An Age Structured Model Of Substance Abuse

by

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Declaration

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Chapter 1

Literature Review

The age structured models capture the effects of demographic behaviour of individuals Liu *et al.* (2015). According to Alexanderian *et al.* (2011) age structured models are the most appropriate in the study of cholera since both vaccine efficacy as well as the risk of contracting cholera depends on the age of an individual. Li and Brauer (2008) emphasised on the fact that even the simplest models do cater for some separate groupings in the population by dividing the population into different disease categories. They also highlighted that age may have some influence on reproduction, survival rates and behaviours. Here they noted that behavioural changes are very crucial as they are the major focus in the control and prevention of many infectious diseases. Let us also note that behavioural changes are an important aspect in the control and prevention of substance abuse.

Age structured models are used in studying various diseases and conditions. Shim *et al.* (2006) used to model the transmission of rotaries infection in the presence of maternal antibodies and vaccination. They made the assumption that contacts between individuals are influenced by age-class activity levels as well age densities. Their population is divided into 6 age dependent classes where the age dependent mixing contacts structure is modelled via the mixing density $p(t, a, a')$ which gives the proportion of contacts between individuals of age a and those aged a' supposing that they have had some contact at time t. The basic reproduction and the control adjusted number are computed from the Lotka characteristic equation.

Castillo-Chavez and Huang (2002) discusses an age structured core group model and its impact on STD dynamics. An assumption of proportionate mixing is made and also assumes that the population has reached its stable distribution.In their work they apply the theorem that checks the two conditions, namely that $R_0 < 1$ and B is uniformly Lipshitz continuous on R^+ to show that the disease free equilibrium is locally asymptotically stable. This is proved by applying the semi-group approach.

Chapter 2

Model

In formulating the age structured model of substance abuse we introduce the following parameters :

Parameter	Description
N(a,t)	Expected population size of age a at time t
$\beta(a,t)$	The effective contact rate of age a at time t.
$\mu(a)$	The per capita death rate as a function of age
$\sigma(a)$	The rate of movement into rehabilitation
$\gamma(a)$	The rate of relapsing while in rehab
$\rho(a)$	The recovery rate
$\omega(a)$	The relapse rate of the recovereds
$\alpha(a)$	The imitation coefficient

Table 2.1: The table gives the model parameters and their description.

2.1 Model Formulation

Here we formulate a model that monitors four populations in relation to substance abuse. The first group consist of individuals susceptible to substance abuse denoted by $S(a,t)$. These are individuals that are recruited into the population through birth. The second group is a group of drug users that are not in rehabilitation and that is denoted by $D(a,t)$. These individuals are initiated into drug use from the susceptible group. The third compartment consists of individuals who are in rehabilitation having been in compartment $D(a,t)$. The last compartment $Q(a,t)$ consists of individuals who have stopped using drugs and they are called quitters. The total population is thus given by

$$N = N(a,t) = S(a,t) + D(a,t) + R(a,t) + Q(a,t).$$

The main assumption of this model is that the population is approximately constant within the modelling time. We also assume that that there is a homogeneous mixing among all individuals from different compartments. The class responsible for initiation is $D(a,t)$ which consists of drug users not in treatment. Drug users who are undergoing treatment in compartment $R(a,t)$ are assumed not to initiate new cases since we assume that the rehabilitation is inpatient. We also assume that the susceptible class $S(a,t)$ consists of people who have never been involved in abusing substances before. Thus those who quit do not necessary move back to $S(a,t)$ but there is a possibility that they can still relapse and move straight back into $D(a,t)$. We now give the description of the dynamics of all compartments. The description is aided by figure 2.1.

