$$dX_t = (a(t)X_t + b(t))\, dt + (c(t)X_t + d(t))\, dB_t$$

Assume $a(t), b(t), c(t), d(t)$ are bounded and continuous on $[0,T]$. Then the drift and diffusion functions

$$\mu(t,x) = a(t)x + b(t), \quad \sigma(t,x) = c(t)x + d(t)$$

are globally Lipschitz in x (with time-dependent coefficients), and satisfy the linear growth condition. Hence, the existence and uniqueness theorem applies, and there exists a unique strong solution to the linear SDE.

5.3 Geometric Brownian Motion

The **Geometric Brownian Motion** (GBM) is a stochastic process commonly used in financial mathematics to model the evolution of asset prices.

Definition. A process $(S_t)_{t\geq 0}$ is said to follow a Geometric Brownian Motion if it satisfies the stochastic differential equation (SDE):

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t, \quad S_0 > 0,$$

where:

- $\mu \in \mathbb{R}$ is the *drift* coefficient,
- $\sigma > 0$ is the *volatility* coefficient,
- $(B_t)_{t\geq 0}$ is a standard Brownian motion.

Solution. To solve the SDE, we apply Itô's formula to the function $\log S_t$. Set $X_t = \log S_t$, so that $S_t = e^{X_t}$. Using Itô's lemma:

$$dX_t = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2.$$

Substituting dS_t from the original SDE:

$$dX_t = \frac{1}{S_t}(\mu S_t\, dt + \sigma S_t\, dB_t) - \frac{1}{2}\sigma^2\, dt = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

Thus:

$$X_t = \log S_t = \log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t,$$

and exponentiating both sides yields the explicit solution:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

Mean and Variance. Since $B_t \sim \mathcal{N}(0, t)$, then $\sigma B_t \sim \mathcal{N}(0, \sigma^2 t)$, and thus $\log S_t \sim \mathcal{N}(\log S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$.

Hence, using properties of the log-normal distribution:

$$\mathbb{E}[S_t] = S_0 e^{\mu t},$$
$$\mathrm{Var}(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right).$$

These formulas show that the expected value of the asset grows exponentially at rate μ, and the variance increases both with time and volatility.

Interpretation. GBM ensures that $S_t > 0$ almost surely, making it a natural model for asset prices. The multiplicative noise models proportional random fluctuations, and the log-normal distribution is consistent with the empirical skewed distribution of returns.

5.4 Expected Value of the Exponential of an Itô Process

Consider the Itô process

$$dY(t) = \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma \, dW(t), \quad Y(0) = 0.$$

Define the process

$$X(t) = e^{Y(t)}.$$

Characterization of $X(t)$

We apply Itô's formula to the function $f(Y) = e^Y$, which yields:

$$dX(t) = f'(Y(t)) \, dY(t) + \frac{1}{2} f''(Y(t)) \, (dY(t))^2.$$

Since $f'(y) = e^y$ and $f''(y) = e^y$, we obtain:

$$\begin{aligned} dX(t) &= e^{Y(t)} \left(\left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma \, dW(t) \right) + \frac{1}{2} e^{Y(t)} \sigma^2 \, dt \\ &= X(t) \left(\left(\alpha - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\right) dt + \sigma \, dW(t) \right) \\ &= X(t) \left(\alpha \, dt + \sigma \, dW(t) \right). \end{aligned}$$

Therefore, $X(t)$ satisfies the stochastic differential equation

$$dX(t) = \alpha X(t) \, dt + \sigma X(t) \, dW(t), \quad X(0) = 1,$$

which is the defining equation for a *Geometric Brownian Motion (GBM).*

Expected Value of $X(t)$

Note that $Y(t) \sim \mathcal{N}\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t, \, \sigma^2 t\right)$. Then the expectation of $X(t) = e^{Y(t)}$ is given by

$$\mathbb{E}[X(t)] = \mathbb{E}[e^{Y(t)}] = \exp\left(\mu_Y + \frac{1}{2} \mathrm{Var}(Y(t)) \right),$$

where

$$\mu_Y = \left(\alpha - \frac{1}{2}\sigma^2\right) t, \quad \mathrm{Var}(Y(t)) = \sigma^2 t.$$

