Cox Huang 2

$$
\begin{aligned}
\frac{ds}{s} &= \mu dt + \sigma dZ \\
\frac{dB}{B} &= rdt \\
& \text{all zero divided securities satisfy.} \\
0 &= E\left(d(\wedge X)\right) \\
&= E\left(\frac{\partial\wedge}{\wedge} + \frac{\partial X}{X} + \frac{\partial\wedge}{\wedge}\frac{\partial X}{X}\right) \\
& \text{Notationally Specify, } \frac{\partial\wedge}{\wedge} = -\alpha dt - \beta dZ \\
\text{Bond}: 0 &= -\alpha + r \Rightarrow \alpha = r \\
\text{Stock}: 0 &= -r + \mu - \beta\sigma \Rightarrow \beta = \left(\frac{\mu - r}{\sigma}\right) \\
\frac{\partial\wedge}{\wedge} &= -rdt - \left(\frac{\mu - r}{\sigma}\right) dZ \\
\wedge_t W_t &= E_t\left(\wedge_t W(t)\right) \\
MaxE\left(\frac{1}{1-r} W_t^{1-r}\right) & \quad s.t. \quad \wedge_t W_t = E_t\left(\wedge_t W_t\right) \\
L &= \int dw D(w) \frac{W(w_t)^{1-r}}{1-r} + +\beta\left[\wedge_t w_t - \int \partial w D(w) \wedge (w) W(w)\right] \\
\frac{\partial Z}{\partial W(w_T)} = 0 &= D(w)W(w)^{-r} - \beta D(w) \wedge (w) \\
\Rightarrow W(w)^{-r} &= \beta \wedge (w) \\
\Rightarrow W_T &= \beta^{-\frac{1}{r}} \wedge_t^{-\frac{1}{r}} \\
\wedge_t w_t &= E\left[\wedge \beta^{-\frac{1}{r}} \wedge^{-\frac{1}{r}}\right] \\
&= \beta^{-\frac{1}{r}} E\left[\wedge^{-\frac{r-1}{r}}\right] \\
\wedge(t) &= \wedge(t) e^{\left(-r - \frac{\theta^2}{r}\right)(T-t) - \theta(Z_t \cdot Z_t)} \\
\wedge_t w_t &= \beta^{-\frac{1}{r}} \wedge_t^{\frac{r-1}{r}} e^{-\left(r + \frac{\theta^2}{r}\right)(T-t)\left(\frac{r-1}{r}\right)} e^{\frac{1}{2}\left(\frac{r-1}{r}\right)^2 \theta^2 (1-r)} \\
\beta^{-\frac{1}{r}} &= \wedge_t w_t \wedge_t^{-1+\frac{1}{r}} e^{\left(r + \frac{\theta^2}{r}\right)(T-t)\left(\frac{r-1}{r}\right)} e^{-\frac{\theta^2}{2}\left(\frac{r-1}{r}\right)^2 (T-t)} \\
&= w_t \wedge_t^{\frac{1}{r}} e^{\left(r + \frac{\theta^2}{2}\right)(T-t)\left(\frac{r-1}{r}\right) - \frac{\theta^2}{2}\left(\frac{r-1}{r}\right)^2 (T-t)} \\
W(w) &= \wedge(w)^{-\frac{1}{r}} w_t \wedge_t^{\frac{1}{r}} e^{\left(r + \frac{\theta^2}{2}\right)(T-t)\left(\frac{r-1}{r}\right) - \frac{\theta^2}{2}\left(\frac{r-1}{r}\right)^2 (T-t)} \\
&= w_t e^{\left(\frac{1}{r}\right)\left(r + \frac{\theta^2}{2}\right)(T-t) + \left(\frac{\theta}{r}\right)(Z_T Z_t)} e^{\left(r + \frac{\theta^2}{2}\right)(T-t)\left(\frac{r-1}{r}\right) - \frac{\theta^2}{2}\left(\frac{r-1}{r}\right)^2 (T-t)} \\
&= w_t e^{\left(\frac{\theta}{r}\right)(Z_T Z_t)} e^{\left(r + \frac{\theta^2}{2}\right)(T-t) - \left(\frac{\theta^2}{2}\right)\left(\frac{r-1}{r}\right)^2 (T-t)} \\
&= w_t e^{\left(\frac{\theta}{r}\right)(Z_T Z_t)} e^{(T-t)} e^{\left(\frac{\theta^2}{2r^2}\right)(T-t)\left(r^2 - \left(r^2 - 2r + 1\right)\right)} \\
