Fractal Growth

Computational Physics

Benedikt Sauer, Alexander Schroer

March 2011

1 Introduction

In 1981, Witten and Sander discovered that complex dendritic structures could be created by having 'particles' perform a random walk on a lattice and stick together on contact (fig. 1). This can be considered a rather surprising result, since without any preferred direction in the motion of particles one would naturally expect to end up with some sort of compact 'blob'. In fact, structures grown according to this or a similar method turn out to be *natural fractals*, i.e. they exhibit a certain scaling behaviour within an apropriate range. The result is all the more remarkable, since Witten and Sander modelled it on natural phenomena – the random walk corresponding to Brownian motion and the sticky particles representing some adhesive molecules or sedimenting colloids.

The prospect that the formation of structures which had up to then been believed to be dominated by complicated interactions could be reduced to such easy principles together with the connection to the then just emerging theory of fractal geometry sparked a remarkable interest and was the basis for many follow-up publications.

Focussing on two particular growth models developed by P. Meakin using the results of Witten and Sander and M. Eden, we give a brief introduction to the basics of *fractal growth* and present the current status of a universal toolkit capable of simulating discrete growth processes in a very general way. It is applied to check some earlier heurestical results.

Further reading Since this text is composed in a review-like fashion and most of the information presented is considered to be well-known today, we will give hints for further reading at the end of each section instead of cluttering the text with citations.

As for Witten and Sander's original publication and Meakin's enhancements, refer to [1] and [3].

2 Some Mathematics

2.1 Why talk of fractal dimensionality?

A possible description of geometrical objects is their treatment as subsets of $\mathbb{R}^n$, e.g. for three-dimensional
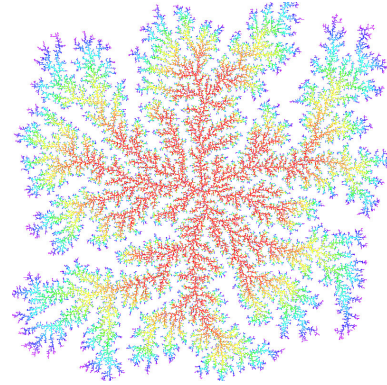

Figure 1: **A typical dendritic structure** grown similar to the original model of Witten and Sander (text). Colour-coded is the age of each particle. The cluster consists of about 60,000 particles in total.

space consider a line, a square and a cube of length, area and volume (all of which we will refer to as generalized volumes hereafter) 1 respectively. It is straightforward to represent the line by $[0, 1]$ whereas $[0, 1]^2$ and $[0, 1]^3$ do nicely for square and cube – intuitively, the volumes work out just as expected. One would now of course like to assign a volume to all possible sets and can in fact do so for quite a lot – though not for all. This idea is extensively treated in *measure theory*.

However, while volume is an important aspect to classify geometric objects we would also like to introduce some notion of *dimensionality*. Clearly, line, square and cube should be of dimension 1, 2 and 3. For our example this can easily be achieved (up to the border) by resorting to topology and basically identify a set with a manifold which is locally equivalent to a vector space $\mathbb{R}^m$, where $m < n$, and call m the dimension of the set. Manifolds, while having a great deal of interesting properties, are hardly a typical example of arbitrary subsets of $\mathbb{R}^n$, though.

A more general approach, which works for any topological (sub-)space and thus trivially for any open subset of $\mathbb{R}^n$, is given by the *Lebesgue covering dimension*: If always some points of the set are contained in at least $n + 1$ elements of an arbitrarily fine open cover, the set is of dimension n.

There is a bunch of other (non-equivalent) approaches to assign sets an integer which is called *dimension* and coincides with what one would expect for our examples and other 'everyday objects'. However, at some point, quite another issue arises. As an example, consider the *Koch curve* (fig. 2), which starting with the interval $[0, 1] \subset \mathbb{R}^2$ can be constructed as follows:

- divide the interval into three parts of equal length,

- replace the middle part by two parts of the same length which together with the removed part form an equilateral triangle,

- repeat these steps for each of the smaller parts.