Hence,

$$\mathbb{E}[X(t)] = \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t + \frac{1}{2}\sigma^2 t \right) = e^{\alpha t}.$$

Conclusion

The process $X(t) = e^{Y(t)}$ is a geometric Brownian motion with drift α and volatility σ, and satisfies

$$\mathbb{E}[X(t)] = e^{\alpha t}.$$

5.5 Applications of Linear SDEs: Interest Rate Models

In this section, we discuss two classical models for the evolution of interest rates, both based on linear stochastic differential equations (SDEs): the **Vasicek model** and the **Cox–Ingersoll–Ross (CIR) model**. In both cases, we analyze the dynamics, interpret the parameters, and compute the mean, variance, and their asymptotic behavior.

5.5.1 The Vasicek Model

The Vasicek model describes the short-term interest rate $r(t)$ by the linear SDE:

$$dr(t) = a(b - r(t))\,dt + \sigma\,dW(t), \quad r(0) = r_0,$$

where:

- $a > 0$: speed of mean reversion,
- $b \in \mathbb{R}$: long-term mean level,
- $\sigma > 0$: volatility,
- $W(t)$: standard Brownian motion.

This is an example of an *Ornstein–Uhlenbeck process.*

Solution. The solution to this linear SDE is given by:

$$r(t) = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)}\,dW(s).$$

Mean. Taking expectations:

$$\mathbb{E}[r(t)] = r_0 e^{-at} + b(1 - e^{-at}).$$

As $t \to \infty$, we get:

$$\lim_{t\to\infty} \mathbb{E}[r(t)] = b.$$

Variance. Using the isometry property of the Itô integral:

$$\operatorname{Var}(r(t)) = \sigma^2 \int_0^t e^{-2a(t-s)}\, ds = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Hence,

$$\lim_{t\to\infty} \operatorname{Var}(r(t)) = \frac{\sigma^2}{2a}.$$

Comments. The Vasicek model is mean-reverting around b, with the parameter a controlling the speed of reversion. However, it allows for negative interest rates, which is a limitation in practice.

5.5.2 The CIR Model

The Cox–Ingersoll–Ross model defines the interest rate $r(t)$ through the SDE:

$$dr(t) = a(b - r(t))\, dt + \sigma\sqrt{r(t)}\, dW(t), \quad r(0) = r_0 \geq 0,$$

where:

- $a > 0$: speed of mean reversion,
- $b > 0$: long-term mean level,
- $\sigma > 0$: volatility coefficient,
- $r(t) \geq 0$: ensures non-negativity under the Feller condition $2ab \geq \sigma^2$.

Mean. The expected value satisfies the ODE:

$$\frac{d}{dt}\mathbb{E}[r(t)] = a(b - \mathbb{E}[r(t)]), \quad \mathbb{E}[r(0)] = r_0,$$

with solution:

$$\mathbb{E}[r(t)] = r_0 e^{-at} + b(1 - e^{-at}).$$

As in the Vasicek model, we have:

$$\lim_{t\to\infty} \mathbb{E}[r(t)] = b.$$

Variance. The variance has the following expression (without proof):

$$\operatorname{Var}(r(t)) = \frac{\sigma^2}{2a}\left(r_0 e^{-at}(1 - e^{-at}) + b(1 - e^{-at})^2\right).$$

As $t \to \infty$:

$$\lim_{t\to\infty} \operatorname{Var}(r(t)) = \frac{b\sigma^2}{2a}.$$

Comments. The CIR model maintains mean reversion and guarantees non-negativity of rates, which is a major advantage over the Vasicek model. It is widely used in term structure modeling and credit risk.

5.5.3 Comparison

- **Mean behavior:** Both models exhibit mean reversion to the level b.
- **Variance:** Both models have finite long-term variance. CIR's variance depends on the square root of the state variable.
- **Positivity:** CIR ensures $r(t) \geq 0$ under the Feller condition, whereas Vasicek does not.
- **Analytical tractability:** Both models have explicit formulas for bond prices under affine term structure models.

x

5.6 The Feynman–Kac Theorem

Statement (Feynman–Kac Theorem). Let $u(t,x)$ be a function defined on $[0,T] \times \mathbb{R}$, and suppose it solves the backward parabolic PDE:

$$\frac{\partial u}{\partial t} + \mu(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2} - r(x)u = -f(t,x), \quad u(T,x) = \phi(x),$$

for given functions $\mu(x), \sigma(x), r(x), f(t,x), \phi(x)$, under suitable regularity and growth conditions.

