Basak Cuoco

Exchange economy, with exogenous dividned process

$$d\delta = \mu_\delta dt + \sigma_\delta dw$$

Bond and Stock process :

$$\begin{aligned}
\frac{dB}{B} &= rdt \\
\frac{ds + \delta dt}{s} &= \mu dt + \sigma \partial w
\end{aligned}$$

(r, μ, σ) to be determined endogenously in equilibrium.

Define:

- $\alpha \equiv$ amount in Bond
- $\theta \equiv$ amount in Stock
- $W \equiv \alpha + \theta =$ Wealth

$$\begin{aligned}
dW + cdt &= (\alpha r + \theta\mu)dt + \theta\sigma d\vec{w} \\
U_i(\{c\}) &= E\left[\int_0^T e^{-\rho t} U_i(c_t)dt\right]
\end{aligned}$$

Throughout assume $U_2(c_2) = Lc_2$

Endowments:
Agent 1 : Long 1 share stock, short β shares of bond
Agent 2 : Long β shares of bond

Equilibrium:
1) $w_i = \alpha_i + \theta_i = E_t[\int_0^T ds \frac{\zeta^i s}{\zeta^i t} c(s)]$ 2) Market clearing

(i) $c_1 + c_2 = \delta$

(ii) $\alpha_1 + \alpha_2 = 0$

(iii) $\theta_1 = s$

Unrestricted Case: - Dynamically complete markets
Introduce representative agent $U[c, \lambda] = E[\int_0^T e^{-\rho t} u[c_t, \lambda]dt]$ where λ is a constant and where $u[c, \lambda] = \max_{c_1 + c_2 = c} u_1(c_1) + \lambda u_2(c_2)$

In equal, $(c = \delta)$, so SDF follows

$$\begin{aligned}
\zeta(t) &= e^{-\rho t} \frac{U_c \delta_{t,\lambda}}{U_c \delta_{0,\lambda}} \\
\frac{\partial \zeta}{\zeta} &= -rdt - K\partial w \\
K &= \frac{\mu - r}{\sigma} \\
-r = \frac{1}{dt} E\left[\frac{d\zeta}{\zeta}\right] &= \frac{1}{dt} E\left[-\rho dt + \frac{u_{cc}}{u_c} d\delta + \frac{1}{2}\frac{u_{ccc}}{u_c} d\delta^2\right]
\end{aligned}$$

$$r = \rho + A\mu_\delta - \frac{1}{2}A\rho\sigma_\delta^2 \tag{12}$$

$$\begin{aligned}
\zeta_t S_t &= E_t\left[\int_t^T \zeta(u)\delta(u)du\right] \\
&\Rightarrow \zeta_t S_t + \int_t^T \zeta(u)\delta(u)du = \text{ P martingale} \\
&\Rightarrow 0 = E\left[\frac{d\zeta}{\zeta} + \frac{\partial s}{s} + \frac{d\zeta}{\zeta}\frac{\partial s}{s} + \frac{\delta}{s}dt\right] \\
&\Rightarrow E\left[\frac{ds + \delta dt}{s}\right] - rdt = -E\left[\frac{\partial\zeta}{\zeta}\frac{\partial s}{s}\right]
\end{aligned}$$

$$\mu - r = K\sigma \tag{13}$$

where, K is defined via, $-K = \frac{d\zeta}{\zeta}|_{stoch} = \frac{U_{cc}}{U_c}\sigma_\delta$

$$K = A\sigma_\delta \tag{14}$$

1
Worth noting that

$$\begin{aligned}
u[c(w_t),\lambda] &= \max_{c_1} u_1[c_1] + \lambda u_2(c_2 = (c - c_1)) \\
FOC : 0 &= u_1'[c_1'(w_t)] - \lambda u_2'[c_2'(w_t) = c'(w_t) - c_1'(w_t)] \\
\lambda &= \frac{u_1'[c_1'(w_t)]}{u_2'[c_2'(w_t)]} \\
& \quad \lambda \text{ is a constant in complete markets case.}
\end{aligned}$$

Now, for a given λ market eventually needs to be determined, we have 2 eqns.