The catch is this – what dimension do you *want* to assign to this set? It should probably be one dimensional. After all, has been constructed from lines. But than again, its length is infinite (precisely $\lim_{n \to \infty} \left(\frac{4}{3} \right)^n$) whilst its extent is finite. Surely, such an object should cover some finite area? Unfortunately, the Koch curve being two-dimensional is clearly ruled out by our topological definition. In fact, the topological dimension of the Koch curve can easily be verified to be 1, which seems somewhat unfitting.

The Koch curve is just an example of many sets which can be defined using a so-called *iterated function system*: Given a set of affine functions which are contractive on the average, the set of points which is invariant under the action of the functions has similiar properties as the Koch curve. Another famous set which can be constructed by an IFS is the Sierpinski triangle (fig. 3).

Alternatively, consider an infinite random walk in two dimensions. The resulting path is one dimensional by construction, but since every point in space has a finite probability of being reached, after an infinite number of steps, the whole space should be covered. So is the path one- or two-dimensional?

It turns out that it is a fruitful idea to resort the obvious, though unintuitive solution: define fractional dimensions. By constructions which will be discussed in a second, one can assign the dimensions 1.26, 1.58, and 1.33 to the Koch curve, the Sierpinski triangle and the random walk respectively. They all are embedded in two-dimensional space, so their dimension has to be less than or equal to 2, and their topological dimension is 1, which serves as a lower boundary.

Structures of non-integer dimension are called *fractals*.

2.2 The fractal (Hausdorff) dimension

One can put these observations on a firm mathematical footing by introducing the outer Hausdorff measure

$$H_\varepsilon^d(S) = \inf_{\substack{\bigcup_{i=1}^\infty U_i \supset S \\ \mathrm{diam}(U_i) < \varepsilon}} \sum_{i=1}^\infty \mathrm{diam}(U_i)^d$$

and define the Hausdorff dimension for a measurable set S as

$$d_H = \sup\{d \in \mathbb{R}_0^+ \mid H^d(S) = \infty\}.$$

While this looks rather intimidating at first sight, it boils down to the same idea which was behind the covering dimension, although being much more versatile and applicable to any set with a well-defined volume, which includes some rather pathological cases. Again, this is dealt with in greater detail in measure theory.

Fortunately, in well-behaved cases there is an easy implication. Note that we will not define the precise meaning of 'well-behaved'. In general, however, fractals which can be constructed explicitly, e.g. by an IFS, tend to be 'well-behaved'. The implication is this: Connected to the dimensionality of a set is a certain scaling behaviour concerning the number N of spheres of radius R covering the set, namely for $R \to 0$, $N \propto R^{-d_H}$. While this is just a reformulation of what we have already had before (provided spheres are suited as covering sets), writing it this way allows for further modification to yield a very useful expression:

$$d_H = -\lim_{R \to 0} \frac{\log N}{\log R}. \tag{1}$$

By this, you can easily confirm, that indeed, the dimension of the Koch curve is $\frac{\log 4}{\log 3} \approx 1.26$. The expected results for geometric primitives are also easily recoverable.

This scaling behaviour can be viewed as an illustration for the connection between fractal dimensionality and *self-similarity*. Self-similarity is a striking aspect of fractals and describes the fact that viewed on different scales, fractals look essentially the same. In the case of IFS fractals this similarity is exact. But there are as well true fractals with only approximate self-similarity, the most famous example probably being the Mandelbrot set.

2.3 Improper fractals

Up to now he have only talked about sets the construction of which involved some limiting procedure. This is clearly not apt to describe natural phenomana which are always limited and finite. Nevertheless, when looking at the right scale, fractal characteristics may emerge. In terms of the example of dendritic growth discussed in the introduction, clearly it is neither useful to study the structure at the atomic scale, nor at a macroscopic one. But a a certain *mesoscopic* scale, the dendritic structure exhibits a large amount of self-similiarity (fig. 4). Our immidiate aim therefore is to further generalize our notion of fractal dimensionality to encompass these kinds of structures as well.

We can do so in a somewhat ad-hoc fashion: Again consider the minimal number of spheres of radius R, $N(R)$, required to cover the structure. But now, instead of taking the limit of a small radius, define the fractal dimension d at the scale given by R as
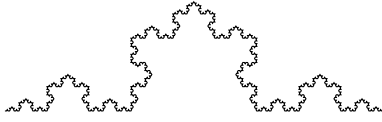
Figure 2: **The Koch curve** is a well-known example for a fractal structure ($d_H = \frac{\log 4}{\log 3}$)

$$d(R) = -\frac{\log N(R)}{\log R}. \qquad (2)$$

Note that this is almost identical to eq. 1. The value of d obtained by this conecpt is called *box-counting dimension* and it does not neccessarily equal the Hausdorff dimension. In fact, in most cases it does not – remember, we are not talking about true fractals any longer. d becomes a meaningful quantity if it does not change (much) over the scales under consideration. This information is usually extracted from log-log-plots (fig. 5).