Then, the Feynman–Kac Theorem states that:

$$u(t,x) = \mathbb{E}^x\left[\int_t^T e^{-\int_t^s r(X_u)\,du} f(s,X_s)\,ds + e^{-\int_t^T r(X_u)\,du}\phi(X_T)\right],$$

where $(X_s)_{s\geq t}$ is the solution of the stochastic differential equation:

$$dX_s = \mu(X_s)\,ds + \sigma(X_s)\,dW_s, \quad X_t = x.$$

Interpretation. The theorem provides a bridge between: - **Partial Differential Equations (PDEs)** of parabolic type - **Stochastic Differential Equations (SDEs)** and expectations of functionals of their solutions.

Instead of solving a PDE directly, one can compute an expected value over a stochastic process (which is often easier numerically or conceptually).

Special case (pricing of European options). In the Black–Scholes model:

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t,$$

a European option with terminal payoff $\phi(S_T)$ has price at time t given by:

$$V(t, S_t) = \mathbb{E}^{S_t}\left[e^{-r(T-t)}\phi(S_T)\right].$$

This is a direct application of Feynman–Kac with:

$$\mu(x) = rx, \quad \sigma(x) = \sigma x, \quad r(x) = r, \quad f \equiv 0.$$

Example. Solve the PDE:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0, \quad u(T, x) = e^x.$$

This is a heat equation (no drift, no potential, no source), and we want to find $u(t, x)$.

Here, the corresponding SDE is simply:

$$dX_s = dW_s, \quad X_t = x.$$

So the Feynman–Kac representation gives:

$$u(t, x) = \mathbb{E}^x[e^{X_T}] = \mathbb{E}[e^{x+(W_T - W_t)}] = e^x\,\mathbb{E}[e^{W_{T-t}}].$$

Since $W_{T-t} \sim \mathcal{N}(0, T-t)$, we have:

$$\mathbb{E}[e^{W_{T-t}}] = e^{\frac{1}{2}(T-t)}.$$

Therefore:

$$u(t, x) = e^{x+\frac{1}{2}(T-t)}.$$

How to Use Feynman–Kac Theorem to Solve Problems:

1. Identify the PDE and check that it fits the Feynman–Kac form (parabolic, final value problem).
2. Write down the associated SDE:
$$dX_s = \mu(X_s)\,ds + \sigma(X_s)\,dW_s, \quad X_t = x.$$
3. Translate the PDE solution into an expectation over the diffusion process:
$$u(t, x) = \mathbb{E}^x\left[\int_t^T e^{-\int_t^s r(X_u)\,du} f(s, X_s)\,ds + e^{-\int_t^T r(X_u)\,du}\phi(X_T)\right].$$
4. Evaluate the expectation (analytically if possible, or numerically via Monte Carlo).

Conclusion.

5.7 Feynman–Kac Theorem and Its Applications

The Feynman–Kac Theorem provides a powerful tool to solve parabolic PDEs by probabilistic means. It is the theoretical foundation of risk-neutral pricing in financial mathematics and allows solving many problems in option pricing and stochastic control.

5.8 Feynman–Kac Application: Solving a PDE with Logarithmic Terminal Condition

Consider the backward parabolic PDE:

$$F_t(t,x) + \frac{1}{2}xF_x(t,x) + 8x^2F_{xx}(t,x) = 0, \quad x \in \mathbb{R},\ t \in [0,10),$$

with terminal condition:

$$F(10,x) = \ln x.$$

Step 1 – Recognize the Structure

This equation is in the form of a backward Kolmogorov equation:

$$\frac{\partial F}{\partial t} + \mu(x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 F}{\partial x^2} = 0,$$

with:

$$\mu(x) = \frac{1}{2}x, \quad \sigma(x) = 4x.$$

Step 2 – Associated Stochastic Differential Equation (SDE)