2

$$\begin{aligned}
\delta &= c = c_1'(c,\lambda) + c_2'(c,\lambda) \\
u_1'(c_1'(w_t)) &= \lambda u_2'(c_2'(w_t)) \\
& \quad \text{These jointly determine } c_i'(c,\lambda), i = 1, 2, ... \\
\text{Ex: if } u_1(x) &= u_2(x) = \log x \\
\frac{1}{c_1} &= \frac{\lambda}{c_2} \\
c_2 &= \lambda c_1 \\
c &= c_1 + \lambda c_1 \\
c_1 &= \frac{c}{1+\lambda} \\
c_2 &= \frac{\lambda c}{1+\lambda} \\
& \quad \text{Envelope Condition:} \\
1 \ \Rightarrow u_c[c(w_t),\lambda] &= u_1'[c_1'(c,\lambda)] \cdot \frac{dc_1}{dc} + \lambda u_2'[c_2']\frac{dc_2}{dc} \\
2 \ \Rightarrow &= \lambda u_2' \frac{d[c_1 + c_2]}{dc}
\end{aligned}$$

$$u_c = \lambda u_2' = u_1' \tag{16.5}$$

Define

$$f_i(x) \equiv \left([u_i']^{-1}\right)(x)$$

$$c_1'(w_t) = f_1\left[u_c(\delta(w_t),\lambda)\right] \tag{15}$$

$$c_2'(w_t) = f_2\left[\frac{u_c(\delta(w_t),\lambda)}{\lambda}\right] \tag{16}$$

Thus everything is determined if λ is identified. To identify λ, use

$$\begin{aligned}
u_c(w_t) &= \lambda u_2'(c_2') = \lambda \frac{1}{c_2'(w_t)} \\
\zeta & \quad \text{Thus everything is determined if } \lambda \text{ is identified} \\
\zeta(w_t) &= e^{-\rho t}\frac{c_2'(0)}{c_2'(w_t)}, \text{ Note } \zeta(0) = 1 \\
\zeta(0)w_2(0) &= w_2(0) = \beta = E\left[\int_0^T e^{-\rho t}\frac{c_2'(0)}{c_2'(w_t)}c_2'(w_t)dt\right] \\
&= c_2'(0)\frac{1}{\rho}(1 - e^{-\rho t})
\end{aligned}$$

$$\beta = \frac{\lambda}{u_c[\delta(0),\lambda]}\frac{1}{\rho}(1 - e^{-\rho t}) \tag{17}$$

This is one equation for the one unknown λ. Worth noting that the richest agent-2 can be is that she consumes all output $\Rightarrow c_2$. In that case,

$$\frac{u_c(\delta,\lambda)}{\lambda} = u_2'[c_2 = \delta] = \frac{\partial}{\partial c_2}[c_2]|_{c_2=\delta} = \frac{1}{8}$$

Plugging this into 17 we get $\beta = \frac{\delta}{\rho}(1 - e^{-\rho t})$, which is estimation in eq 16.

$$u_1'[c_1'(\delta,\lambda)] = u_c[\delta,\lambda] \tag{A}$$

$$\begin{aligned}
\text{I to LHS } du_i' &= u_1''dc_1 + \frac{1}{2}u_1'''dc_1^2 \\
\text{I to RHS } & \quad u_{cc}d\delta + \frac{1}{2}r_{ccc}d\delta^2
\end{aligned}$$

$$dc_1 = \frac{1}{u_1''}\left[u_{cc}\left(\mu_\delta dt + \sigma_\delta \partial w\right) + \frac{1}{2}u_{ccc}\sigma_\delta^2 dt - \frac{1}{2}u_1'''dc_1^2\right] \tag{B}$$

