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Exchange economy, with exogenous dividend process

$$d\delta = \mu_\delta dt + \sigma_\delta dw$$

Bond and Stock process :

$$\begin{aligned}\frac{dB}{B} &= rdt \\ \frac{ds + \delta dt}{s} &= \mu dt + \sigma dw\end{aligned}$$

(r, μ, σ) to be determined endogenously in equilibrium.

Define:

- $\alpha \equiv$ amount in Bond
- $\theta \equiv$ amount in Stock
- $W \equiv \alpha + \theta =$ Wealth

$$\begin{aligned}dW + cdt &= (\alpha r + \theta \mu)dt + \theta \sigma d\vec{w} \\ U_i(\{c\}) &= E \left[\int_0^T e^{-\rho t} U_i(c_t) dt \right]\end{aligned}$$

Throughout assume $U_2(c_2) = Lc_2$

Endowments:

Agent 1 : Long 1 share stock, short β shares of bond

Agent 2 : Long β shares of bond

Equilibrium:

1) $w_i = \alpha_i + \theta_i = E_t \left[\int_0^T ds \frac{\zeta_i^s}{\zeta_i^t} c(s) \right]$ 2) Market clearing

$$(i) \ c_1 + c_2 = \delta$$

$$(ii) \ \alpha_1 + \alpha_2 = 0$$

$$(iii) \ \theta_1 = s$$

Unrestricted Case: - Dynamically complete markets

Introduce representative agent $U[c, \lambda] = E \left[\int_0^T e^{-\rho t} u[c_t, \lambda] dt \right]$ where λ is a constant and where $u[c, \lambda] = \max_{c_1 + c_2 = c} u_1(c_1) + \lambda u_2(c_2)$

In equal, ($c = \delta$), so SDF follows

$$\begin{aligned}\zeta(t) &= e^{-\rho t} \frac{U_c \delta_{t, \lambda}}{U_c \delta_{0, \lambda}} \\ \frac{\partial \zeta}{\zeta} &= -r dt - K dw \\ K &= \frac{\mu - r}{\sigma} \\ -r &= \frac{1}{dt} E \left[\frac{d\zeta}{\zeta} \right] = \frac{1}{dt} E \left[-\rho dt + \frac{u_{cc}}{u_c} d\delta + \frac{1}{2} \frac{u_{ccc}}{u_c} d\delta^2 \right]\end{aligned}$$

$$r = \rho + A\mu_\delta - \frac{1}{2}A\rho\sigma_\delta^2 \quad (12)$$

$$\begin{aligned} \zeta_t S_t &= E_t \left[\int_t^T \zeta(u) \delta(u) du \right] \\ \Rightarrow \zeta_t S_t + \int_t^T \zeta(u) \delta(u) du &= \text{P martingale} \\ \Rightarrow 0 &= E \left[\frac{d\zeta}{\zeta} + \frac{\partial s}{s} + \frac{d\zeta}{\zeta} \frac{\partial s}{s} + \frac{\delta}{s} dt \right] \\ \Rightarrow E \left[\frac{ds + \delta dt}{s} \right] - r dt &= -E \left[\frac{\partial \zeta}{\zeta} \frac{\partial s}{s} \right] \\ \mu - r &= K\sigma \end{aligned} \quad (13)$$

where, K is defined via, $-K = \frac{d\zeta}{\zeta}|_{stoch} = \frac{U_{cc}}{U_c} \sigma_\delta$

$$K = A\sigma_\delta \quad (14)$$

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Worth noting that

$$\begin{aligned} u[c(w_t), \lambda] &= \max_{c_1} u_1[c_1] + \lambda u_2(c_2 = (c - c_1)) \\ FOC : 0 &= u'_1[c'_1(w_t)] - \lambda u'_2[c'_2(w_t) = c'(w_t) - c'_1(w_t)] \\ \lambda &= \frac{u'_1[c'_1(w_t)]}{u'_2[c'_2(w_t)]} \\ \lambda &\text{ is a constant in complete markets case.} \end{aligned}$$

Now, for a given λ market eventually needs to be determined, we have 2 eqns.