To apply the Feynman–Kac theorem, we first identify the stochastic differential equation (SDE) associated with the given PDE. Comparing the PDE:

$$F_t + \mu(x)F_x + \frac{1}{2}\sigma^2(x)F_{xx} = 0,$$

with:

$$F_t + \frac{1}{2}xF_x + 8x^2F_{xx} = 0,$$

we can read off the drift and diffusion coefficients:

$$\mu(x) = \frac{1}{2}x, \quad \sigma^2(x) = 16x^2 \quad \Rightarrow \quad \sigma(x) = 4x.$$

Thus, the associated SDE for the diffusion process X_t is:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t = \frac{1}{2}X_t\,dt + 4X_t\,dW_t,$$

with initial condition $X_t = x$. This is a multiplicative noise SDE: both drift and diffusion are proportional to X_t.

Reduction to a Geometric Brownian Motion (GBM). We divide both sides by X_t (assuming $X_t \neq 0$) to obtain the logarithmic form:

$$\frac{dX_t}{X_t} = \frac{1}{2}dt + 4\,dW_t.$$

This is the standard form of a geometric Brownian motion (GBM) with constant drift $\frac{1}{2}$ and volatility 4.

Solution of the SDE. We are given the stochastic differential equation (SDE):

$$\frac{dX_t}{X_t} = \mu\,dt + \sigma\,dW_t,$$

which is the standard form of a **Geometric Brownian Motion (GBM)**. The solution to this SDE, with initial condition $X_0 = x$, is well known and given by:

$$X_t = x\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right).$$

Why this formula? We want to solve the SDE:

$$\frac{dX_t}{X_t} = \mu\,dt + \sigma\,dW_t.$$

This is a multiplicative equation, and a standard method to solve it is to apply a logarithmic transformation. Define:

$$Y_t = \ln X_t.$$

The idea is that by working with Y_t instead of X_t, we convert the equation into an additive one, which is easier to integrate.

Now we apply **Itô's formula** to compute dY_t. Recall the general Itô formula for a function $f(X_t)$ when X_t follows an Itô process:

$$d(f(X_t)) = f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2.$$

In our case:

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}.$$

We apply Itô's formula to $Y_t = \ln X_t$ where:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t.$$

So:

$$\begin{aligned}
dY_t &= \frac{1}{X_t} dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)(dX_t)^2 \\
&= \frac{1}{X_t}(\mu X_t \, dt + \sigma X_t \, dW_t) - \frac{1}{2} \cdot \frac{1}{X_t^2} \cdot \sigma^2 X_t^2 \, dt \\
&= \mu \, dt + \sigma \, dW_t - \frac{1}{2}\sigma^2 \, dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \, dW_t.
\end{aligned}$$

This is now a linear SDE in Y_t, which we can integrate directly:

$$Y_t = Y_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Since $Y_0 = \ln X_0 = \ln x$, we have:

$$Y_t = \ln x + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Exponentiating both sides gives:

$$X_t = e^{Y_t} = x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Conclusion: The term $-\frac{1}{2}\sigma^2$ arises from the Itô correction due to the second derivative of $\ln x$, and it is essential for capturing the true mean behavior of the GBM.

Apply to our case. In the PDE we are solving, we found that the corresponding SDE is:

$$dX_t = \frac{1}{2} X_t \, dt + 4 X_t \, dW_t.$$

This matches the GBM form with:

$$\mu = \frac{1}{2}, \quad \sigma = 4.$$

Plugging these values into the general formula:

$$X_t = x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) = x \exp\left(\left(\frac{1}{2} - \frac{1}{2}\cdot 16\right)t + 4W_t\right) = x \exp\left(-\frac{15}{2}t + \right.$$

This gives us the explicit expression for X_t, which will be used to compute the expected value in the Feynman–Kac formula.

We will use this explicit formula for X_t to compute expectations in the next step.

