Methods Lesson 5

(Questions will be numbered like ①, ②, ③ with circles around them in order of the original document)

1.

$a = 3t(2-t)$

a) $\int 3t(2-t)\,dt = \int 6t - 3t^2\,dt$

$= 3t^2 - t^3 + C$

$\because$ Starts from rest

$\therefore$ $C = 0$

$v = 3t^2 - t^3$

b)

$x(t) = \int v\,dt$

$= t^3 - \frac{t^4}{4} + c$

Not given initial position

$\therefore$ Must use integration

$\int_0^4 v(t)\,dt =$ displacement after 4 seconds

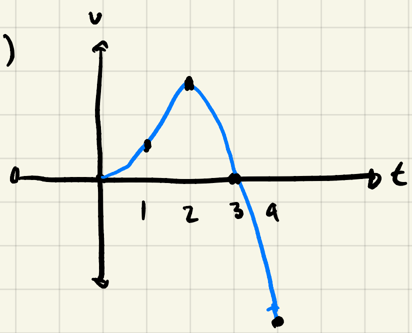
$= \int_0^4 3t^2 - t^3\,dt$

$= \left[t^3 - \frac{t^4}{4}\right]_0^4$

$= 4^3 - \frac{4^4}{4}$

$= 0$

c)



$v = 3t^2 - t^3$

x-int

$= t^2(3-t)$

$\therefore (3,0), (0,0)$

x-int

$v(1) = 3-1$

$= 2$

$v(2) = 12-8$

$= 4$

$v(3) = 0$

$v(4) = -16$

d) Total distance travelled

$$\int_0^3 v(t)dt + \left|\int_3^4 v(t)dt\right|$$

gives distance from 0-3

gives distance from 3-4 (in opposite direction)

$$\int_0^3 v(t)dt = \int_0^3 3t^2 - t^3 dt$$

$$= \left[t^3 - \frac{t^4}{4}\right]_0^3$$

$$= 27 - \frac{81}{4}$$

$$= 27 - (20 + \tfrac{1}{4})$$

$$= 6\tfrac{3}{4}$$

$$\int_3^4 3t^2 - t^3 dt = \left[t^3 - \frac{t^4}{4}\right]_3^4$$

$$= 0 - 6\tfrac{3}{4}$$

$\therefore$

total distance travelled $= 6\tfrac{3}{4} + 6\tfrac{3}{4}$

$$= 13\tfrac{2}{4}$$

$$= 13.5m$$

(2)

1a) $v = e^{-0.5t} - 0.5$

a) $a(t) = -0.5e^{-0.5t}$

b) $x(t) = \int v(t)\,dt$

$$= \frac{-e^{-0.5t}}{0.5} - 0.5t$$

$$= -2e^{-0.5t} - 0.5t$$

($c=0$ as $x=0$ when $t=0$)

c) displacement after 4 seconds $= \int_0^4 v(t)dt$

$$= \left[-2e^{-0.5t} - 0.5t\right]_0^4$$

$$= \left(-2e^{-0.5\times4} - 0.5\times4\right) - \left(-2e^{-0.5\times0} - 0.5\times0\right)$$

$$= (-2e^{-2} - 2) - (-2)$$

$$= \frac{-2}{e^2} \approx -0.2707m$$

③

15) $f(x) = 25 - 0.02x^2$

↑

cross section

$\therefore \int_{-20}^{20} 25 - 0.02x^2 \, dx =$ cross section area of roof

$\approx 893.333 m^2$

$f(20) = 25 - 0.02 \times (20)^2$

$= 17$

$f(-20) = 25 - 0.02 \times (-20)^2$

$= 17$

$\therefore$ Cross section under roof $= 17 \times 40$

$= 680 m^2$

$(680 + 893.333) \times 80 =$ total volume

$\approx 125867 m^3$

④

Q17) ant $a(t) = 2$

Snail $v(t) = 1.4 \ln(1 + t^2)$ $\quad 0 \le t \le 15$

$\therefore$

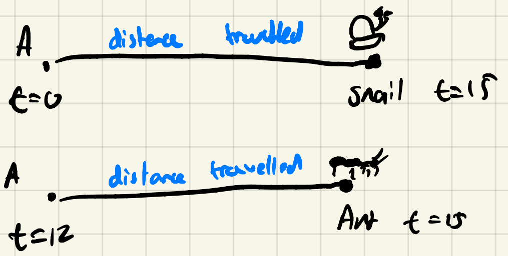
ant $v(t) = 2t + c$

Snail travels from point A at $t=0$

Ant passes point A at $t=12$

Ant passes snail at $t=15$

$\therefore$ Snail displacement = Ant displacement at $t=15$



$$\therefore \int_0^{15} \text{Snail velocity } dt = \int_{12}^{15} \text{ant velocity } dt$$

