Methods Lesson 7

1.

a) $y = x^4 - 5x^3 + x^2 - 9$

$\frac{dy}{dx} = 4x^3 - 15x^2 + 2x$

$\frac{d^2y}{dx^2} = 12x^2 - 30x + 2$

$x = 0$, $\frac{d^2y}{dx^2} = 2$

$\therefore$ Concave up

b) $y = x^3 - 4x^2$

$\frac{dy}{dx} = 3x^2 - 8x$

$\frac{d^2y}{dx^2} = 6x - 8$

$x = 0$, $\frac{d^2y}{dx^2} = -8$

$\therefore$ concave down

c) $y = 4 - x^2$

$\frac{dy}{dx} = -2x$

$\frac{d^2y}{dx^2} = -2$

$x = 0$, $\frac{d^2y}{dx^2} = -2$

$\therefore$ concave down

2.

7.

a) $f(x) = 4\ln(2x-3)$

$f'(x) = \frac{8}{2x-3}$

$= 8(2x-3)^{-1}$

$f''(x) = -8 \times 2(2x-3)^{-2}$

$= -\frac{16}{(2x-3)^2}$

$f''(3) = -\frac{16}{(6-3)^2}$

$= -\frac{16}{9}$

Concave down

b) $f(x) = e^{x^2}$

$f'(x) = 2xe^{x^2}$

$u = 2x \qquad v = e^{x^2}$

$u' = 2 \qquad v' = 2xe^{x^2}$

$f''(x) = 4x^2e^{x^2} + 2e^{x^2}$

$= 2e^{x^2}(2x^2+1)$

$f''(1) = 2e(2+1)$

$= 6e$

concave up

(3) 14) Second derivative test
Concavity

a) $y = e^{x} \sin(x)$

$u = e^{x} \quad v = \sin(x)$

$u' = e^{x} \quad v' = \cos(x)$

$$\frac{dy}{dx} = e^{x}\cos(x) + e^{x}\sin(x)$$

$$= e^{x}(\sin(x) + \cos(x))$$

Stationary point when $\frac{dy}{dx} = 0$

$$0 = e^{x}(\sin(x) + \cos(x))$$

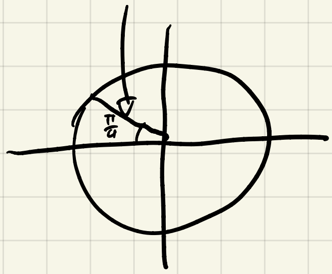
$$= \sin(x) + \cos(x)$$

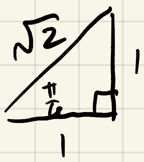
$$-\sin(x) = \cos(x)$$

$$1 = -\frac{\sin(x)}{\cos(x)}$$

$$= -\tan(x)$$

tan is negative in Q2





$\tan\theta = 1$
at $\frac{\pi}{4}$

$\therefore x = \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

$\therefore \frac{3\pi}{4}$ is a stationary point

Test by subbing $\frac{3\pi}{4}$

$$\frac{dy}{dx} = e^{\frac{3\pi}{4}}\left(\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)\right)$$

$\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$\therefore \frac{dy}{dx} = e^{\frac{3\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 0$$

$\therefore$ Stationary point

Concavity

$$\frac{dy}{dx} = e^x(\sin(x) + \cos(x))$$

$u = e^x \qquad v = \sin(x) + \cos(x)$

$u' = e^x \qquad v' = \cos(x) - \sin(x)$

$$\frac{d^2y}{dx^2} = e^x(\cos(x) - \sin(x)) + e^x(\sin(x) + \cos(x))$$

$$= e^x(\cos(x) - \sin(x) + \sin(x) + \cos(x))$$

$$= e^x(2\cos(x))$$

$$= 2e^x\cos(x)$$

at $x = \frac{3\pi}{4}$

$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\therefore \frac{d^2y}{dx^2} = 2 \times e^{\frac{3\pi}{4}} \times -\frac{1}{\sqrt{2}}$