2

$$\begin{aligned} \delta &= c = c'_1(c, \lambda) + c'_2(c, \lambda) \\ u'_1(c'_1(w_t)) &= \lambda u'_2(c'_2(w_t)) \\ \text{These jointly determine } c'_i(c, \lambda), i &= 1, 2, \dots \\ \text{Ex: if } u_1(x) &= u_2(x) = \log x \\ \frac{1}{c_1} &= \frac{\lambda}{c_2} \\ c_2 &= \lambda c_1 \\ c &= c_1 + \lambda c_1 \\ c_1 &= \frac{c}{1 + \lambda} \\ c_2 &= \frac{\lambda c}{1 + \lambda} \\ \text{Envelope Condition:} \\ 1 \Rightarrow u_c[c(w_t), \lambda] &= u'_1[c'_1(c, \lambda)] \cdot \frac{dc_1}{dc} + \lambda u'_2[c'_2] \frac{dc_2}{dc} \\ 2 \Rightarrow &= \lambda u'_2 \frac{d[c_1 + c_2]}{dc} \end{aligned}$$

$$u_c = \lambda u'_2 = u'_1 \quad (16.5)$$

Define

$$f_i(x) \equiv \left([u'_i]^{-1} \right) (x)$$

$$c'_1(w_t) = f_1[u_c(\delta(w_t), \lambda)] \quad (15)$$

$$c'_2(w_t) = f_2\left[\frac{u_c(\delta(w_t), \lambda)}{\lambda}\right] \quad (16)$$

Thus everything is determined if λ is identified. To identify λ , use

$$\begin{aligned} u_c(w_t) &= \lambda u'_2(c'_2) = \lambda \frac{1}{c'_2(w_t)} \\ \zeta &\quad \text{Thus everything is determined if } \lambda \text{ is identified} \\ \zeta(w_t) &= e^{-\rho t} \frac{c'_2(0)}{c'_2(w_t)}, \text{ Note } \zeta(0) = 1 \\ \zeta(0)w_2(0) &= w_2(0) = \beta = E\left[\int_0^T e^{-\rho t} \frac{c'_2(0)}{c'_2(w_t)} c'_2(w_t) dt\right] \\ &= c'_2(0) \frac{1}{\rho} (1 - e^{-\rho t}) \\ \beta &= \frac{\lambda}{u_c[\delta(0), \lambda]} \frac{1}{\rho} (1 - e^{-\rho t}) \end{aligned} \quad (17)$$

This is one equation for the one unknown λ . Worth noting that the richest agent-2 can be is that she consumes all output $\Rightarrow c_2$. In that case,

$$\frac{u_c(\delta, \lambda)}{\lambda} = u'_2[c_2 = \delta] = \frac{\partial}{\partial c_2}[c_2]|_{c_2=\delta} = \frac{1}{8}$$

Plugging this into 17 we get $\beta = \frac{\delta}{\rho}(1 - e^{-\rho t})$, which is estimation in eq 16.

$$u'_1[c'_1(\delta, \lambda)] = u_c[\delta, \lambda] \quad (A)$$

$$\begin{aligned} \text{I to LHS } du'_i &= u''_1 dc_1 + \frac{1}{2} u'''_1 dc_1^2 \\ \text{I to RHS} &u_{cc} d\delta + \frac{1}{2} r_{ccc} d\delta^2 \\ dc_1 &= \frac{1}{u''_1} \left[u_{cc} (\mu_\delta dt + \sigma_\delta \partial w) + \frac{1}{2} u_{ccc} \sigma_\delta^2 dt - \frac{1}{2} u'''_1 dc_1^2 \right] \end{aligned} \quad (B)$$

$$\text{A + RHS1:} = \frac{u_{cc}}{u_c} \frac{u'_1}{u''_1} (\mu_\delta dt + \sigma_\delta \partial w) = \frac{A}{A_1} (\mu_\delta dt + \sigma_\delta \partial w)$$

$$\text{A + RHS2:} = \frac{1}{2} \frac{u_{ccc}}{u_c} \frac{u'_1}{u''_1} \sigma_\delta^2 dt = -\frac{1}{2} A \rho \frac{1}{A_1} \sigma_\delta^2 dt$$

$$\text{RHS3:} -\frac{1}{2} \frac{u'''_1}{u''_1} dc_1^2$$

$$\text{But from B, we have, } dc_1^2 = \left(\frac{u_{cc}}{u''_1} \right)^2 \sigma_\delta^2$$

