DTLZ7 function: Ideal and Nadir points derivation

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1 Introduction

According to the definition [3, 2], the ideal solution is the individual minima of each objective function:

$$\mathbf{z}^\star = \left[\min_{\mathbf{x}\in\mathcal{X}} f_1(\mathbf{x}) \ \min_{\mathbf{x}\in\mathcal{X}} f_2(\mathbf{x}) \ \dots \ \min_{\mathbf{x}\in\mathcal{X}} f_M(\mathbf{x}) \right]^T \tag{1}$$

while the Nadir is the individual maxima, *but restricted to the Pareto-optimal front*:

$$\mathbf{z}^{Nadir} = \left[\max_{\mathbf{x}\in\mathcal{X}^*} f_1(\mathbf{x}) \ \max_{\mathbf{x}\in\mathcal{X}^*} f_2(\mathbf{x}) \ \dots \ \max_{\mathbf{x}\in\mathcal{X}^*} f_M(\mathbf{x}) \right]^T \tag{2}$$

In this notation, $\mathcal{X}$ is the set of feasible solutions, while $\mathcal{X}^*$ is the set of Pareto-optimal solutions.

The DTLZ test suite [1] is such that each problem has dimension $n = (M-1) + k$, wherein M is the number of objectives, and k is an integer parameter with a different value depending on the test function: 5 for DTLZ1, 20 for the DTLZ7, and 10 for the rest. The variable vector $\mathbf{x}$ has the following structure:

$$\mathbf{x} = [\underbrace{x_1 \ x_2 \ \dots \ x_{M-2} \ x_{M-1}}_{M-1 \text{ position variables}} \ \underbrace{x_M \ x_{M+1} \ \dots \ x_{n-1} \ x_n}_{k \text{ distance variables}}]^T \tag{3}$$

that is, the last k variables, called *distance variables*, need to achieve a given value V (which is 0.5 for DTLZ1 to DTLZ5, and 0 for the remaining two) in order to a given point belong to the efficient front. The first $M-1$ components, named *position variables*, are varied from 0 to 1 to map the Pareto front, assuming the last k components have the proper value.

Each test function has an auxiliary function $g(\cdot)$ which depends only on the distance variables. Thus, when we are in the Pareto front, this function is a constant. The appropriate approach for computing the ideal and Nadir points is hence reduced to employing regular calculus strategies.

2 An example: The ideal and Nadir solutions of the DTLZ1 function

Let us begin with a simpler function, the DTLZ1, with the following formulation[1]:

[1] I changed the notation $\mathbf{x}_M$ in [1] to $\mathbf{x}_{M:n}$ to indicate a vector with variables from M to n, which is essentially the same but (in my opinion) easier to understand.

$$
\begin{aligned}
\text{minimize} \quad & f_1(\mathbf{x}) = \frac{1}{2}x_1x_2\dots x_{M-1}(1+g(\mathbf{x}_{M:n})) \\
& f_2(\mathbf{x}) = \frac{1}{2}x_1x_2\dots(1-x_{M-1})(1+g(\mathbf{x}_{M:n})) \\
& \vdots \\
& f_{M-1}(\mathbf{x}) = \frac{1}{2}x_1(1-x_2)(1+g(\mathbf{x}_{M:n})) \\
& f_M(\mathbf{x}) = \frac{1}{2}(1-x_1)(1+g(\mathbf{x}_{M:n})) \\
\text{subject to} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}
$$

The exact formulation of $g(\cdot)$ function is not relevant for this discussion. What matters is that, according to [1], the Pareto-optimal front occurs when $\mathbf{x}_{M:n} = \mathbf{0}$, in which case $g(\mathbf{x}_{M:n}) = 0$ and is thus a constant. The objective functions above become:

$$
\begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_{M-1}(\mathbf{x}) \\ f_M(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1x_2\dots x_{M-1} \\ \frac{1}{2}x_1x_2\dots(1-x_{M-1}) \\ \vdots \\ \frac{1}{2}x_1(1-x_2) \\ \frac{1}{2}(1-x_1) \end{bmatrix}
$$

So, the smallest value of $f_1(\cdot)$ will be 0 when all (or at least one) $x_i = 0$, and the greatest value will be $\frac{1}{2}$, when all $x_i = 1$. For $f_2(\cdot)$, the smallest one will be 0 with the same reasoning, and the largest one will be $\frac{1}{2}$ when $x_i = 1$, $i = 1, \dots, M-2$ and $x_{M-1} = 0$. Extending this explanation, we arrive at the solutions $\mathbf{z}^\star = [0\ 0\ 0 \dots 0\ 0]^T$ and $\mathbf{z}^{Nad} = [0.5\ 0.5\ \dots 0.5\ 0.5]^T$ as used in the code.