(0,0)rectangle(2,2) ; (S) at (1,1) S(a,t); (4,0)rectangle(6,2); (D) at (5,1) D(a,t); (8,0)rectangle(10,2); (R) at (9,1) R(a,t); (12,0)rectangle(14,2); (Q) at (13,1) Q(a,t); [->] (2,1) – (4,1); [->] (6,1) – (8,1); [->] (10,1) – (12,1); [-] (13,0) – (13,-2); [-] (13,-2) – (5,-2); [->] (5,-2)–(5,0); [->](1,2) – (1,4); (2) at (0.9,4.3) $\mu(a)$ S(a,t); (3) at (3,1.3) $\lambda(a,t)$ S(a,t); [->] (5,2) – (5,4); [->] (9,2) – (9,4); [->] (13,2) – (13,4); [->] (8,.5) – (6,.5); (4) at (7,1.3) $\sigma(a)D(a,t)$; (10) at (7,.1) $\gamma(a)R(a,t)$; (5) at (11,1.3) $\rho(a)R(a,t)$; (6) at (4.9,4.3) $(\mu(a)+\psi(a))D(a,t)$; (7) at (8.9,4.3) $\mu(a)R(a,t)$; (8) at (12.9,4.3) $\mu(a)Q(a,t)$; (9) at (9,-1.7) $\omega(a)Q(a,t)$;

Figure 2.1: The diagrammatic representation of the substance abuse model with the compartments densities S(a,t),D(a,t),R(a,t),Q(a,t) and $\mu(a),\gamma(a),\sigma(a),\omega(a),\lambda(a),\rho(a)$ denoting the transition rates which are ≥ 0.

2.1.1 Susceptible Individuals, S(a,t)

This is a group of individuals who are at risk of getting addicted to drugs. Individuals in this compartment have no history of substance abuse. Recruitment into the susceptible compartment is due to births that are denoted by the boundary condition given below. Susceptible individuals initiated into substance abuse with an initiation function $\lambda(a,t)$ that is synonymous to the force of infection in disease models . The susceptible population is depleted through natural death plus disease induced death.

The differential equation explaining the dynamics of the susceptible population is

$$\frac{\partial S(a,t)}{\partial t}+\frac{\partial S(a,t)}{\partial a} = -\mu(a)S(a,t)-\lambda(a,t)S(a,t). \qquad (2.1)$$

2.1.2 Drug Abusing Individuals, D(a,t)

The individuals in this compartment are from the susceptible compartment as a result of contact. They are recruited at a rate $\lambda(a)$. Some of the people in this compartment are from the Rehabilitation compartment who relapse at rate $\gamma(a)$. Some of them come from the quitters compartment at a rate $\omega(a)$. These are the individuals who revert back to abusing drugs after recovering. Individuals exit compartment $D(a,t)$ as a result of death at a rate equal to $\mu(a)+\psi(a)$. Some are recruited into rehabilitation at a rate $\sigma(a)$. The differential equation explaining the dynamics of this compartment are given as

$$\frac{\partial D(a,t)}{\partial t} + \frac{\partial D(a,t)}{\partial a} = \lambda(a)S(a,t) + \gamma(a)R(a,t) + \omega(a)Q(a,t) - (\mu(a)+\sigma(a)+\psi(a))D(a,t) \quad (2.2)$$

2.1.3 Individuals in rehabilitation, R(a,t)

Individuals in this compartment are taking corrective measures to stop abusing drugs. All of them are coming from $D(a,t)$. This compartment decreases due to death and progression to the quitters compartment. Some of them unfortunately revert back to abusing substance due to the process usually referred to as relapsing. The corresponding differential equation is

$$\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} = \sigma(a)D(a,t) - (\mu(a)+\rho(a)+\gamma(a))R(a,t). \quad (2.3)$$

2.1.4 Quitters, Q(a,t)

Through rehabilitation there is a possibility of a complete recovery of individuals abusing substances. These individuals progress to compartment $Q(a,t)$ at a per capita rate $\rho(a)$. We allow these people not to quit permanently but have a chance to reuse drugs. When they relapse they progress to compartment $D(a,t)$ at a per capita rate $\omega(a)$. With a natural mortality rate $\mu(a)$, the dynamics of compartment $Q(a,t)$ is given by

$$\frac{\partial Q(a,t)}{\partial t} + \frac{\partial Q(a,t)}{\partial a} = \rho(a)R(a,t) - (\mu(a)+\omega(a))Q(a,t). \quad (2.4)$$