$$= \frac{1}{2} \rho_1 \left(\frac{u_{cc}/u_c}{u''_1/u'_1} \right)^2 \sigma_\delta^2$$

$$= \frac{1}{2} \rho_1 \left(\frac{A}{A_1} \right)^2 \sigma_\delta^2$$

$$\sigma_{c'} = \frac{u_{cc}}{u''_1} \sigma_\delta = \frac{A}{A_1} \sigma_\delta$$

$$\mu_{c'} = \frac{A}{A_1} \mu_\delta - \frac{1}{2} \frac{A}{A_1} \rho \sigma_\delta^2 + \frac{1}{2} \left(\frac{A}{A_1} \right)^2 \rho_1 \sigma_\delta^2$$

$$\begin{aligned}
w_2(t) &= E_t \left[\int_t^T e^{-\rho t} \frac{u_c[\delta_s, \lambda]}{u_c[\delta_t, \lambda]} c'_2(s) ds \right] \\
&= \frac{1}{u_c(\delta_t, \lambda)} E_t \left[\int_t^T e^{-\rho t} \frac{\lambda}{c'_2(s)} c'_2(s) ds \right] \\
&= \frac{\lambda}{u_c(\delta_t, \lambda)} \frac{1}{\rho} \left(1 - e^{-\rho(T-t)} \right), \text{ using 17, } \frac{\lambda}{\rho} = \frac{\beta u_c(\delta_0, \lambda)}{1 - e^{-\rho t}} \\
&= \beta \frac{u_c(\delta_0, \lambda)}{u_c(\delta_t, \lambda)} \frac{1 - e^{-\rho(T-t)}}{1 - e^{-\rho t}} \\
&= \beta \left[\frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho t}} \right] \frac{1}{\zeta(t)} \\
dw_2|_{stoch} &= w_2 \left(\frac{-ds}{s} \right) |_{stoch} \\
&= w_2 K, \text{ vs } dw_2|_{stoch} = \theta_2 \sigma \text{ from wealth dynamics} \\
\theta_2 &= \frac{K}{\sigma} w_2 = \frac{\mu - r}{\sigma^2} w_2 \\
\alpha_2 &= w_2 - \theta_2 = w_2 \left[1 - \frac{\mu - r}{\sigma^2} \right] \\
\theta_1 &= s - \theta_2 \\
\alpha_1 &= -\alpha_2
\end{aligned}$$

Restricted Case :

When agent-2 restricted from participating in markets, equilibrium not pareto efficient, so solution is not same as social planner/representative agent with a constant weight λ . However, can still introduce representation agent with stochastic weight $\lambda(w_t) \Rightarrow$ This reduces search for an equilibrium to the specification of the weighting process $\alpha \lambda$ [and $\lambda(0)$].

Representative agent:

$$\begin{aligned}
U[c, \lambda] &= E_0 \left[\int_0^T e^{-\rho t} u[c, \lambda] dt \right] \\
&\text{where, } u[c, \lambda] = \max_{c_1} \{u_1[c_1] + \lambda(w_t) u_2[c_2 = c - c_1]\} \\
\text{FOC : } &u'_1[c'_1] = \lambda(w_t) u'_2[c'_2] \\
&\text{A1 optimum, must be indifferent to small, affordable, consumption changes}
\end{aligned}$$

$$u[c'_0, \lambda_0] + D_0(w_t) e^{-\rho t} u[c(w_t), \lambda(w_t)] = u[c'_0 - \Delta c_0] + D_0(w_t) e^{-\rho t} u[c' + \Delta c(w_t), \lambda(w_t)]$$