$\int_0^{15} 1.4 \ln(1+t^2) dt = 76.0431 m$ (using calculator)

$$\therefore \int_{12}^{15} 2t + c \, dt = 76.0431 m$$

using calculator

nSolve$\left(76.0431 = \int_{12}^{15} 2t + c \, dt, c\right)$

$c = -1.65231$

$\therefore$ Ant $v(t) = 2t - 1.65231$

Ant $v(12) = 24 - 1.65231$

$= 22.35$ cm/min

(velocity at $t=12$)
(Point A)

Exam Preperation

5

1)

a) $\ln x + \ln 25 = \ln x^3$

$\ln 25x = \ln x^3$

$\therefore 25x = x^3$

$25 = x^2$

$x = 5$

$(x > 0 \therefore x \neq -5)$

b) $3 \times 3^{2x} + 3^x - 10 = 0$

let $3^x = a$

$\therefore$

$3a^2 + a - 10 = 0$

$\underline{6} + \underline{-5} = 1$

$\underline{6} \times \underline{-5} = -30$

$3a^2 + 6a - 5a - 10 = 0$

$3a(a+2) - 5(a+2) = 0$

$(3a-5)(a+2) = 0$

$a = -2, \frac{5}{3}$

$3^x > 0$

$\therefore a \neq -2$

$3^x = \frac{5}{3}$

$x = \log_3(\frac{5}{3})$

$= \log_3 5 - \log_3 3$

$= \log_3 5 - 1$

c) $\log_{16}(3x-1) = \log_4(3x) + \log_4(\frac{1}{2})$

$\frac{\log_4(3x-1)}{\log_4 16} = \log_4(3x \times \frac{1}{2})$

$\frac{\log_4(3x-1)}{2} = \log_4(\frac{3x}{2})$

$\log_4(3x-1)^{\frac{1}{2}} = \log_4(\frac{3x}{2})$

$\sqrt{3x-1} = \frac{3x}{2}$

$3x - 1 = \frac{9x^2}{4}$

$12x - 4 = 9x^2$

$9x^2 - 12x + 4 = 0$

$\underline{-6} + \underline{-6} = -12$

$\underline{-6} \times \underline{-6} = 36$

$9x^2 - 6x - 6x + 4 = 0$

$3x(3x-2) - 2(3x-2) = 0$

$(3x-2)^2 = 0$

$3x - 2 = 0$

$x = \frac{2}{3}$

6

5)

a) $\int_0^1 \frac{2}{5x+5}\,dx$

$\int_0^1 \frac{2}{5x+5}\,dx = \left[\frac{2}{5}\ln(5x+5)\right]_0^1$

$= \frac{2}{5}\ln(5+5) - \frac{2}{5}\ln(5)$

$= \frac{2}{5}(\ln(10) - \ln(5))$

$= \frac{2}{5}\ln 2$

b) $\int_1^4 2x + \frac{2}{x}\,dx$

$\int_1^4 2x + \frac{2}{x}\,dx = \left[x^2 + 2\ln x\right]_1^4$

$= (4^2 + 2\ln 4) - (1^2 + 2\ln 1)$

$= (16 + 2\ln 4) - (1)$

$= 15 + 2\ln 4$

⑦

4) $\log_2(7x^2+8x+3) = \log_2(x^2)+1$

$= \log_2(x^2) + \log_2 2$

$= \log_2(2x^2)$

$7x^2+8x+3 = 2x^2$

$5x^2+8x+3=0$

$\underline{5} \times \underline{3} = 15$

$\underline{5} + \underline{3} = 8$

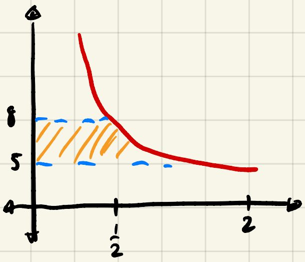
$5x^2+5x+3x+3=0$

$5x(x+1)+3(x+1)=0$

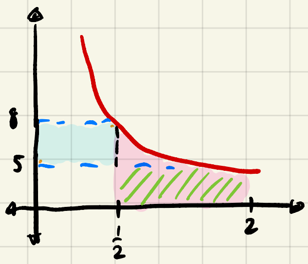
$(5x+3)(x+1)=0$

$x=-1$ or $x=-\frac{3}{5}$

⑧



$f(x) = 4+\frac{2}{x}$



Shaded area $= (8-5)\times\frac{1}{2} + \left(\int_{\frac{1}{2}}^{2} f(x)\,dx - 5\times\frac{3}{2}\right)$