Step 3 – Feynman–Kac Representation

The Feynman–Kac formula tells us that:

$$F(t,x) = \mathbb{E}^x\left[\ln(X_{10})\right],$$

where X_t starts at x at time t. We write:

$$X_{10} = x \cdot \exp\left(-\frac{15}{2}(10-t) + 4(W_{10} - W_t)\right),$$

and so:

$$F(t,x) = \mathbb{E}\left[\ln\left(x \cdot \exp\left(-\frac{15}{2}(10-t) + 4(W_{10} - W_t)\right)\right)\right].$$

Using the properties of logarithms and expectation:

$$F(t,x) = \ln x - \frac{15}{2}(10-t) + \mathbb{E}[4(W_{10} - W_t)].$$

Since $W_{10} - W_t \sim \mathcal{N}(0, 10-t)$, we have $\mathbb{E}[W_{10} - W_t] = 0$, hence:

$$F(t,x) = \ln x - \frac{15}{2}(10-t).$$

Final Answer

$$\boxed{F(t,x) = \ln x - \frac{15}{2}(10-t)}$$

This solution is valid for $x > 0$ and $t \in [0, 10)$. It illustrates a direct application of the Feynman–Kac theorem to a backward parabolic PDE with logarithmic final condition.

Chapter 6

Financial Applications

6.1 Introduction to Geometric Brownian Motion and Risk-Neutral Pricing

In the Black–Scholes framework, the price of a stock S_t is modeled as a *geometric Brownian motion*, satisfying the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

with S_0 given, where W_t is a standard Brownian motion, μ is the (constant) drift rate, and σ is the (constant) volatility. The solution of this SDE is

$$S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\Big),$$

so that S_t is log-normally distributed. In particular, $\mathbb{E}[S_t] = S_0 e^{\mu t}$. Under this real-world measure $\mathbb{P}$, the drift μ represents the expected return of the asset.

For pricing derivatives, we work under a *risk-neutral measure* $\mathbb{Q}$. In a no-arbitrage market with a constant risk-free rate r, the discounted asset price is a martingale under $\mathbb{Q}$. Equivalently, under $\mathbb{Q}$ the stock's drift is r rather than μ. Thus the risk-neutral dynamics of S_t are

$$dS_t = rS_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}},$$

where $W_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. Then the (arbitrage-free) price $V(S,t)$ of a derivative with payoff $\Phi(S_T)$ at maturity T is given by the risk-neutral pricing formula

$$V(S,t) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \mid S_t = S].$$

In other words, the price is the discounted expected payoff under the risk-neutral measure. This expectation is equivalent to solving a partial differential equation, as described below.

6.2 The Black–Scholes PDE and Terminal Condition

By applying Itô's Lemma to $V(S_t, t)$ under the risk-neutral dynamics $dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}$, one obtains

$$dV = \Big(\partial_t V + rS\,\partial_S V + \tfrac{1}{2}\sigma^2 S^2\,\partial_{SS} V\Big)\,dt + \sigma S\,\partial_S V\,dW_t^{\mathbb{Q}}.$$

Set $\Delta = \partial_S V$ and form the portfolio $\Pi = V - \Delta S$. Using $d\Pi = dV - \Delta\,dS$, the stochastic terms cancel and one finds

$$d\Pi = \Big(\partial_t V + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} V\Big)\,dt.$$

Since Π is riskless (no dW term), it must earn the risk-free rate r: $d\Pi = r\Pi\,dt = r(V - S\partial_S V)\,dt$. Equating coefficients yields

$$\partial_t V + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} V = r(V - S\partial_S V).$$

Rearranging gives the Black–Scholes partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

or equivalently,

$$\partial_t V + \frac{1}{2}\sigma^2 S^2\,\partial_{SS} V + rS\,\partial_S V - rV = 0$$

valid for $0 \le t < T$ and $S > 0$. This PDE, together with the terminal (boundary) condition at maturity $t = T$, fully characterizes the option price. The terminal condition is simply the payoff of the derivative. For a European call option with strike K, for example,

$$V(S, T) = \Phi(S) = \max\{S - K, 0\}.$$

For a put, $\Phi(S) = \max\{K - S, 0\}$.

Using the Feynman–Kac theorem or by change of variables, one can solve this PDE explicitly. For a European call $C(S, t)$ and put $P(S, t)$, the closed-form Black–Scholes formulas are:

$$C(S,t) = S\,\Phi(d_1) - Ke^{-r(T-t)}\,\Phi(d_2), \quad P(S,t) = Ke^{-r(T-t)}\,\Phi(-d_2) - S\,\Phi(-d_1),$$

where

$$d_{1,2} = \frac{\ln(S/K) + \big(r \pm \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}},$$

and $\Phi(x)$ is the cumulative distribution function of a standard normal random variable. These formulas depend on the model parameters r, σ and on time to maturity $T - t$. They form the basis for computing the *Greeks*, as shown next.