$$\begin{aligned}
0 &= -\Delta c_0 u'[c'_0, \lambda_0] + \Delta c(w_t) u'[c'(w_t), \lambda(w_t)] D_0(w_t) e^{-\rho t} \\
&\text{where, } [\Delta c(w_t), -\Delta c(0)] \text{ have zero cost via A/D prices:} \\
0 &= -\Delta c(0) + \Delta c(w_t) \cdot AD_0(w_t) \\
AD_0(w_t) u'[c'_0, \lambda_0] &= D_0(w_t) e^{-\rho t} u'[c'(w_t), \lambda(w_t)] \\
\zeta_0(w_t) &\equiv \frac{AD_0(w_t)}{D_0(w_t)} = e^{-\rho t} \frac{u'[c'(w_t), \lambda(w_t)]}{u'[c'(0), \lambda(0)]} = e^{-\rho t} \frac{u'[\delta(w_t), \lambda(w_t)]}{u'[\delta(0), \lambda(0)]} \\
&\text{Also, from bottom of page 6,} \\
u_c(c, \lambda) &= u'_1[c'_1] \frac{\partial c'_1}{\partial c} + \lambda u'_2(c'_1) \frac{\partial c'_1}{\partial c}, \text{ use, } \mu'_1 = \lambda u'_2 \\
&= u'_1(c'_1) \frac{d(c'_1 + c'_2)}{dc} = u'_1(c'_1), \text{ since, } c'_1 + c'_2 = c = \delta \\
u_c(c, \lambda) &= u'_1(c'_1) = \lambda \mu'_2(c'_2)
\end{aligned}$$

Implications

i

$$c'_1[\delta(w_t), \lambda(w_t)] = [u'_1]^{-1}(u_c(\delta(w_t), \lambda(w_t))) \equiv f_1[u_c(\delta(w_t), \lambda(w_t))] \quad (18)$$

ii :

$$c'_2[\delta(w_t), \lambda(w_t)] = [u'_2]^{-1} \left(\frac{u_c(\delta(w_t), \lambda(w_t))}{\lambda} \right) \equiv f_2 \left[\frac{u_c(\delta(w_t), \lambda(w_t))}{\lambda} \right] \quad (19)$$

iii :

$$\lambda(w_t) = \frac{u'_1[c'_1(w_t)]}{u'_2[c'_2(w_t)]} \quad (20)$$

Important!!! When they wrote $u[c] = u_1[c_1] + \lambda u_2[c_2]$, they already assumed agent-1 was unrestricted.

More generally, need to write $u[c] \equiv \max_{c_1+c_2=c} \lambda_1(w_t) u_1[c_1] + \lambda_2(w_t) u_2[c_2]$ and will need to identify $(\lambda_1(w_t), \lambda_2(w_t))$ separately.

Have shown above that,

$$\begin{aligned}
u'_1[c'_1(\delta(w_t), \lambda(w_t))] &= u_c[\delta(w_t), \lambda(w_t)] = e^{-\rho t} \zeta(w_t) u_c[\delta_0, \lambda_0] \\
&\Rightarrow e^{-\rho t} u'_1[c'_1] \propto \zeta(w_t)
\end{aligned}$$

B/C interpret, “Since agent-1 facing a dynamically complete market, optimality of c'_1 , equivalent to $e^{-\rho t} u'_1(c'_1)$ being proportional to $\zeta(t)$ ”

Also shown above that,

$$\begin{aligned}
\lambda u'_2[c'_1] &= u_c = e^{-\rho t} \zeta u_c[\delta_0, \lambda_0] \\
e^{-\rho t} u'_2[c'_2(w_t)] &\propto \frac{\zeta(w_t)}{\lambda(w_t)} \\
&\propto B(w_t)^{-1}
\end{aligned}$$

B/C interpret, “Since agent-2 facing a dynamically incomplete market, marginal utility is not proportional to $\zeta(w_t)$. Since agent-2 has log preferences and can only invest in bond, then,

$$e^{-\rho t} u'_2[c'_2(w_t)] \propto B(w_t)^{-1}$$

My interpretation : Without access to stocks, $\frac{\partial \zeta_2}{\zeta_2} = -r dt$. By definition, $\frac{\partial B}{B} = \delta dt \Rightarrow \zeta_2 \propto \frac{1}{B} \Rightarrow \frac{\partial \zeta_2}{\zeta_2} = -\frac{\partial B}{B} = -r dt$. Here $\zeta_2 \neq \zeta$ in paper, but pricing kernel for restricted agent. More generally

we have for 2nd agent,

$$\begin{aligned} sw_2 + c_2 dt &= rw_2 dt \\ \zeta_2(t)w_2(t) &= E_t\left[\int_t^T ds \zeta_2(s)c_2(s)\right] \\ \rightarrow 0 &= E_t\left[\frac{\partial \zeta_2}{\zeta_2} + \frac{\partial w_2}{w_2} + \frac{\partial \zeta_2}{\zeta_2} \frac{\partial w_2}{w_2} + \frac{c_2}{w_2} dt\right] \end{aligned}$$

Notationally specify,

$$\frac{\partial \zeta_2}{\zeta_2} = -\alpha dt - \beta dw \quad (9A)$$

Use 9A

$$\begin{aligned} 0 &= -\alpha + \frac{1}{w_2}[rw_2 - c_2] + 0 \text{ (since } \partial w_2 \text{ deterministic)} + \frac{c_2}{w_2} \\ &= -\alpha + r \\ \alpha &= r \end{aligned}$$

But β not determined.