$= \frac{3}{2} + \left[4x+2\ln x\right]_{\frac{1}{2}}^{2} - \frac{15}{2}$

$= \frac{3}{2} + \left((8+2\ln 2)-(2+2\ln\frac{1}{2})\right) - \frac{15}{2}$

$= \frac{3}{2} + (6+2\ln 4) - \frac{15}{2}$

$= -6+6+2\ln 4$

$= 2\ln 4$ units^2

⑨

4) $x(t) = e^t\sin(t),\ 0\le t\le 2\pi$

$u=e^t \quad v=\sin(t)$

$u'=e^t \quad v'=\cos(t)$

$v(t) = e^t\cos(t) + e^t\sin(t)$

$= e^t(\cos(t)+\sin(t))$

at rest when $v(t)=0$

$\therefore \cos(t)+\sin(t)=0$

$\cos(t) = -\sin(t)$

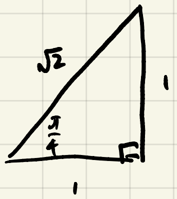
$1 = -\dfrac{\sin(t)}{\cos(t)}$

$= -\tan(t)$

$\tan(t) = -1$

$\tan = -ve$ in $Q2, Q4$

$\tan = 1$ at $\frac{\pi}{4}$



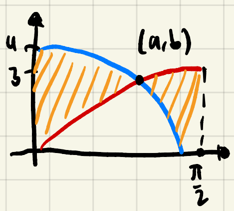
$Q_2 = \frac{3\pi}{4}$

$Q_4 = \frac{7\pi}{4}$

at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$ particle is at rest

(10)

1) $y = 3\sin x$ $\quad y = 4\cos(x)$



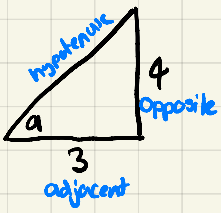
a) at point a, $3\sin(a) = 4\cos(a)$

i) $\therefore$ $3\sin(a) = 4\cos(a)$

$\dfrac{\sin(a)}{\cos(a)} = \dfrac{4}{3}$

$\tan(a) = \dfrac{4}{3}$

ii) $\sin(a) = \dfrac{O}{H}$



$\therefore$ $3^2 + 4^2 = H^2$

$H = 5$

$\sin(a) = \frac{4}{5}$

iii) $\cos(a) = \frac{3}{5}$

b) Area $= \displaystyle\int_0^a 4\cos(x) - 3\sin(x)\,dx + \int_a^{\frac{\pi}{2}} 3\sin(x) - 4\cos(x)\,dx$

$= \Big[4\sin(x) + 3\cos(x)\Big]_0^a + \Big[-3\cos(x) - 4\sin(x)\Big]_a^{\frac{\pi}{2}}$

$= \Big(\big(4\sin(a) + 3\cos(a)\big) - \big(4\sin(0) + 3\cos(0)\big)\Big) + \Big(\big(-3\cos(\tfrac{\pi}{2}) - 4\sin(\tfrac{\pi}{2})\big) - \big(-3\cos(a) - 4\sin(a)\big)\Big)$

$= \Big(\big(\tfrac{16}{5} + \tfrac{9}{5}\big) - (3)\Big) + \Big((-4) - \big(-\tfrac{9}{5} - \tfrac{16}{5}\big)\Big)$

$= \frac{25}{5} - 3 - 4 + \frac{25}{5}$

$= 5 - 3 - 4 + 5$

$= 3$

(11)

Q5)