The model diagram and assumptions leads to the following system of differential equations;

$$\begin{aligned}
\frac{\partial S(a,t)}{\partial t} + \frac{\partial S(a,t)}{\partial a} &= -\mu(a)S(a,t) - \lambda(a,t)S(a,t),\\
\frac{\partial D(a,t)}{\partial t} + \frac{\partial D(a,t)}{\partial a} &= \lambda(a,t)S(a,t) + \gamma(a)R(a,t) + \omega(a)Q(a,t) - (\mu(a)+\sigma(a)+\psi(a))D(a,t),\\
\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} &= \sigma(a)D(a,t) - (\mu(a)+\rho(a)+\gamma(a))R(a,t),\\
\frac{\partial Q(a,t)}{\partial t} + \frac{\partial Q(a,t)}{\partial a} &= \rho(a)R(a,t) - (\mu(a)+\omega(a))Q(a,t).
\end{aligned} \quad (2.5)$$

We assume that the number of births and deaths are balanced and also that the number of births is equal to the number of people aged zero such that we have

$$N(0,t) = S(0,t) \tag{2.6}$$

As a result we have the following boundary conditions:

$$S(0,t) = \theta(t) \quad \text{and} \quad D(0,t) = R(0,t) = Q(0,t) = 0 \tag{2.7}$$

Adding the boundary conditions we get:

$$N(0,t) \quad = \quad S(0,t) + D(0,t) + R(0,t) + Q(0,t) = \theta(t). \tag{2.8}$$

If we add up equations in (2.5) we get

$$\frac{\partial N(a,t)}{\partial t} + \frac{\partial N(a,t)}{\partial a} = -\mu N(a,t) - \psi(a)D(a,t) \tag{2.9}$$

$$\frac{\partial N(a,t)}{\partial t} + \frac{\partial N(a,t)}{\partial a} \leq -\mu N(a,t) \tag{2.10}$$

We can see that equation (2.10) is a von McKendrick Form with a solution of the form

$$N(a,t) \leq \begin{cases} \theta(t)e^{-\int\limits_0^a \mu(\epsilon)\mathrm{d}\epsilon} & \text{if } a \leq t \\ N(a-t,0)e^{-\int\limits_{a-t}^a \mu(\epsilon)\mathrm{d}\epsilon} & \text{if } a \geq t \end{cases}$$

2.2 Rescaling The Parameters

We introduce the scaled state variables: $s(a,t) = \frac{S(a,t)}{N(a,t)}$, $d(a,t) = \frac{D(a,t)}{N(a,t)}$, $r(a,t) = \frac{R(a,t)}{N(a,t)}$ and $q(a,t) = \frac{Q(a,t)}{N(a,t)}$.

Below is the rescaled version of equation (2.5)

$$\begin{aligned}
\frac{\partial s(a,t)}{\partial t} + \frac{\partial s(a,t)}{\partial a} &= -\lambda(a,t)s(a,t), \\
\frac{\partial d(a,t)}{\partial t} + \frac{\partial d(a,t)}{\partial a} &= \lambda(a,t)s(a,t) + \gamma(a)r(a,t) + \omega(a)q(a,t) - (\sigma(a) + \psi(a))d(a,t), \\
\frac{\partial r(a,t)}{\partial t} + \frac{\partial r(a,t)}{\partial a} &= \sigma(a)d(a,t) - (\rho(a) + \gamma(a))r(a,t), \\
\frac{\partial q(a,t)}{\partial t} + \frac{\partial q(a,t)}{\partial a} &= \rho(a)r(a,t) - \omega(a)q(a,t).
\end{aligned} \tag{2.11}$$

With the following boundary conditions

$s(0,t) = 1$ and $d(0,t) = r(0,t) = q(0,t) = 0$

And the following initial conditions

$s(a,0) = \varphi_s(a)\ d(a,0) = \varphi_d(a)\ r(a,0) = \varphi_r(a)\ q(a,0) = \varphi_q(a)$

$\lambda(t,a) = \int_0^a \beta(a,\upsilon)N(\upsilon)d(t,\upsilon)d\upsilon$

2.3 Existence and Uniqueness of the solution

Let $X = L^1(0,a)^4$ with the norm $\|\varphi\|\|_X = \sum_{i=1}^4 \|\varphi_i\|_{l^1}$ where $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X)$. $(X, \|.\|_X)$ is clearly a Banach Space.