Markes sense in that any β will price bond correctly with α set to r.

My guess is that 2nd agent will not correctly price the stock, since if they did I assume they would price everything correctly.

Apply I/O's lemma to

$$\begin{aligned} \lambda(t) &= \frac{u'_1(c'_1)}{u'_2(c'_2)} = \frac{\Psi_1 \zeta(w_t)}{\Psi_2 B(w_t)^{-1}} = \frac{\Psi_1}{\Psi_2} \zeta B \\ \frac{d\lambda}{\lambda} &= \frac{d\zeta}{\zeta} + \frac{\partial B}{B} \\ &= (-rdt - K\partial w) + rdt = -K\partial w = \frac{\partial \zeta}{\zeta}|_{stoch} \\ d\lambda &= -\lambda K\partial w, \text{ Q? what is K?} \\ \text{We have, } e^{-\rho t} u'_1 &= e^{-\rho t} u_c = \Psi_1 \zeta \\ \frac{d(e^{-\rho t} u'_1)|_{stoch}}{e^{-\rho t} u'_1} &= \frac{d(e^{-\rho t} u_c)|_{stoch}}{e^{-\rho t} u_c} = \frac{d\zeta}{\zeta}|_{stoch} \\ \frac{u''_1}{u'_1} dc_1|_{stoch} &= \frac{u_{cc}}{u_c} dc|_{stoch} = -K d\vec{w} \end{aligned}$$

In corollary 2, BC show that $dc_1|_{stoch} = dc|_{stoch} = \sigma_s \partial w$

Another way to see this is that $\delta = c_1 + c_2 = d\delta|_{stoch} = dc_1|_{stoch} + dc_2|_{stoch}$

$$u''_2 dc_2|_{stoch} = -\frac{\partial B}{B}|_{stoch} = 0 \Rightarrow \sigma_\delta = dc_1|_{stoch} \quad (22)$$

$$\begin{aligned} -K &= \frac{u''_1}{u'_1} \sigma_\delta = \frac{u_{cc}}{u_c} \sigma_\delta, \text{ and since } u'_1 = u_c \\ &= \frac{u''_1}{u_c} \sigma_\delta \end{aligned}$$

$$d\lambda = \lambda \frac{u''_1}{u_c} \sigma_\delta d\vec{w} \quad (23)$$

Theorem 1 Taking,

$$e^{-\rho t} u'_2(c'_2) = \Psi_2 B(t)^{-1}$$

as given, initial width of agent 2 is

$$\begin{aligned} \beta &= w_2(0) = E_0 \left[\int_0^T ds e^{-\rho s} \frac{u'_2 c_2(s)}{u'_2 c_2(0)} c_2(s) \right] \\ &= \frac{1}{u'_2(c_2(0))} \frac{1}{\rho} (1 - e^{-\rho T}), \text{ use, } \lambda(0) u'_2(c'_2(0)) = u_c(\delta(0), \lambda(0)) \\ \beta &= \frac{\lambda(0)}{u'_2(c_2(0))} \frac{1}{\rho} (1 - e^{-\rho T}) \end{aligned} \tag{24}$$

Everything is correctly priced by marginal utility of representative agent wealth of economy $w_1 + w_2 = (s - w_2) + w_2 = s$

$$\begin{aligned} w_2(w_t) &= E_t \left[\int_t^T ds e^{-\rho(s-t)} \frac{u'_2[c(0)]}{u'_2[c(w_t)]} c_2(w_s) \right] \\ &= \frac{1}{u'_2 c(w_t)} \frac{1}{\rho} (1 - e^{-\rho(T-t)}) \text{ use, } \lambda(w_t) u'_1(c'_2(w_t)) = u_c(\delta(w_t), \lambda) \\ &= \end{aligned}$$