$f(x) = \frac{\ln(2x)}{x}$

$u = \ln(2x) \quad v = x$

$u' = \frac{1}{x} \quad v' = 1$

$f'(x) = \frac{vu' - uv'}{v^2}$

$= \frac{1 - \ln(2x)}{x^2}$

Stationary points at

$f'(x) = 0$

$0 = \frac{1 - \ln(2x)}{x^2}$

$= 1 - \ln(2x)$

$1 = \ln(2x)$

$\ln e = 1$

$\ln e = \ln 2x$

$e = 2x$

$x = \frac{e}{2}$

$f(\frac{e}{2}) = \frac{\ln(2 \times \frac{e}{2})}{\frac{e}{2}}$

$= \frac{2 \ln e}{e}$

$= \frac{2}{e}$

Stationary point $= (\frac{e}{2}, \frac{2}{e})$

(12)

Q6)

$P(t) = \frac{100}{1 + e^{b-t}}$

$P(3) = 50$

$50 = \frac{100}{1 + e^{b-3}}$

$1 = \frac{2}{1 + e^{b-3}}$

$1 + e^{b-3} = 2$

$e^{b-3} = 1$

$b - 3 = 0$

$b = 3$

$\therefore P(t) = \frac{100}{1 + e^{3-t}}$

$= 100(1 + e^{3-t})^{-1}$

$P'(t) = 100(1 + e^{3-t})^{-2} \times -1 \times -1$

$= \frac{100}{(1 + e^{3-t})^2}$

$P'(3) = \frac{100}{(1 + e^0)^2}$

$= \frac{100}{4}$

$= 25$ students / day

(13)

$n = 92e^{-0.01p}$

16)

revenue = selling price × number sold

$= p \times 92e^{-0.01p}$

$R(p) = 92e^{-0.01p}p$

$\therefore$ $u = p$ $v = 92e^{-0.01p}$

$u' = 1$ $v' = -0.92e^{-0.01p}$

$$\frac{dR}{dp} = -0.92e^{-0.01p}p + 92e^{-0.01p}$$

maximum revenue at $\frac{dR}{dp} = 0$ using graphics

$\longrightarrow p = 100$

$\therefore$ Should sell at $p = 100$

$n = 92e^{-0.01 \times 100}$

$= 33.84$

$\approx$ 34 handbags / week

(14)

11)

find x when $H = 1$ Km

$H(x) = 1.8e^{-x} + 0.43$

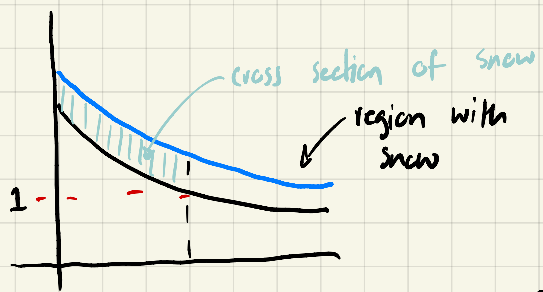
$1 = 1.8e^{-x} + 0.43$

using GC

$x = 1.1499$ Km

2m of snow

$\therefore$



Function with snow added

$$H_2(x) = \underbrace{1.8e^{-x} + 0.43}_{\text{original}} + \underbrace{0.002}_{\text{2m snow}}$$

Area between curve

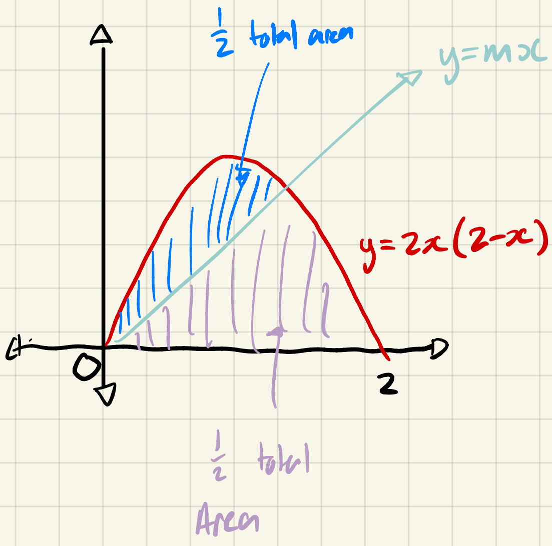
$$\int_0^{1.1499} H_2 - H \, dx = 0.0023 \text{Km}^2$$

Volume = Area × width (300m wide)