$$\begin{aligned}
\text{A + RHS1:} &= \frac{u_{cc}}{u_c}\frac{u_1'}{u_1''}\left(\mu_\delta dt + \sigma_\delta \partial w\right) = \frac{A}{A_1}\left(\mu_\delta dt + \sigma_\delta \partial w\right) \\
\text{A + RHS2:} &= \frac{1}{2}\frac{u_{ccc}}{u_c}\frac{u_1'}{u_1''}\sigma_\delta^2 dt = -\frac{1}{2}A\rho\frac{1}{A_1}\sigma_\delta^2 dt \\
\text{RHS3:} & \quad -\frac{1}{2}\frac{u_1'''}{u_1''}dc_1^2 \\
& \quad \text{But from B, we have, } dc_1^2 = \left(\frac{u_{cc}}{u_1''}\right)^2\sigma_\delta^2 \\
&= \frac{1}{2}\rho_1\left(\frac{u_{cc}/u_c}{u_1''/u_1'}\right)^2\sigma_\delta^2 \\
&= \frac{1}{2}\rho_1\left(\frac{A}{A_1}\right)^2\sigma_\delta^2 \\
\sigma_{c'} &= \frac{u_{cc}}{u_1''}\sigma_\delta = \frac{A}{A_1}\sigma_\delta \\
\mu_{c'} &= \frac{A}{A_1}\mu_\delta - \frac{1}{2}\frac{A}{A_1}\rho\sigma_\delta^2 + \frac{1}{2}\left(\frac{A}{A_1}\right)^2\rho_1\sigma_\delta^2
\end{aligned}$$

$$
\begin{aligned}
w_2(t) &= E_t\left[\int_t^T e^{-\rho t}\frac{u_c[\delta_s,\lambda]}{u_c[\delta_t,\lambda]}c_2'(s)ds\right] \\
&= \frac{1}{u_c(\delta_t,\lambda)}E_t\left[\int_t^T e^{-\rho t}\frac{\lambda}{c_2'(s)}c_2'(s)ds\right] \\
&= \frac{\lambda}{u_c(\delta_t,\lambda)}\frac{1}{\rho}\left(1-e^{-\rho(T-t)}\right)\ \text{, using 17, } \frac{\lambda}{\rho}=\frac{\beta u_c(\delta_0,\lambda)}{1-e^{-\rho t}} \\
&= \beta\frac{u_c(\delta_0,\lambda)}{u_c(\delta_t,\lambda)}\frac{1-e^{-\rho(T-t)}}{1-e^{-\rho t}} \\
&= \beta\left[\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho t}}\right]\frac{1}{\zeta(t)} \\
dw_2|_{stoch} &= w_2\left(\frac{-ds}{s}\right)|_{stoch} \\
&= w_2K,\ \text{vs } dw_2|_{stoch}=\theta_2\sigma \text{ from wealth dynamics} \\
\theta_2 &= \frac{K}{\sigma}w_2=\frac{\mu-r}{\sigma^2}w_2 \\
\alpha_2 &= w_2-\theta_2=w_2\left[1-\frac{\mu-r}{\sigma^2}\right] \\
\theta_1 &= s-\theta_2 \\
\alpha_1 &= -\alpha_2
\end{aligned}
$$

Restricted Case :
When agent-2 restricted from participating in markets, equlibrium not pareto efficient, so solution is not same as social planner/representative agent with a constant weight λ. However, can still introduce representation agent with stochastic weight $\lambda(w_t)\Rightarrow$ This reduces search for an equilibrium to the specification of the weighting process $\alpha\lambda$ [and $\lambda(0)$].

