UAI Formula Sheet

Handy Theorems

Chain Rule: $P(X_1, X_2) = P(X_1|X_2)P(X_2)$

Marginalization: $P(X_1) = \sum_{x \in X_2} P(X_1, X_2 = x)$

Bayes Rule: $P(X_1|X_2) = \frac{P(X_2|X_1)P(X_1)}{P(X_2)}$

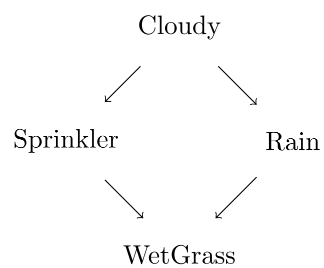
1. $P(X_1|X_2) = \frac{P(X_2|X_1) \cdot P(X_1)}{\sum_{x \in X_1} P(X_2|X_1=x) \cdot P(X_1=x)}$

2. $P(X_1|X_2, X_3) = \frac{P(X_2|X_1, X_3) \cdot P(X_1|X_3)}{P(X_2|X_3)}$

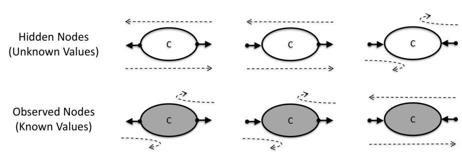
3. $P(X_1 = x|X_2) = \frac{P(X_2|X_1=x) \cdot P(X_1=x)}{P(X_2)}$

Bayesan Networks

Bayesan Network: a DAG where Nodes represent random variables and edges represent direct influence.



Conditional Independence: $P(X_1|X_2, X_3) = P(X_1|X_3)$ then X_1 and X_2 are conditionally independent given X_3



Two (sets of) nodes A and B are conditionally independent (d-separated) given C if and only if all the path from A to B are shielded by C.

Joint Distribution Factorization: $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i|\mathrm{pa}(X_i))$

Explaining Away: describes two variable which become dependent because you observe a third one.

Markov Chains

(first Order) Markov Chain: $P(X_{t+1}|X_t, X_{t-1}, ..., X_1) = P(X_{t+1}|X_t)$

Stationary Event: $P(X_{t+1} = j|X_t = i) = p_{ij} \ \forall t$

Transition Matrix: $P = \begin{bmatrix} p_{11} & ... & p_{1n} \\ ... & ... & ... \\ p_{n1} & ... & p_{nn} \end{bmatrix}$ $\sum_{j=1}^{n} p_{ij} = 1$

Probability at step n: $P_{ij}(n) = P^n(i,j)$

Reachability: A state j is reachable from i if there exists a path from i to j

Communicability: States i and j communicate if each is reachable from the other

Absorbing State: A state i is absorbing if $p_{ii} = 1$

Transient State: A state i is transient if it is reachable from another state j but not vice versa.

Recurrent State: A state i is recurrent if it is not transient.

Ergodic Markov Chain: A Markov Chain is ergodic if it is:

- Recurrent
- Aperiodic
- All states communicate

Steady State Distribution: $\pi = \lim_{n \to \infty} P^n$

Expected Transient Time: $m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$

Absorbing Markov Chain: $P = \begin{bmatrix} Q & R \\ 0 & 1 \end{bmatrix}$ Q transient states, R absorbing states.

Hidden Markov Models

Hidden Markov Models: A HMM is (S, E, P, A, B):

- S is the set of hidden states
- E is the set of observations
- P is the distribution of the initial state
- A is the transition probability matrix
- B emission probability matrix

HMM: Viterbi Algorithm

F rom observations, compute the most likely sequence of hidden states:

$$\arg\max P(X_{1:t}|e_{1:t}) = \arg\max \frac{P(X_{1:t}, e_{1:t})}{P(e_{1:t})} = \arg\max P(X_{1:t}, e_{1:t}) \tag{1}$$

Lets apply bayesan factorization:

$$P(X_{1:t}, e_{1:t}) = P(X_0) \prod_{i=1}^{t} P(X_i|X_{i-1}) P(e_i|X_i) \tag{2}$$

The solution is the one that minimize:

$$-\log P(X_{1:t}, e_{1:t}) = -\log P(X_0) + \sum_{i=1}^{t} (-\log P(X_i|X_{i-1}) - \log P(e_i|X_i)) \tag{3}$$

Graph Construction

Construct a graph with $1 + t \cdot n$ nodes, where n is the number of hidden states with one initial node and n nodes at time i where j^{th} node is $X_i = s_j$.
Find the shortest path from the initial node to the final node considering the weights of the edges.