W(w_T) &= W_t e^{\frac{\theta}{r}(Z_T Z_t)} e^{r(T-t)} e^{\frac{\theta^2}{2r^2}(T-t)(2r-1)} \\
\wedge(w_s)W(w_s) &= E_s\left(\wedge(w_T)W(w_t)\right) \\
\wedge(w_T) &= \wedge(w_s) e^{-\left(r + \frac{\theta^2}{2}\right)(T-s) - \theta(Z_T Z_s)} \\
\wedge(w_s)W(w_s) &= w_t \wedge (w_s) e^{-\left(r + \frac{\theta^2}{2}\right)(T-s)} e^{r(T-t)} e^{\frac{\theta^2}{2r^2}(T-t)(2r-1)} E_s\left[e^{-\theta(Z_T Z_s)} e^{\frac{\theta}{r} Z_T Z_t} e^{\frac{\theta}{r} Z_s Z_t}\right] \\
W(w_s) &= w_t e^{r(s-t)} e^{-\frac{\theta^2}{2}(t-s)} e^{\frac{\theta^2}{2r^2}(T-t)(2r-1)} e^{\frac{1}{2}\left(\frac{\theta}{r}(1-r)\right)^2 (T-s)} e^{\frac{\theta}{r}(Z_s - Z_t)}
\end{aligned}
$$

$$
\begin{aligned}
&= W_t e^{r(s-t)} e^{\frac{\theta}{r}(Z_s - Z_t)} e^{\frac{\theta^2}{2r^2}(T-s)\left(2r-1-r^2+(1-r)^2\right)} e^{\frac{\theta^2}{2r^2}(s-t)} \\
&= w_t e^{r(s-t)} e^{\frac{\theta}{r}(Z_s Z_t)} e^{\frac{\theta^2}{2r^2}(s-t)(2r-1)} \\
\Rightarrow dW_s &= W\left(rds + \frac{\theta^2}{2r^2}(2r-1)\,ds + \frac{\theta}{r}dZ + \frac{1}{2}\left(\frac{\theta}{r}\right)^2 ds\right) \\
&= W\left(\left[r + \frac{\theta^2}{2r^2}2r\right]ds + \frac{\theta}{r}dZ\right) \\
&= W\left(\left[r + \frac{\theta^2}{r}\right]ds + \frac{\theta}{r}dZ\right) \\
&\text{Compare with,} \\
W &= n_B B + n_S S \Rightarrow w \equiv \frac{n_S S}{w}, 1 - w = \frac{n_B B}{w} \\
dW &= n_B dB + n_S dS \\
\frac{dw}{w} &= \frac{n_B B}{w}\frac{\partial B}{B} + \frac{n_S S}{w}\frac{\partial S}{S} \\
&= (1-w)\,rdt + w\,(\mu dt + \sigma dZ) \\
&= (r + w\,(\mu - r))\,dt + w\sigma dZ \\
\text{vs} & \\
&= \left(r + \frac{\theta^2}{r}\right)dt + \frac{\theta}{r}dZ \\
w\sigma &= \frac{\theta}{r} \Rightarrow w = \frac{\mu - r}{r\sigma^2} \\
\text{Check, } w\,(\mu - r) &= \frac{\theta^2}{r} \\
\frac{1}{r}\left(\frac{\mu - r}{\sigma}\right)^2 &= \frac{\left(\mu - \frac{r}{\sigma}\right)^2}{r} \\
\wedge(w_t)W(w_t) &= \left(\wedge(0)e^{\left(r+\frac{\theta^2}{2}\right)t - \theta Z_t}\right)\left(w_0 e^{rt} e^{\frac{\theta^2}{2r^2}(2r-1)t} e^{\frac{\theta}{r}Z_t}\right) \\
&= \wedge_0 w_0 e^{\left(\frac{\theta}{r}\right)(1-r)Z_t} e^{\frac{1}{2}\left(\frac{\theta}{r}\right)^2 t\left(2r-1-r^2\right)} \\
&= \wedge_0 w_0 e^{\frac{\theta(1-r)}{r}Z_t} e^{-\frac{1}{2}t\left(\frac{\theta(1-r)}{r}\right)^2} \\
&\equiv \text{P-Martingale} \\
d(\wedge w) &= \wedge w\frac{\theta\,(1-r)}{r}dZ \\
\text{vs} & \\
d(\wedge w) &= \wedge w\left(\frac{\partial\wedge}{\wedge} + \frac{\partial w}{w} + \frac{\partial\wedge}{\wedge}\frac{\partial w}{w}\right) \\
&= \wedge w\,(-\theta dZ + X_s\sigma dZ) \\
\Rightarrow X_s\sigma &= \theta + \frac{\theta\,(1-r)}{r} \\
\text{Note: m log case, } r &= 1, X_s\sigma = \theta \\
\Rightarrow X_s &= \frac{\mu - r}{\sigma^2}
\end{aligned}
$$