Representative agent:

$$
\begin{aligned}
U[c,\lambda] &= E_0\left[\int_0^T e^{-\rho t}u[c,\lambda]dt\right] \\
&\quad \text{where, } u[c,\lambda]=\max_{c_1}\{u_1[c_1]+\lambda(w_t)u_2[c_2=c-c_1]\} \\
&\quad \text{FOC : } u_1'[c_1']=\lambda(w_t)u_2'[c_2''] \\
&\quad \text{A1 optimum, must be indifferent to small, affordable, consumption changes}
\end{aligned}
$$

$$
u[c_0',\lambda_0]+D_0(w_t)e^{-\rho t}u[c(w_t),\lambda(w_t)] = u[c_0'-\Delta c_0]+D_0(w_t)e^{-\rho t}u(c'+\Delta c(w_t),\lambda(w_t))
$$

$$\begin{aligned}
0 &= -\Delta c_0 u'[c_0', \lambda_0] + \Delta c(w_t) u'[c'(w_t), \lambda(w_t)] D_0(w_t) e^{-\rho t} \\
&\quad \text{where, } [\Delta c(w_t), -\Delta c(0)] \text{ have zero cost via A/D prices:} \\
&\quad 0 = -\Delta c(0) + \Delta c(w_t).AD_0(w_t) \\
AD_0(w_t) u'[c_0', \lambda_0] &= D_0(w_t) e^{-\rho t} u'[c'(w_t), \lambda(w_t)] \\
\zeta_0(w_t) &\equiv \frac{AD_0(w_t)}{D_0(w_t)} = e^{-\rho t} \frac{u'[c'(w_t), \lambda(w_t)]}{u'[c'(0), \lambda(0)]} = e^{-\rho t} \frac{u'[\delta(w_t), \lambda(w_t)]}{u'[\delta(0), \lambda(0)]} \\
&\quad \text{Also, from bottom of page 6,} \\
u_c(c, \lambda) &= u_1'[c_1'] \frac{\partial c_1'}{\partial c} + \lambda u_2'(c_1') \frac{\partial c_1'}{\partial c}, \text{ use, } \mu_1' = \lambda u_2' \\
&= u_1'(c_1') \frac{d(c_1' + c_2')}{dc} = u_1'(c_1'), \text{ since, } c_1' + c_2' = c = \delta \\
u_c(c, \lambda) &= u_1'(c_1') = \lambda \mu_2'(c_2')
\end{aligned}$$

Implications

i

$$c_1'[\delta(w_t), \lambda(w_t)] = [u_1']^{-1} \left(u_c(\delta(w_t), \lambda(w_t)) \right) \equiv f_1[u_c(\delta(w_t), \lambda(w_t))] \tag{18}$$

ii :

$$c_2'[\delta(w_t), \lambda(w_t)] = [u_2']^{-1} \frac{(u_c(\delta(w_t), \lambda(w_t)))}{\lambda} \equiv f_2 \left[\frac{u_c(\delta(w_t), \lambda(w_t))}{\lambda} \right] \tag{19}$$

iii :

$$\lambda(w_t) = \frac{u_1'[]c_1'(w_t)}{u_2'[c_2'(w_t)]} \tag{20}$$

Important!!! When they wrote $u[c] = u_1[c_1] + \lambda u_2[c]_2]$, they already assumed agent-1 was unrestricted.

More generally, need to write $u[c] \equiv \max_{c_1+c_2=c} \lambda_1(w_t) u_1[c_1] + \lambda_2(w_t) u_2[c_2]$ and will need to identify $(\lambda_1(w_t), \lambda_2(w_t))$ separately.