3 The ideal and Nadir solutions of the DTLZ7 function

We can use a similar analysis to derive the ideal and Nadir solutions of all DTLZ functions. For the DTLZ7, which is more intricate, consider first its formulation:

$$
\begin{aligned}
\text{minimize} \quad & f_1(\mathbf{x}) = x_1 \\
& f_2(\mathbf{x}) = x_2 \\
\vdots \quad & \\
& f_{M-1}(\mathbf{x}) = x_{M-1} \\
& f_M(\mathbf{x}) = (1+g(\mathbf{x}_{M:n}))h(f_1, f_2, \dots, f_{M-1}, g) \\
\text{subject to} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}
$$

in which

$$
h(f_1, f_2, \dots, f_{M-1}, g) = M - \sum_{i=1}^{M-1} \left[\frac{f_i}{1+g}(1+\sin(3\pi f_i)) \right]
$$

To begin with, notice that the individual minima and maxima of the first $M-1$ functions are immediate from the formulation, since each f_i equals x_i, $i = 1, \ldots, M-1$. Thus, the first $M-1$ coordinates of $\mathbf{z}^\star$ are 0, and those of the Nadir $\mathbf{z}^{Nad}$ are, *in principle*, 1. We will see later that this may not be the case.

For the M-th objective function, as usual, the Pareto-optimal front is obtained when $\mathbf{x}_{M:n} = \mathbf{0}$, which, in this case, results in $g(\mathbf{x}_{M:n}) = 1$ (see [1]). In addition to that, since each $f_i(\cdot)$ is equal to x_i for $i = 1, \ldots, M-1$, the auxiliary $h(\cdot)$ function can be written as a function of $\mathbf{x}$:

$$h(\mathbf{x}) = M - \sum_{i=1}^{M-1} \left[\frac{x_i}{2} (1 + \sin(3\pi x_i)) \right]$$

and then the M-th objective function in the Pareto front becomes

$$f_M(\mathbf{x}) = 2 \left\{ M - \sum_{i=1}^{M-1} \left[\frac{x_i}{2} (1 + \sin(3\pi x_i)) \right] \right\}$$

With a quick glance, we can see that this function will have its greatest value when the sum is at its minimum, which happens when $x_i = 0$. In that case, f_M becomes $2M$, which is the last coordinate of the Nadir solution.

For the last coordinate of the ideal point, let us compute its gradient with the usual product rule. The differentiation of the inner element of the sum becomes

$$\begin{aligned} \frac{d}{dx_i} \left[\frac{x_i}{2} (1 + \sin(3\pi x_i)) \right] &= \frac{1}{2} \times (1 + \sin(3\pi x_i)) + \frac{x_i}{2} \times 3\pi \cos(3\pi x_i) \\ &= \frac{1}{2} \left[1 + \sin(3\pi x_i) + 3\pi x_i \cos(3\pi x_i) \right] \end{aligned}$$

and then

$$\nabla f = \begin{bmatrix} -\left[1 + \sin(3\pi x_1) + 3\pi x_1 \cos(3\pi x_1)\right] \\ -\left[1 + \sin(3\pi x_2) + 3\pi x_2 \cos(3\pi x_2)\right] \\ \vdots \\ -\left[1 + \sin(3\pi x_{M-1}) + 3\pi x_{M-1} \cos(3\pi x_{M-1})\right] \\ -\left[1 + \sin(3\pi x_M) + 3\pi x_M \cos(3\pi x_M)\right] \end{bmatrix}$$

To find stationary points, we need to equal each term to zero. The solution of equation

$$1 + \sin(3\pi x_i) + 3\pi x_i \cos(3\pi x_i) = 0 \tag{4}$$

can be obtained numerically as $x_i^* \approx 0.85940$, which corresponds to `xaux` in the code. At this point, each term of the sum is at its largest, and thus $f_M(\cdot)$ is at its smallest. This value corresponds to `faux`, which is then computed as

$$\begin{aligned} \texttt{faux} &= 2 \left\{ M - \sum_{i=1}^{M-1} \left[\frac{\texttt{xaux}}{2} (1 + \sin(3\pi \texttt{xaux})) \right] \right\} \\ &= 2 \left\{ M - (M-1) \times \left[\frac{\texttt{xaux}}{2} (1 + \sin(3\pi \texttt{xaux})) \right] \right\} \end{aligned}$$

thus explaining the code for `faux`. Therefore, the ideal solution becomes $\mathbf{z}^\star = [0\ 0 \ldots\ 0\ \texttt{faux}]^T$.