There are also other ways to determine the fractal dimension from scaling laws. In fact, one should rather call them methods to find a (sic) fractal dimension, because just as with box-counting and Hausdorff, equivalences break down due to finite-size effects. Then again, there are some scaling laws which give approximately the right dimensionality (i.e. in accordance to other established recipies) for no good reason at all.

Two methods which are often applied in $\mathbb{R}^n$ with Lebesgue metric and which will be particularly suited to our ends are

- $R_{\text{gyr}} \propto m^{1/d_H}$,

  where $R_{\text{gyr}} = \frac{1}{m}\left(\int_V \mathrm{d}^n x \rho(x)(x - x_0)^2\right)^{\frac{1}{2}}$ is the radius of gyration around the center of mass x_0, V the volume and $m = \int_V \rho$ the mass and

- $C(r) \propto r^{D-d}$,
  where $C(r) = \rho \star \rho$ is the density-density correlation.

The density $\rho(x)$ is usually defined as 1 if x belongs to the fractal and 0 else.

Structures which exhibit a fractal-like scaling behaviour over certain ranges are common in nature. Typical examples include coast lines, veins and cracks.

Further reading The rough sketches of topology and measure theory can be reviewed in any decent book on advanced analysis. For a comprehensive introduction to fractal geometry, have a look at [7]. Different methods to measure the fractal dimension of finite structures can be found e.g. in [2] and [1].

3 The Meakin model

The Meakin model comprises two aspects: *diffusion limited aggregation* and *number limited aggregation.* Both deal with particles perfoming a random walk and
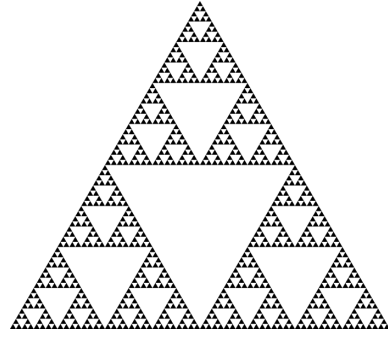

Figure 3: **The Sierpinski triangle** is another famous example for a strictly self-similar fractal which can be defined as the solution of an iterated function system. Wikimedia Commons.
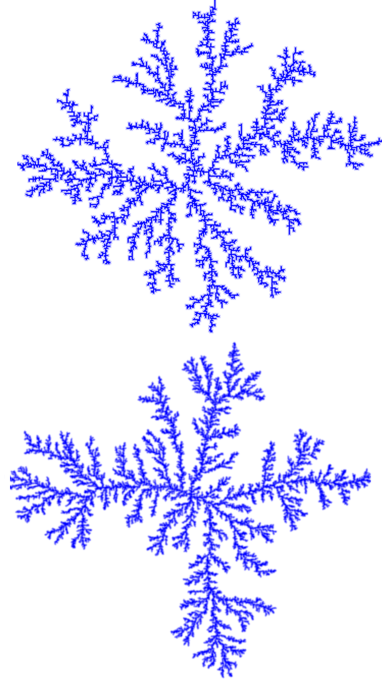

Figure 4: **Self-similarity** is a form a scale invariance. The upper cluster consists of 10,000 particles, whereas the lower one contains 100,000 particles. Note that the larger cluster just appears to have a slightly more detailled boundary. Actually, it has grown much larger; the size of the individual particles is unchanged.
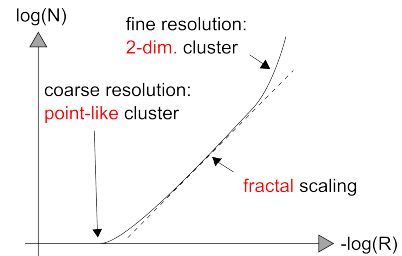