6.3 The Greeks

The *Greeks* are partial derivatives of the option price that measure its sensitivity to various parameters. We denote by $\Delta, \Gamma, \Theta, \mathrm{Vega}, \rho$ the most commonly used Greeks, defined as follows:

$$\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad \Theta = \frac{\partial V}{\partial t}, \quad \mathrm{Vega} = \frac{\partial V}{\partial \sigma}, \quad \rho = \frac{\partial V}{\partial r}.$$

Each Greek has an intuitive financial interpretation: - *Delta* (Δ) measures the sensitivity of the option price to small changes in the underlying price. - *Gamma* (Γ) measures the sensitivity of delta itself to changes in the underlying (the convexity). - *Theta* (Θ) measures the sensitivity to the passage of time (time decay). - *Vega* measures the sensitivity to volatility σ. - ρ measures the sensitivity to the interest rate r.

Using the Black–Scholes formulas above, we can write explicit expressions for the Greeks of a European call (and similarly for a put). Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ denote the standard normal density, and define d_1, d_2 as above. Then:

6.3.1 Delta (Δ)

$$\Delta_C = \frac{\partial C}{\partial S} = \Phi(d_1), \qquad \Delta_P = \frac{\partial P}{\partial S} = \Phi(d_1) - 1,$$

for a call and put, respectively. Note that $0 < \Phi(d_1) < 1$, so $\Delta_C \in (0, 1)$ and $\Delta_P \in (-1, 0)$. Intuitively, Δ is the hedge ratio: owning one call is roughly equivalent to holding Δ_C shares of stock. Key points:

- For a call, $\Delta_C = \Phi(d_1)$ is positive and increases with S; deep in-the-money calls have Δ near 1.
- For a put, $\Delta_P = \Phi(d_1) - 1$ is negative (in $(-1, 0)$); deep in-the-money puts have Δ near -1.
- A *delta-neutral* position has total $\Delta = 0$, meaning its first-order sensitivity to small moves in S is zero.

6.3.2 Gamma (Γ)

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}},$$

which is the same for calls and puts. Gamma is always positive and measures the curvature of V with respect to S. Important facts:

- $\Gamma > 0$ for both calls and puts, reflecting the convexity of option prices in S.
- A large Γ means the option's delta will change rapidly for small moves in S; it indicates higher curvature.
- Gamma is highest when the option is at-the-money and when time to maturity is short; it decreases for deep in/out-of-the-money options or long-dated options.

6.3.3 Theta (Θ)

Theta measures the sensitivity to time (how the option price decays as t increases). For a European call,

$$\Theta_C = \frac{\partial C}{\partial t} = -\frac{S\sigma\phi(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2).$$

Similarly, for a European put,

$$\Theta_P = -\frac{S\sigma\phi(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}\Phi(-d_2).$$

Typically, Θ is negative, reflecting that as time passes (with everything else fixed), the option loses value (time decay). Key observations:

- Long options (calls or puts) have $\Theta < 0$ in most cases: they lose value as expiration approaches.
- The first term $-\frac{S\sigma\phi(d_1)}{2\sqrt{T-t}}$ represents decay due to the underlying's volatility and time; the second term involves interest and time-to-maturity.
- Near expiration, theta can become large in magnitude for at-the-money options, meaning very rapid decay of value.

6.3.4 Vega

Vega measures sensitivity to volatility σ. For both calls and puts,

$$\text{Vega} = \frac{\partial V}{\partial \sigma} = S\,\phi(d_1)\,\sqrt{T-t}.$$

Thus Vega is positive: an increase in volatility always raises the value of (European) options. It is largest when the option is at-the-money and when time to maturity is moderate. Key points:

- Vega > 0 for both calls and puts.
- Options with longer time to maturity have larger Vega (more exposure to volatility).
- Traders monitor Vega because volatility is a key input; changes in implied volatility move option prices through Vega.