$= 0.0023 \times 0.3$

$= 0.00069 \text{Km}^3$

(15) Q10)



$$\int_0^2 2x(2-x)\,dx = \int_0^2 4x - 2x^2\,dx$$

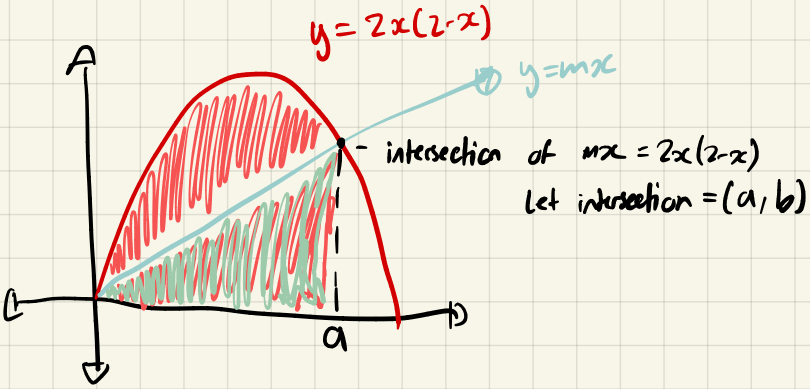
$$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$$

$$= 8 - \frac{16}{3}$$

$$= \frac{8}{3} \text{ units}^2$$

$$\frac{1}{2}\text{ Area} = \frac{8}{6} \text{ units}^2$$

$$= \frac{4}{3} \text{ units}^2$$



$$\int_0^a 2x(2-x) - \int_0^a mx = \frac{4}{3}$$

finding a

$$mx = 2x(2-x)$$

$$ma = 2a(2-a)$$

$$m = 2(2-a)$$

$$= 4-2a$$

$$2a = 4-m$$

$$a = \frac{4-m}{2}$$

$$\therefore \int_0^{\frac{4-m}{2}} 2x(2-x)\,dx - \int_0^{\frac{4-m}{2}} mx\,dx = \frac{4}{3}$$

$$= \int_0^{\frac{4-m}{2}} 2x(2-x) - mx\,dx = \frac{4}{3}$$

$$= \left[2x^2 - \frac{2x^3}{3} - \frac{mx^2}{2}\right]_0^{\frac{4-m}{2}}$$

$$= 2\left(\frac{4-m}{2}\right)^2 - \frac{2\left(\frac{4-m}{2}\right)^3}{3} - \frac{m\left(\frac{4-m}{2}\right)^2}{2} = \frac{4}{3}$$

using nSolve

$$m \approx 0.825198$$

16

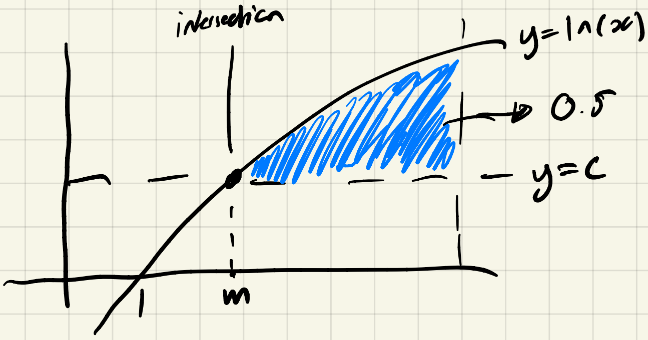
Q20)

a) $y = x\ln x - x$

$u = x \quad v = \ln x$
$u' = 1 \quad v' = \frac{1}{x}$

$y' = 1 + \ln x - 1$
$= \ln x$

b)



let intersection $= m$

$c = \ln m$
$m = e^c$
at e^c $\ln x = c$

$\int_1^e \ln(x)\,dx = 1$

$\therefore \frac{1}{2}$ area $= 0.5$

$\therefore \int_{e^c}^{e} \ln(x) - \int_{e^c}^{e} c = 0.5$

$= \left[x\ln x - x - cx\right]_{e^c}^{e} = 0.5$

$= (e\ln e - e - ce) - (e^c\ln e^c - e^c - ce^c)$

$= (-ce) - (ce^c - e^c - ce^c)$

$= -ce + e^c = 0.5$

using nSolve

$c \approx 0.325$

Q9a) $f(x) = 2e^x - 3x$
$g(x) = -3e^{-x} + x^2 + 3$

using gc, intersection

at $x = 0.662273, 1.54731$

$g(x)$ on top $\therefore$

Area $= \int_{0.662273}^{1.54731} g(x) - f(x) = 0.298659$