$= -ve$

$\therefore$ concave down

$\therefore$ Maximum

4 a) $y = x^3 + 2x^2 - 3x + 1$

$\frac{dy}{dx} = 3x^2 + 4x - 3$

$\frac{d^2y}{dx^2} = 6x + 4$

concave up when $0 > 6x + 4$

$-4 > 6x$

$-\frac{4}{6} > x$

$x < -\frac{2}{3}$

concave down when $0 < 6x + 4$

$-4 < 6x$

$-\frac{2}{3} < x$

$x > -\frac{2}{3}$

point of inflection when $f''(x) = 0$

$0 = 6x + 4$

$x = -\frac{2}{3}$ point of inflection

5 $f(x) = \frac{1}{2}x^2 - 3x^4$

a) $f'(x) = x - 12x^3$

$f''(x) = 1 - 36x^2$

$0 = 1 - 36x^2$

$36x^2 = 1$

$x^2 = \frac{1}{36}$

$\therefore x = \pm\sqrt{\frac{1}{36}}$

$= \pm \frac{1}{6}$

$x = \frac{1}{6}$, $x = -\frac{1}{6}$

points of inflection

b) concave down

$f''(x) < 0$

$0 > 1 - 36x^2$

$36x^2 > 1$

$x^2 > \frac{1}{36}$

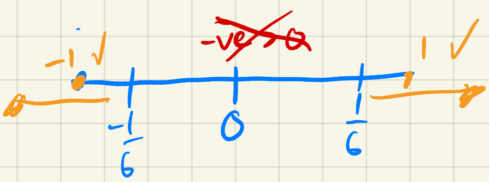
$x > \frac{1}{6}$, $x < -\frac{1}{6}$

concave down

Inequality is flipped for the negative sqrt

$x^2 - \frac{1}{36} > 0$

$(x - \frac{1}{6})(x + \frac{1}{6}) > 0$



at $x = 0$, $(-\frac{1}{6}) \times (\frac{1}{6}) = -ve$ $\therefore$ x not between $-\frac{1}{6}, \frac{1}{6}$

at $x = 1$, $(1 - \frac{1}{6})(1 + \frac{1}{6}) = +ve$ $\therefore$ $x > \frac{1}{6}$

at $x = -1$, $(-1 - \frac{1}{6})(-1 + \frac{1}{6}) = +ve$ $\therefore$ $x < -\frac{1}{6}$