Have shown above that,

$$\begin{aligned}
u_1'[c_1'(\delta(w_t), \lambda(w_t))] &= u_c[\delta(w_t), \lambda(w_t)] = e^{-\rho t} \zeta(w_t) u_c[\delta_0, \lambda] \\
\Rightarrow e^{-\rho t} u_1'[c_1'] &\propto \zeta(w_t)
\end{aligned}$$

B/C interpret,"Since agent-1 facing a dynamically complete market, optimality of c_1', equvivalent to $e^{-\rho t} u_1'(c_1')$ being proportional to $\zeta(t)$"

Also shown above that,

$$\begin{aligned}
\lambda u_2'[c_1'] &= u_c = e^{-\rho t} \zeta u_c[\delta_0, \lambda_0] \\
e^{-\rho t} u_2'[c_2'(w_t)] &\propto \frac{\zeta(w_t)}{\lambda(w_t)} \\
&\propto B(w_t)^{-1}
\end{aligned}$$

B/C interpret, "Since agent-2 facing a dynamically incomplete market, marginal utility is not proportional to $\zeta(w_t)$. Since agent-2 has log preferences cand can only invest in bond, then,

$$e^{-\rho t} u_2'[c_2'(w_t)] \propto B(w_t)^{-1}$$

My interpretation : Without access to stocks, $\frac{\partial \zeta_2}{\zeta_2} = -rdt$. By definition, $\frac{\partial B}{B} = \delta dt \Rightarrow \zeta_2 \propto \frac{1}{B}$ $\Rightarrow \frac{\partial \zeta_2}{\zeta_2} = \frac{-\partial B}{B} = -rdt$. Here $\zeta_2 \neq \zeta$ in paper, but pricing kernel for restricted agent. More generally we have for 2nd agent,

$$
\begin{aligned}
sw_2 + c_2dt &= rw_2dt \\
\zeta_2(t)w_2(t) &= E_t[\int_t^T ds\zeta_2(s)c_2(s)] \\
\to 0 &= E_t\left[\frac{\partial\zeta_2}{\zeta_2} + \frac{\partial w_2}{w_2} + \frac{\partial\zeta_2}{\zeta_2}\frac{\partial w_2}{w_2} + \frac{c_2}{w_2}dt\right]
\end{aligned}
$$

Notationally specify,

$$
\frac{\partial\zeta_2}{\zeta_2} = -\alpha dt - \beta dw \tag{9A}
$$

Use 9A

$$
\begin{aligned}
0 &= -\alpha + \frac{1}{w_2}[rw_2 - c_2] + 0(\text{ since } \partial w_2 \text{ deterministic }) + \frac{c_2}{w_2} \\
&= -\alpha + r \\
\alpha &= r \\
&\quad \text{But } \beta \text{ not determined.}
\end{aligned}
$$

Markes sense in that any β will price bond correctly with α set to r.

My guess is that 2nd agent will not correctly price the stock, since if they did I assume they would price everything correctly.

Apply I/O's lemma to

$$
\begin{aligned}
\lambda(t) &= \frac{u_1'(c_1')}{u_2'(c_2')} = \frac{\Psi_1\zeta(w_t)}{\Psi_2 B(w_t)^{-1}} = \frac{\Psi_1}{\Psi_2}\zeta B \\
\frac{d\lambda}{\lambda} &= \frac{d\zeta}{\zeta} + \frac{\partial B}{B} \\
&= (-rdt - K\partial w) + rdt = -K\partial w = \frac{\partial\zeta}{\zeta}|_{stoch} \\
d\lambda &= -\lambda K\partial w, \text{ Q? what is K?} \\
\text{We have, } e^{-\rho t}u_1' &= e^{-\rho t}u_c = \Psi_1\zeta \\
\frac{d(e^{-\rho t}u_1')}{e^{-\rho t}u_1'}|_{stoch} &= \frac{d(e^{-\rho t}u_c)}{e^{-\rho t}u_c}|_{stoch} = \frac{d\zeta}{\zeta}|_{stoch} \\
\frac{u_1''}{u_1'}dc_1|_{stoch} &= \frac{u_{cc}}{u_c}dc|_{stoch} = -Kd\vec{w}
\end{aligned}
$$