Figure 5: **Improper fractals** exhibit fractal scaling behaviour only over a certain range. In this example, the fractal is composed of two-dimsensional objects and has a finite size which induces different scaling behaviours at large and small scales.

stick together on contact just like Witten and Sander proposed. In the case of *diffusion limited aggregation* (DLA), one follows the pattern

1. create a single particle at the origin and call it the cluster (seeding);

2. create a new particle at a random position, uniformly chosen on the n-sphere with radius $r = r_C + 10$ particle diameters around the unweighted center of the cluster. r_C labels the 'cluster radius', which is defined as the minimum radius of an n-sphere which completely contains the whole cluster;

3. let the new particle perform a random walk. It becomes part of the cluster in the instant they touch;

4. go back to step 2.

Creating particles shortly beyond the cluster radius simulates diffusion from infinite distance, simply because at some point, that boundary has to be crossed. Outside this region, the influence of the internal structure of the cluster on the random walk trajectory is assumed to be negligible. To speed things up, particles have to be removed again if they wander to far from the cluster. Usually, a distance of $2r_C$ from the center of the cluster is chosen as a border beyond which particles are removed and a new one is created. This is discussed in more detail in the section on the implementation.

One of Meakin's most important results is that the model can be maximally discretized without changing its properties. Here, maximum discretization means to perform the random walk on a lattice with a lattice spacing equal to the particle diameter. Two particles can be considered to be touching if they are located at nearst-neighbour lattice sites. Moreover, the particular choice of a lattice does not have measurable effects either. Therefore, all simulations can be performed on n-dimensional cubic lattices, which obviously is a huge advantage from the implementation point of view.

Meakin investigated the fractal dimension for embedding dimensions $D = 2$ to 6, mainly via radius of gyration and density-density-correlation, and came to the conclusion that very roughly $d_H = \frac{5}{6}D$. On a qualitative level, the resulting structures are approximately spherical and strongly dendritic, much like patterns of ice on a window. They can be seen in fig. 4.

Number limited aggregation consists of the following steps

1. create a given number of particles on a lattice with periodic boundary conditions;

2. particles which touch are joined to clusters;

3. particles which touch clusters are added to them;

4. cluster which touch are merged;

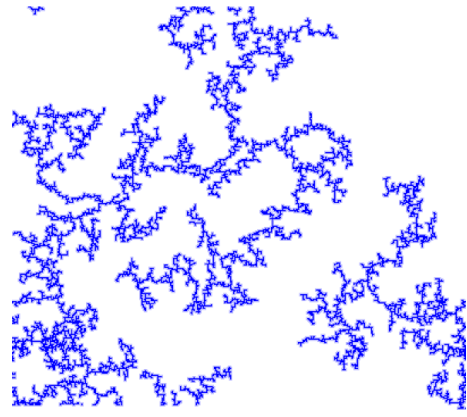5. move all particles and clusters randomly;

Figure 6: **Number limited growth** with 16,000 particles on a 2-dimensional 400x400 lattice.

6. go back to step 2.

Optionally, clusters have a certain probability not to move which is proportional to the number of particles they consist of, thereby simulating 'mass'. This results in net-like structures (fig. 6) with a fractal dimension of about 1.5 in 2-dimensional embedding space. Again, Meakin could not find discretization artifacts caused by the lattice.

Further reading Meakin's models are presented in [3] and [4].

4 The Eden-Meakin model

In contrast to the Meakin-model discussed before, the Eden-Meakin model is inspired by organic rather than anorganic processes. In 1961, M. Eden investigated the following growth model on a lattice:

1. start with a single particle;

2. add a particle at a random lattice site with a probability proportional to the number of its occupied neighbours;

3. go back to step 2.

Unsurprisingly, what forms very much looks like a cell culture or a fungus. Eden asked the question: To what extend do random effects influence the development of biologic systems? Rephrased in a fancy way this is: Why do monozygotic twins have different finger prints? Eden's approach was purely analytic (or combinatoric); he did some simulations, though rather as a last resort and without much enthusiasm. But then again, this was in 1961. When Meakin build his model of adaptive growth on top of Eden's model in 1991, he took a purely numeric approach instead. The boundary of Eden's cell culture had already been discovered to have a fractal dimension, but Meakin was interested in something different: He introduced a scoring system to determine, which cells (particles) are 'important' for

the whole cluster and remove the irrelevant ones. This works as follows:

- whenever a new particle is added, assign it an initial score of 0 and a 'parent', which is one of its neighbours (e.g. the one with the highest score or a random choice),

- the new particle and all of its ancestors (forming a unique chain back to the seed of the cluster) are awarded a score $\Delta S = (1 + l)^{-\eta}$, where l is the length of the chain and $\eta \in \mathbb{R}_+$ is a parameter,

- reduce the score of each particle in the cluster by $\frac{1}{N_m}$, $N_m \in \mathbb{N}$, and remove particles of negative score.