If we consider the operator
$A : D(a) \subset X \longrightarrow X$ defined by

$$A\varphi = (\frac{-d\varphi_1}{da}, \frac{-d\varphi_1}{da}, \frac{-d\varphi_2}{da}, \frac{-d\varphi_3}{da}, \frac{-d\varphi_4}{da})^T$$

where $D(A) = \left\{ \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in X; \varphi_i \in W_1^1(0,a) \quad and \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

and the function $F : \overline{D(A)} \longrightarrow X$ defined by $F \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \begin{pmatrix} -\lambda(.,\upsilon)\varphi_1 \\ -\lambda(.,\upsilon)\varphi_1 + \gamma\varphi_3 + \omega\varphi_4 - (\sigma+\psi)\varphi_2 \\ \sigma\varphi_2 - (\rho+\gamma)\varphi_3 \\ \rho\varphi_3 - \omega\varphi_4 \end{pmatrix}$

The non linear operator F is defined on the whole space X where $\lambda(.,\upsilon) \in L^1(0,a)$ such that

$$\lambda(a,\upsilon) = \int_0^a \beta(a,\upsilon)N(\upsilon)\varphi_2(\upsilon)d\upsilon$$

where $\beta(a,\upsilon) \in L^\infty((0,a) \times (0,a))$

Let $u(t) = (s(.,t), d(.,t), r(.,t), q(.,t))^T \in X$. We can rewrite the initial boundary value problem (2.11) as the abstract semi linear problem in X.

$$\frac{du(t)}{dt} = Au(t) + F(u(t)) \quad u(0) = u_0 \in X \tag{2.12}$$

where $u_0(a) = (s_0(a), d_0(a), r_0(a), q_0(a))^T$

Thus A is the infinitesimal generator of a C_0 semi-group $T(t), \quad t \geq 0$ and F is continuous and locally Lipschitz. Then for each $u_0 \in X$ there exists a maximal interval of existence $[0, t_0]$ and a unique continuous (mild) solution $t \longrightarrow u(t; u_0)$ from $[0, t_0] to \quad X$ such that

$$u(t; u_0) = T(t)u_0 + \int_0^t T(t-s)F(u(s; u_0))ds \tag{2.13}$$

2.4 Computation of the Reproduction Number

The drug free equilibrium of our normalised model is given as

$$E_0 = (1, 0, 0, 0)$$

Below we linearise the $d(a,t)$ and $r(a,t)$ about the drug free equilibrium and make the assumption that the solutions initially change exponentially to obtain the characteristic equation.

The characteristic equation will thus be analysed to obtain the formula for the reproductive number.

If we assume the following solutions

$s(a,t) = 1 + s(\bar{a})e^{\kappa t} \quad d(a,t) = d(\bar{a})e^{\kappa t} \quad r(a,t) = r(\bar{a})e^{\kappa t} \quad \lambda(a,t) = \lambda_0 e^{\kappa t} + 0(e^{2\kappa t})$
Where

$$\lambda_0 = \int_0^\infty d(\bar{a})B_\infty(a)da$$

Below we linearise equation (2.11)
The linear form of the equation for $d(a,t)$ is

$$te^{\kappa t}\bar{d}(a) + e^{\kappa t}\frac{d\bar{d}(a)}{da} = \lambda_0 e^{\kappa t}(1 + \bar{s}(a)) + \gamma(a)\bar{r}(a)e^{\kappa t} - (\sigma(a) + \psi(a))\bar{d}(a)e^{\kappa t}$$

Dividing by $e^{\kappa t}$ and considering the linear part of the equation we get

$$t\bar{d}(a) + \frac{d\bar{d}(a)}{da} = \lambda_0 + \gamma(a)\bar{r}(a) - (\sigma(a) + \psi(a))\bar{d}(a) \tag{2.14}$$

We do the same for the equation of the compartment $r(a,t)$ to get

$$t\bar{r}(a) + \frac{d\bar{r}(a)}{da} = \sigma(a)\bar{d}(a) - (\rho(a) + \gamma(a))\bar{r}(a) \tag{2.15}$$