In corollary 2, BC show that $dc_1|_{stoch} = dc|_{stoch} = \sigma_s\partial w$
Another way to see this is that $\delta = c_1 + c_2 = d\delta|_{stoch} = dc_1|_{stoch} + dc_2|_{stoch}$

$$
u_2''dc_2|_{stoch} = -\frac{\partial B}{B}|_{stoch} = 0 \Rightarrow \sigma_\delta = dc_1|_{stoch} \tag{22}
$$

$$
\begin{aligned}
-K &= \frac{u_1''}{u_1'}\sigma_\delta = \frac{u_{cc}}{u_c}\sigma_\delta, \text{ and since } u_1' = u_c \\
&= \frac{u_1''}{u_c}\sigma_\delta
\end{aligned}
$$

$$
d\lambda = \lambda\frac{u_1''}{u_c}\sigma_\delta d\vec{w} \tag{23}
$$

Theorem 1 Taking,

$$e^{-\rho t}u_2'(c_2') = \Psi_2 B(t)^{-1}$$

as given, initial width of agent 2 is

$$\begin{aligned}\beta &= w_2(0) = E_0\left[\int_0^T dse^{-\rho s}\frac{u_2'c_2(s)}{u_2'c_2(0)}c_2(s)\right]\\ &= \frac{1}{u_2'(c_2(0))}\frac{1}{\rho}(1-e^{-\rho t}),\ \text{use, } \lambda(0)u_2'(c_2'(0)) = u_c(\delta(0),\lambda(0))\end{aligned}$$

$$\beta = \frac{\lambda(0)}{u_2'(c_2(0))}\frac{1}{\rho}(1-e^{-\rho t}) \tag{24}$$

Everything is correctly priced by marginal utility of representative agent wealth of economy $w_1+w_2 = (s-w_2)+w_2 = s$

$$\begin{aligned}w_2(w_t) &= E_t\left[\int_t^T dse^{-\rho(s-t)}\frac{u_2'[c(0)]}{u_2'[c(w_t)]}c_2(w_s)\right]\\ &= \frac{1}{u_2'c(w_t)}\frac{1}{\rho}(1-e^{-\rho(T-t)})\ \text{use, } \lambda(w_t)u_1'(c_2'(w_t)) = u_c(\delta(w_t),\lambda)\\ &= \frac{1}{\rho}(e^{-\rho t}-e^{-\rho t})\frac{1}{\Psi_2}B(w_t)\end{aligned}$$

$$w_2(w_t) = \frac{1-e^{-\rho(T-t)}}{\rho}\frac{\lambda(w_t)}{u_c(\delta(w_t),\lambda(w_t))} \tag{26B}$$

$$\begin{aligned}w_2(w_t) &= c_2(w_t)\frac{1-e^{-\rho(T-t)}}{\rho}\\ w_2(0) &= \beta = \frac{1}{\rho}(1-e^{-\rho t})\frac{1}{\Psi_2}\\ w_2(w_t) &= \alpha_2(w_t) = \beta\frac{e^{-\rho t}-e^{-\rho t}}{1-e^{-\rho t}}B(w_t)\\ &\ \text{Lemma 2, Pg 331:}\\ u(c(w_t),\lambda(w_t)) &= \max_{c_1}\{u_1[c_1(w_t)]+\lambda(w_t)u_2[c_2(w_t) = c(w_t)-c_1(w_t)]\}\\ FOC: &= 0 = u_1'[c_1'(w_t)] - \lambda(w_t)u_2'[c_2'(w_t) = c'(w_t).c_1'(w_t)]\\ \lambda(w_t) &= \frac{u_1'(c_1'(w_t))}{u_2'(c_2'(w_t))}\\ &\ \text{Envelope Condition :}\\ u_c &= u_1'\frac{\partial c_1}{\partial c}+\lambda u_2'\frac{\partial c_2}{\partial c}\\ &= u_1'\frac{\partial(c_1+c_2)}{\partial c}\\ &= u_1' = \lambda u_2'\\ u_1'(c_1') &= u_c(c) \Rightarrow c_1' \equiv f_1(u_c(c))\\ u_2'(c_2') &= \frac{1}{\lambda}u_c(c) \Rightarrow c_2' \equiv f_2\frac{(u_c(c))}{\lambda}\end{aligned}$$