Amazingly, a dendritic backbone of high-scored particles develops (fig. 7), which has a fractal dimension, the exact value of which depends on the cut-off score. There are other properties, however, which might be considered even more interesting.

First of all, the cluster only reaches a finite size. This is conceivable, because the larger it becomes, the more possibilities there are to add new particles and the smaller is the score a new particle is initially assigned. Therefore, new particles will be removed before they become parents to another particle themselves. Meakin gives a crude but easy estimate about the final size depending on η and N_m in his publication.

Secondly, the model is adaptive. The meaning of this can most easily be demonstrated by an example. Fig. 8 shows the number of particles in two clusters grown with different parameters. At the time marked by the vertical bar, the parameter sets are swapped. Soon after, the cluster sizes have swapped as well, thus adapting to the new conditions. This stands in clear contrast to the strictly irreversible growth models discussed so far.

Third, the cluster has a memory. It can also be seen from fig. 8 that the total score of the clusters keeps on growing all the time. Because most of it is alotted to the backbone, it takes an increasingly large number of steps to remove a particle from it, once conditions change. Another way to say this is that as parts of the cluster become older, they become less adaptive as well.

We can therefore safely state that the Eden-Meakin model has not only been *inspired* organically but also exhibits behaviour commonly attributed to living entities.

Further reading   The original model as presented by M. Eden in 1961 is [5]. Meakin's modifications are covered in [6].

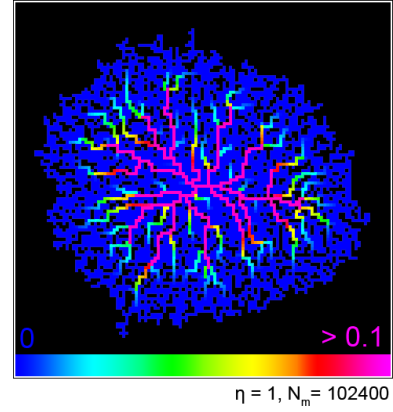# 5 `trivial` − **a fractal growth toolkit**

bla

Figure 7: **The Eden-Meakin model** produces different dendritic structures depending on the cut off-score (colour-coded).
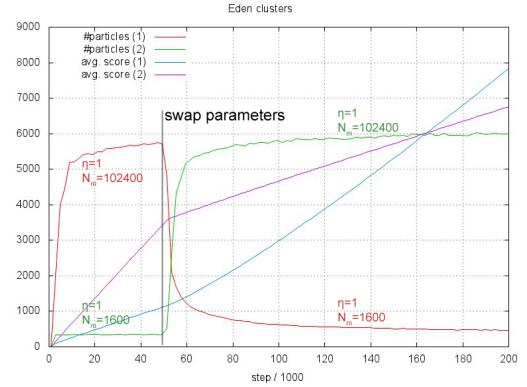

Figure 8: **The adaptive nature of the Eden-Meakin model** can be seen, if two clusters are grown with different parameter sets. If the parameter sets are exchanged, the cluster sizes swap as well. The total score increases monotonically. This renders the cluster more passive towards changed over time and can be interpreted as a memory.
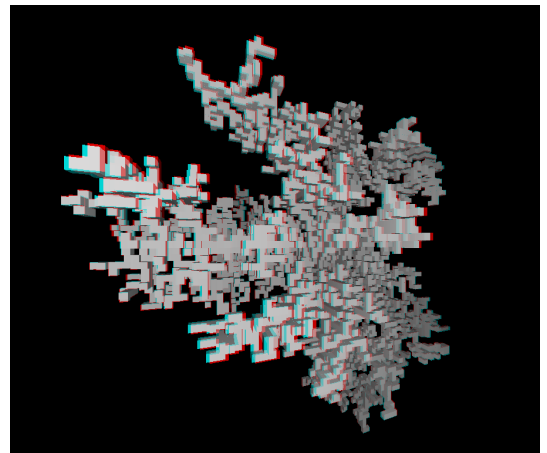