Equation (2.14) is integrated as follows:

$$(t + \sigma(a) + \psi(a))\bar{d}(a) + \frac{d\bar{d}(a)}{da} = \lambda_0 + \gamma(a)\bar{r}(a) \tag{2.16}$$

Using the integrating factor we obtain the following:

$$\bar{d}(a) = e^{-(\sigma(a)+\psi(a)+\kappa)a} \int_o^a e^{(\sigma(\alpha)+\psi(\alpha)+\kappa)}[\lambda_0 + \gamma(\alpha)\bar{r}(\alpha)]d\alpha \tag{2.17}$$

Similarly we get an expression of $\bar{r}a$ by using the integrating factor on equation (2.15) as follows:

First we rearrange the equation (2.15) to get

$$(t + \rho(a) + \gamma(a))\bar{r}(a) + \frac{d\bar{r}(a)}{da} = \sigma(a)\bar{d}(a) \tag{2.18}$$

After applying the integrating factor to equation (2.18) we get the following expression for $\bar{r}a$

$$\bar{r}(a) = e^{-(\rho(a)+\gamma(a)+\kappa)a} \int_0^a e^{(\rho(\alpha)+\gamma(\alpha)+\kappa)}\sigma(\alpha)\bar{d}(\alpha)d\alpha \tag{2.19}$$

Substituting equation (2.19) into equation (2.17) we get the following expression of $\bar{d}a$

$$\bar{d}(a) = e^{-(\sigma(a)+\psi(a)+\kappa)a} \int_o^a e^{(\sigma(\alpha)+\psi(\alpha)+\kappa)}[\lambda_0+\gamma(\alpha)]e^{-(\rho(\alpha)+\gamma(\alpha)+\kappa)\alpha} \int_0^a e^{(\rho(b)+\gamma(b)+\kappa)}\sigma(b)\bar{d}(b)dbd\alpha \tag{2.20}$$

2.5 Numerical Solution

To discretize our model we make use of the forward difference where

$$\begin{aligned} \frac{\partial X(a,t)}{\partial a} + \frac{\partial X(a,t)}{\partial t} &\approx \frac{X(a_i,t_j) - X(a_{i-1},t_j) + X(a_{i-1},t_j) - X(a_{i-1},t_{j-1})}{h} \\ &= \frac{X(a_i,t_j) - X(a_{i-1},t_{j-1})}{h} \\ &= \frac{X_i^j - X_{i-1}^{j-1}}{h} \end{aligned} \tag{2.21}$$

After using equation (2.21) our discretized system is given as

$$\begin{aligned} \frac{S_i^j - S_{i-1}^{j-1}}{h} &= -(\mu_i + \lambda_i^j)S_i^j \\ \frac{D_i^j - D_{i-1}^{j-1}}{h} &= \lambda_i^j S_i^j + \gamma_i R_i^j + \omega_i Q_i^j - (\mu_i + \sigma_i + \psi_i) \\ \frac{R_i^j - R_{i-1}^{j-1}}{h} &= \sigma_i D_i^j - (\mu_i + \rho_i + \gamma_i)R_i^j \\ \frac{Q_i^j - Q_{i-1}^{j-1}}{h} &= \rho_i R_i^j - (\mu_i - \omega_i)Q_i^j \end{aligned} \tag{2.22}$$

Solving equations (2.22) we get

$$
\begin{aligned}
S_i^j &= \frac{S_{i-1}^{j-1}}{1+h(\mu_i+\lambda_i^j)} \\
D_j^j &= \frac{h(\lambda_i^j S_i^j+\gamma_i R_i^j+\omega_i Q_i^j)+D_i^j}{1+h(\mu_i+\sigma_i+\psi_i)} \\
R_i^j &= \frac{R_{i-1}^{j-1}+h\sigma_i D_i^j}{1+h(\mu_i+\rho_i+\gamma_i)} \\
Q_i^j &= \frac{Q_{i-1}^{j-1}+h\rho_i R_i^j}{1+h(\mu_i+\omega_i)}
\end{aligned}
\tag{2.23}
$$

Appendices

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