6.3.5 Rho (ρ)

Rho measures sensitivity to the interest rate r. For a call,

$$\rho_C = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}\Phi(d_2),$$

which is positive since higher rates increase the call value. For a put,

$$\rho_P = \frac{\partial P}{\partial r} = -K(T-t)e^{-r(T-t)}\Phi(-d_2),$$

which is negative (higher rates decrease put values). Rho is usually small compared to other Greeks for short maturities or at-the-money options, but can be significant for long-dated or deeply in/out-of-the-money options. In summary:

- $\rho_C > 0$, $\rho_P < 0$.
- Longer time to maturity amplifies $|\rho|$.
- Typically, Rho is less critical than Delta, Gamma, Vega, Theta, except in interest-rate-sensitive contexts.

6.4 Delta-Neutral and Long-Gamma Strategies

A *delta-neutral* portfolio has total delta zero. For example, if a trader is long one call (with Δ_C) and wants to hedge against small moves in S, they would short Δ_C shares of stock, making the combined delta zero. Key ideas of delta-neutral and gamma-related strategies:

- **Delta-neutral hedging:** By rebalancing the stock position to offset the option's delta, a trader eliminates first-order price risk. This requires continuous (or frequent) rebalancing since Δ changes with S and t.
- **Long gamma:** Holding options (long calls or puts) gives positive gamma. A delta-neutral long-gamma position profits if the underlying moves (because you buy low and sell high while re-hedging). In effect, frequent re-hedging "captures" volatility: large moves generate gains.
- **Long gamma vs short gamma:** A *short-gamma* position (e.g. short options) will lose money if realized volatility exceeds the implied volatility, since one must re-hedge at an unfavorable price. Conversely, a long-gamma (long volatility) strategy profits from realized volatility above the market's expectations.
- **Gamma scalping:** This is a strategy where a trader maintains a delta-neutral position while being long gamma. When the underlying price moves, the trader adjusts the hedge and locks in small gains. Over time, if movements are sufficient, the accumulated gains exceed hedging costs.
- **Trade-off with Theta:** Long-gamma strategies benefit from volatility but suffer from negative theta (time decay). Traders must consider both effects: they earn from volatility while paying a cost each day as the option loses time value.

6.5 Model Assumptions and Practical Behavior

The Black–Scholes model is based on several key assumptions. Understanding these assumptions and how real markets deviate from them is crucial:

- **Lognormal returns:** The model assumes the underlying asset price is continuous and log-normally distributed, with constant volatility σ.
- **No arbitrage, frictionless markets:** Trading is continuous, there are no transaction costs or taxes, and one can borrow and lend unlimited amounts at the constant risk-free rate r.
- **No dividends:** The basic model assumes the stock pays no dividends during the option's life (or a known continuous yield).
- **Continuous hedging:** It assumes one can continuously rebalance the hedge portfolio.

- **Constant parameters:** The volatility σ and interest rate r are taken as constant.

In practice, these assumptions are idealizations, and market behavior often departs from the model:

- **Volatility smiles/skews:** Empirical option prices imply that volatility is not constant across strikes. At-the-money options often have lower implied volatilities than deep in/out-of-the-money options, producing a volatility "smile" or "skew." This contradicts the model's assumption of constant σ.
- **Stochastic volatility and jumps:** Real asset prices can exhibit volatility that changes over time (stochastic volatility) and can experience jumps or discontinuities. Models like Heston or Merton jump-diffusions extend Black–Scholes to handle these.
- **Fat tails and kurtosis:** Actual return distributions typically have heavier tails than the log-normal, meaning extreme moves are more likely than the model predicts.
- **Discrete hedging and costs:** In reality, hedging must be done in discrete time and incurs transaction costs. Continuous rebalancing is impossible; thus, perfect delta-hedging is only an approximation.
- **Parameter estimation:** The model requires inputs (volatility, interest rate) which must be estimated or implied from market prices. Misestimation can lead to pricing or hedging errors.

Despite these limitations, the Black–Scholes model serves as a fundamental benchmark. Traders often quote and interpret option prices in terms of *implied volatility*: the value of σ that, plugged into the Black–Scholes formula, matches the market price. Observed patterns in implied volatilities then guide adjustments or the use of more advanced models. Understanding the Black–Scholes assumptions and Greeks provides essential insight into option behavior, hedging, and risk management in financial markets.