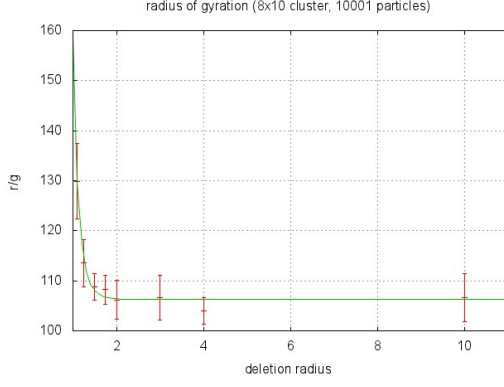Figure 9: **An anaglyph of a 3-d Meakin cluster** of 8,000 particles

Figure 10: **The dependence of the radius of gyration on the 'kill-radius'**, i.e. the distance from the center of the cluster at which diffusion particles are removed. The line is a simple power law and meant for illustrational purposes only.

5.1 Details on the implementation

blubb

5.2 Examples

5.2.1 Higher statistics on DLA clusters

Limited by the computational resources available at his time, Meakin based his conclusions about the properties of diffusion limited aggregation on a handful of clusters. He used those to estimate their fractal dimension given by the radius of gyration and the density-density-correlation in 2- to 6-dimensional space and concluded $d \approx \frac{5}{6}D$ where D is the topological, i.e. here also the embedding, dimension. This is only a heurestic result, e.g. you can easily see that it does not hold for $D = 1$, where $d = 1$ as well. Additionally, there are strong fluctuations in dimensionality even for particle numbers as large as 10,000, so there is some uncertainty left. Being unaware of any theoretical results for DLAs settling that issue, we checked Meakin's results on 196 clusters in two and three dimensions up to 100,000 particles respectively. Care was taken to reduce autocorrelations: all growth processes were uniquely seeded and entered the statistics only once, i.e. a cluster with a final size of 50,000 particles has not been used as a cluster of 20,000 particles at some earlier stage of its development. Applying the radius of gyration method to determine the clusters' dimensionality we obtained $d \pm \Delta d = 1.70 \pm 0.02$ for $D = 2$ and $d \pm \Delta d = 2.50 \pm 0.03$ for $D = 3$ in perfect accordance to Meakin's results. Determining the dimension by the density-density-correlation can not be automated as easily, as range in which it scales in a fractal fashion is less obvious. Nevertheless, it was checked by hand in about ten cases for each dimensionality giving no significant deviation from the values obtained via the radius of gyration, as already stated by Meakin. Some seven-dimensional cluster have been grown, but since dense storage soon becomes unfeasible, the main

speed advantage of our implementation is lost. While the results required are not statistically significant and may suffer from finite-size artifacts, they still do not rule out the 'law of $\frac{5}{6}$'. The large fluctuations in fractal dimensionality Meakin observed at 10,000 particles, unfortunately do not decrease when going to clusters of 100,000 particles. It seems likely that this is not due to a finite-size effect, but rather an artifact inherent in the methods of measurement (radius of gyration and density-density-correlation). Remember – the structures are *not* true fractals in the mathematical sense. Interestingly, our data cannot rule out that the fractal dimension d is in fact a random variable with expectation value of about $\frac{5}{6}D$ even in the limit $N \to \infty$.

5.2.2 Cluster diffusion

Instead of growing cluster from diffusing particles, one could as well grow a cluster from diffusing clusters. One could take several of cluster-diffusion-grown clusters to grow a higher-order cluster still. One could go on like this. Where is the point? Earlier, we have seen that the fractal dimension of a structure is connected to its scaling behaviour. If we now compose clusters from smaller clusters of a given size, we impose an additional scaling behaviour onto the system. In general, the overall dimension will result from the original, statistically induced scaling behaviour and this new one in some complicated way we do not want do investigate further here. Instead, we just grow 70 clusters by the following recipie:

- build a cluster of n particles according to Meakin's original model

- take n of these cluster to assemble a new cluster

- go on like this until the currently assembled cluster has more than 10,000 particles

Changing n from 5 to 50 in increments of 5, dimensions between 1.7 and 1.3 can be created. Moreover $dim(n)$ appears to be monotonic. Other than just the final size, the average coordination number remains unchanged. So we can change the fractal dimension while maintaining other properties. Most important, we could stay with our original growth principle of Brownian motion and adhesion, which, after all, has been physically motivated.