⑥ $f(x) = 2x^3 - kx^2 + 3x$

$f'(x) = 6x^2 - 2kx + 3$

$f''(x) = 12x - 2k$

$f''(3) = 0$

$0 = 12 \times 3 - 2k$

$2k = 36$

$k = 18$

$x^2 > \frac{1}{36}$

$x < -\sqrt{\frac{1}{36}}$, $x > \sqrt{\frac{1}{36}}$

⑦ $x(t) = 2\cos(3t-1) + 3$

a) max displacement: 5

min displacement: 1

b) $x'(t) = v(t)$

$v(t) = -6\sin(3t-1)$

$0 = -6\sin(3t-1)$

$= \sin(3t-1)$

$\sin\theta = 0, \pi, 2\pi$

when $\theta = 0$

$\therefore \quad 3t - 1 = 0$

$t = \frac{1}{3}$ seconds

c) at rest when

$v(t) = 0$

$\sin(3t-1) = 0$

$\theta = \pi$

$3t - 1 = \pi$

$t = \frac{\pi+1}{3}$ seconds

$\therefore$ elapsed time

$= \frac{\pi+1}{3} - \frac{1}{3}$

$= \frac{\pi}{3} + \frac{1}{3} - \frac{1}{3}$

$= \frac{\pi}{3}$ seconds

d) $x''(t) = -18\cos(3t-1)$

8 Optimisation

let x, y = numbers

$x + y = 3$

$y = 32 - x$

$x + 32 - x = 32$

let $P(x)$ = Product

$P(x) = xy$

$= x(32 - x)$

$= 32x - x^2$

→ To find max, need to find stat. point

$P'(x) = 32 - 2x$

$0 = 32 - 2x$

$2x = 32$

$x = 16$

$\therefore x = 16, \; y = 32 - 16$

$= 16$

Confirm it is a max

$P''(x) = -2$

$P''(16) = -2$

$\therefore$ concave down

$\therefore$ maximum

9 $x + y = 8$

$y = 8 - x$

$S(x) = x^3 + y^2$

$= x^3 + (8-x)^2$

$= x^3 + 64 - 16x + x^2$

$S'(x) = 3x^2 - 16 + 2x$

$= 3x^2 - 16 + 2x$

$0 = 3x^2 + 2x - 16$

$= 3x^2 - 6x + 8x - 16$

$= 3x(x-2) + 8(x-2)$

$= (3x+8)(x-2)$

$x = -\frac{8}{3}, \; x = 2$

x is the

$\therefore x \neq -\frac{8}{3}$

$x = 2$

$y = 6$

Check if minimum

$S''(x) = 6x + 2$

$S''(2) = 12 + 2$

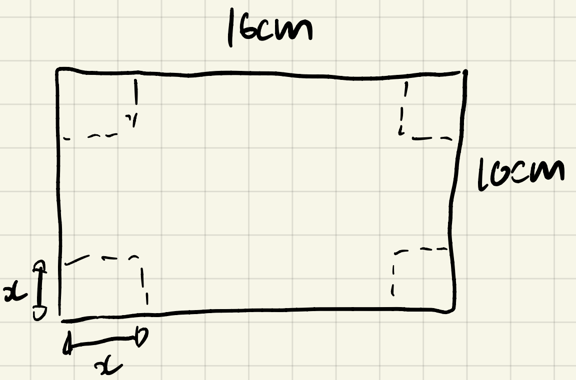
$= 14$

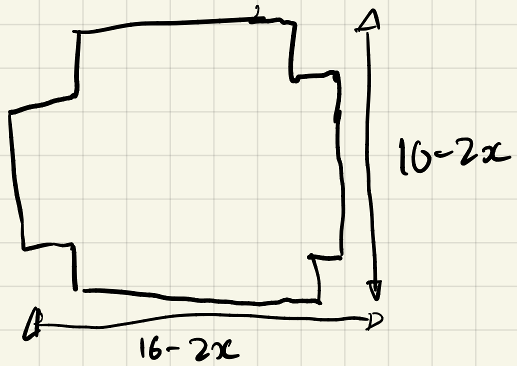
$S''(2) > 0$

$\therefore$ concave up

(minimum)

(10)





height $= x$

a) $V(x) = L \times W \times H$

$= (10-2x)(16-2x)x$

$= (160 - 20x - 32x + 4x^2)x$

$= (4x^2 - 52x + 160)x$

$= 4x^3 - 52x^2 + 160x$

b) max at stat pt.

$V(x) = 4x^3 - 52x^2 + 160x$

$V'(x) = 12x^2 - 104x + 160$

$0 = 12x^2 - 104x + 160$

$= 3x^2 - 26x + 40$

$= 3x^2 - 6x - 20x + 40$

$= 3x(x-2) - 20(x-2)$

$= (3x-20)(x-2)$

$x = \frac{20}{3},\ x = 2$

$24 \times 20 = 480$

$\frac{480}{3} = 160$

sub into $V''(x)$

$V''(x) = 24x - 104$

$V''\left(\frac{20}{3}\right) = \frac{24 \times 20}{3} - 104$

$= 160 - 104$

> 0

$\therefore$ minimum

$V''(2) = 48 - 104$

$= -ve$

$\therefore$ maximum

$\therefore$ greatest volume at $x = 2$

$10 - 2 \times 2 = 6$

$16 - 2 \times 2 = 12$

$6 \times 12 \times 2$: dimensions

$6 \times 12 \times 2 = 144\text{cm}^3$ volume