So far we believe the Nadir is $\mathbf{z}^{Nad} = [1\ 1\ \ldots\ 1\ 2M]^T$, which is the individual maxima of each function. However, remember that these maxima must be *restricted* to the Pareto-optimal front. Take the DTLZ7 with 2 objectives. According to equation (3), there is $M-1=1$ position variable, and $k=20$ distance variables, all of those with a fixed value in the Pareto-optimal front. Thus, we can map this front by varying x_1 from 0 to 1, which results in the graph of Figure 1.

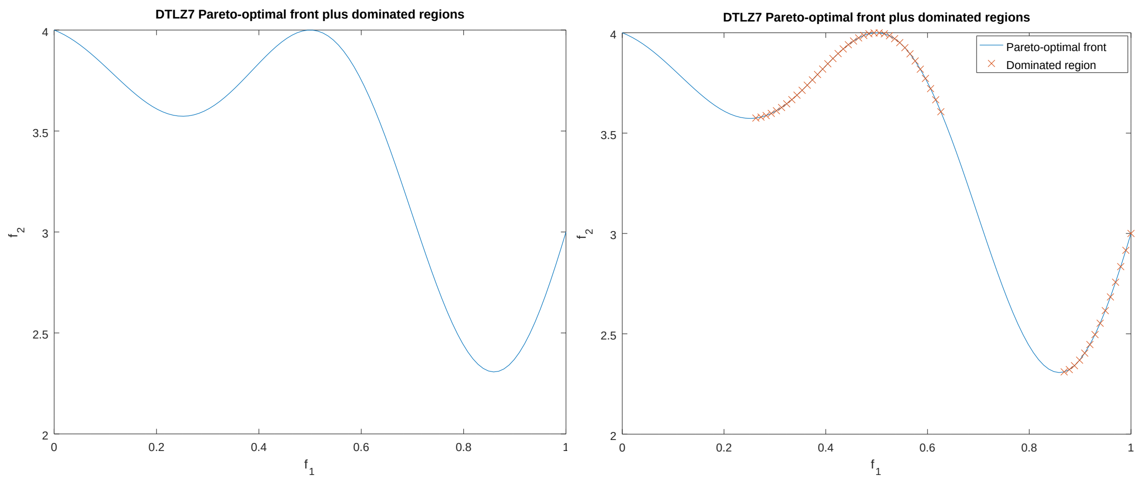

Figure 1: DTLZ7 Pareto-optimal front. Notice some solutions become non-dominated, and thus not all values of x_1 generate a proper optimal point.

With 2 objectives, we can see that the point $[1\ 2M]^T = [1\ 4]^T$ is clearly not the actual Nadir solution as the points from $f_1 \approx 0.85940$ to $f_1 = 1$ are dominated. This minimum value of ≈ 0.85940 corresponds to `xaux` as computed before in equation (4). Therefore, the "true" Nadir point for two objectives is $\mathbf{z}^{Nad} = [\texttt{xaux}\ 4]^T$.

We can generalize this with a quick reasoning. Looking at the definition of the DTLZ7 function, we can see that all of the first $M-1$ functions grow as each x_i, $i = 1, \ldots, M-1$ also grow, while the last M-th objective function decreases, but only to a certain point. As proved before, $f_M(\cdot)$ has its smallest value when all $x_i = \texttt{xaux}$ and, from this point on, $f_M(\cdot)$ starts to increase as well. Well, if all functions are growing together, these new solutions start to be dominated. Therefore, the biggest value that each $f_i(\cdot)$, $i = 1, \ldots, M-1$ can acquire *while remaining in the Pareto-optimal front* is `xaux`, which leads to the Nadir solution $\mathbf{z}^{Nad} = [\texttt{xaux}\ \ \texttt{xaux}\ \ldots\ \texttt{xaux}\ \ 2M]^T$, as used in the code.

References

[1] Kalyanmoy Deb, Lothar Thiele, Marco Laumanns, and Eckart Zitzler. Scalable test problems for evolutionary multiobjective optimization. Technical Report 1990, 2005.

[2] Fillipe Goulart. Preference-guided Evolutionary Algorithms for Optimization with Many Objectives. Master's thesis, Universidade Federal de Minas Gerais, 2014.

[3] K. Miettinen. *Nonlinear Multiobjective Optimization*. International Series in Operations Research & Management Science. Springer US, 1999.