5.2.3 Dendrites

Dendrites, as commonly found in mineralogy, are fern-like structures on rocks. They are easily confused with plant fossils (*dendron* even is Greek for *tree*) although they are entirely anorganic in origin. Usually, they are formed by sedimenting metal oxides. In fact, this is the very process which inspired Witten and Sander in the first place. The important difference to the Meakin clusters we have discussed so far is that in this case there is a preferred orientation. In nature, it is induced by the flow of water carrying the sediments (and maybe
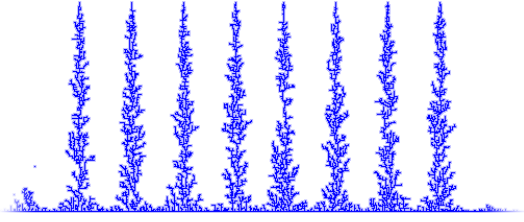
Figure 11: **Dendrites as found in mineralogy** develop
if the random walk has a preferred direction.
This example was grown with a high flow rate,
which leads to pine-shaped structures.

to some extend also by the geometry of the rock, the
seed in the language of the growth model). The Meakin
model can be adapted to describe this process in the
following straight-forward way:

- create particles not uniformly on the n-sphere, but
  with a fixed x_0 coordinate. All other coordinates
  can be distributed uniformly.

- increase the probability for stepping along the 0-
  direction in the random walk. The more probable
  it becomes, the stronger is the 'flow'.

- seed with a solid plane of particles perpendicular
  to the 0-direction not too close to where particles
  are created.

Depending on the strength of the 'flow', you can grow
pine-like (strong) or bush-like (light) dendrites. The
distribution of the other coordinates upon creation
does not seem to have much influence as long as it is
continuous. Otherwise, several dendrites can be grown
simultaneously. The influence of the seeding geometry
has not been studied systematically, but we conjecture
that apart from determining the initial position of the
dendrite, it has no observable effect, because as with
the original Meakin model new particles are nearly ex-
clusively added to the outer regions of the dendrite.
Other interesting modifications would be the introduc-
tion of vortices in the 'flow' or sticking probabilities.
Some dendrites grown with this model at a high flow
rate are displayed in fig. 11.

5.2.4 A toy model: tubes

The idea of preferred directions and boundary con-
ditions can also lead to quite a different set-up: By
the techniques developed for dendrites, you can also
model particles flowing through a tube. In this case,
an n-dimensional tube shall be a hollow n-dimensional
cuboid the 0-extend of which is much larger than all
other. The direction of flow is chosen to be the 0-
direction. Depending on the size of the tube, the flow
rate and the incoming particle density, dendrites grow-
ing from the boundaries will at some point block the
tube entirely. This is of course hardly a useful model to
describe real world phenoma or even useful in engineer-
ing. On can, however, learn some things about the be-
haviour of the Meakin model if many free particles are

involved. One non-obvious result is that the tube will
stay clear of obstacles for a longer time if more parti-
cles are injected. This is because the higher the density,
the more likely it is for middle-sized clusters to form
during transport. According to the concept of mass in-
troduced by Meakin, the probability for those cluster
to touch the tube before leaving it, become smaller.

5.2.5 Dendrites and crystals

5.2.6 Growing towards the sun

This final example is based on the Eden-Meakin model.
Again, the basic idea is to introduce a preferred direc-
tion. Additionally, as we know of the adaptive nature
of this particular model, we shall try to change the pre-
ferred direction over time. For the sake of concreteness
let us attempt to model a plant growing towards to
sun. We will try to do so by only changing the scoring
algorithm. The creation of new particles shall still be
possible at each nearest-neighbour site. That way, we
want to emulate to process of natural selection on a mi-
crobiologic level. In altering the scoring procedure we
have to take care not to create a situation in which a
particle has a higher score than any of its ancestors be-
cause we do not want the cluster to be split. This leaves
only a limited number of possibilities one of which is to
introduce an additional factor for the chain score, such
that $\Delta S = \gamma(\vec{x})/(1+l)^\eta$, where $\vec{x}$ is the position of the
new particle. Since our new scoring should somehow
reflect the influence of the sun, seed the cluster at the
origin and choose

$$\gamma(\vec{x}) = \begin{cases} \frac{\vec{x} \cdot \hat{s}}{|x|} & \text{if } x_0 > 0 \\ 0 & \text{if } x_0 \leq 0, \end{cases}$$

where $\hat{s}$ is a unit-vector pointing into the direction
of the sun. This reflects the fact that sunlight reaches
the earth essentially as parallel rays. The 0-direction
is interpreted as the 'up-direction'. It is known that
the growth of plants is also crucially influenced by the
direction of gravity, but we shall neglect that here and
forbid growth downwards manually.

Fig. 12 depicts in 2 dimensions the extreme case,
where the sun has been situated at $(0, 1)$ during the
first 10,000 steps and at $(0, -1)$ afterwards. It can
clearly be seen, how the plant follows the sun. The
'old' branches gradually degenerate because the sun
never returns.

In a more sophisticated model, where the sun moves
continuously, one encounters the following behaviour:
depending on the rate of growth determined by the
parameters η and N_m, the plant follows the sun for
a few 'days'. At some point however, the memory ef-
fect described before inhibits a timely adaptation and
the plant only 'sees' the average illumination becoming
symmetric and time-invariant in the upper half-space.
As with the original Eden-Meakin-model, a stationary
phase is reached. There also is another possible out-
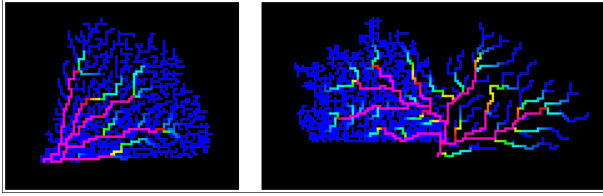come: if the 'night' is too long, the plant may 'die', i.e.

Figure 12: **A tree** is what one might see in this particularly scored Eden model. During the first 10,000 steps, the direction of the sun is $\hat{s} = (0, 1)$ and $\hat{s} = (0, -1)$ later (text).

a particles are removed. Again, due to the memory effect this will only happen during the first night or not at all.

Instead of only talking about living and dying plants, one could now implement some kind of scoring for a plant as a whole, describing its adaptivity with respect to environmental constraints. Together with an enlarged parameter set for the clusters, being determined by some genetic algorithm based on this scoring, one would obtain a toy-model of evolution.

While this is in a direct line with Eden's original aims, we are now admittedly abandoning the realm of fractal growth – and so the scope of this text.

Further reading Again, Meakin's results on DLA are summarized in [3]. The Eden-Meakin model is based on [5] and [6]. A development version of `trivial` can freely be obtained from `github.com` as the repository `comp_phys` of *filmor*'s. Please note that at the time of writing, it is still heavily under construction and some parts may not be functional or only implemented in a 'hackish' way.

6 Conclusion

Even 30 years after its advent, fractal growth still is an exciting topic and not completely unterstood. There are many links to other disciplines besides mathematics and physics, be it biology, engineering, computer science or art. In fact, there is one additional striking feature about fractals in general: They are considered beautiful. We have seen that there is a deep connection between dimensionality and scaling laws, i.e. self-similarity. The question whether the perceived beauty of fractals is due to this special kind of symmetry (the human brain seeming to react to symmetry on a very fundamental level), or the perceived beauty of symmetries is due to fractals being a phenomenon common in nature remains for the reader to solve or maybe psychology.

References

[1] T. A. Witten and L. M. Sander, Phys. Rev. Lett. 47, 1400 (1981)

[2] H. E. Stanley, J. Phys. A 10, L211 (1977)

[3] P. Meakin, Phys. Rev. A 27, 1495 (1983)

[4] P. Meakin, Phys. Rev. Lett. 51, 13 (1983)

[5] M. Eden, Proc. 4th Berkeley Symp. on Mathematics, Statistics and Probability, vol. 4, F. Neyman, ed., (1961)

[6] P. Meakin, Physica A, 179 (1991)

[7] The Fractal Geometry of Nature, Benoit B. Mandelbrot