$$c = c_1'+c_2' = f_1(u_c(c)) + f_2\frac{u_c(c)}{\lambda} \tag{31}$$

$$
\begin{aligned}
1 &= f_1'[u_c(c,\lambda)]u_{cc} + f_2' \frac{u_c(c,\lambda)}{\lambda}\frac{u_{cc}}{\lambda} \\
&\quad \text{Note that, by definition of the f-functions,} \\
u_i[f_i(x)] &= x \\
u_i''[f_i(x)].f_i'(x) &= 1 \\
f_i'(x) &= \frac{1}{u_i''[f_i(x)]} \\
&\quad \text{Plug into 12-A,} \\
1 &= \frac{u_{cc}}{u_1''[f_1(u_c(c,\lambda))]} + \frac{1}{\lambda}\frac{u_{cc}}{u_2''[f_2\frac{u_c}{\lambda}]}
\end{aligned}
$$

$$
\frac{1}{u_{cc}(c,\lambda)} = \frac{1}{u_1''[f1(u_c(c,\lambda))]} + \frac{1}{\lambda}\frac{1}{u_2''[f_2\frac{u_c(c,\lambda)}{\lambda}]} \tag{32}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\delta}[eq31]:0 &= f_1'[u_c(c,\lambda)]u_{c\lambda}(c,\lambda) + f_2'\frac{u_c(c,\lambda)}{\lambda}\left[\frac{1}{\lambda}u_{c\lambda}(c,\lambda) - \frac{u_c(c,\lambda)}{\lambda^2}\right] \\
&= \frac{u_{c\lambda}(c,\lambda}{u_1''[f_1(u_c(c,\lambda))]} + \frac{1}{\lambda}\frac{1}{u_2''[f_2\frac{u_c(c,\lambda)}{\lambda}]}\frac{\lambda u_{c\lambda} - u_c(c,\lambda)}{\lambda} \\
&\quad \text{use 32,} \\
&= \frac{u_{c\lambda}(c,\lambda}{u_1''[f_1(u_c(c,\lambda))]} + \left[\frac{1}{u_{cc}(c,\lambda)} - \frac{1}{u_1'[f_1(u_c(c,\lambda))]}\right]\frac{\lambda u_{c\lambda} - u_c(c,\lambda)}{\lambda} \\
&= \frac{u_c(c,\lambda)}{\lambda u_1''[f1(u_c(c,\lambda))]} + \frac{\lambda u_{c\lambda} - u_c(c,\lambda)}{\lambda u_{cc}(c,\lambda)}
\end{aligned}
$$

$$
u_1''[f_1(u_c(c,\lambda))] = \frac{u_c(c,\lambda)u_{cc}(c,\lambda)}{u_c(c,\lambda) - \lambda u_{c\lambda}(c,\lambda)} \tag{33}
$$

Proof of theorem 1 : 2 steps:

(i) show $\{c_i\}$ financed by $\{\alpha,\theta\}$

(ii) show $\{c_i\}$ optimal

They seem to be assuming eq 25 in step 1 ...

Clearly from 25, $d[e^{-rhot}u_c] = -re^{-\rho t}u_c dt + ()dw$
The stochastic term is $e^{-\rho t}[u_{cc}dc|_{stoch} + u_{c\lambda}d\lambda|_{stoch}]$ using eq 23
$= e^{-\rho t}[u_{cc}\sigma_\delta\partial w + u_{c\lambda}\frac{u_1''}{u_c}\lambda\sigma_\delta\partial w]$, using eq 33.

$$
\begin{aligned}
\Rightarrow d(e^{-\rho t}u_c) &= e^{-\rho t}u_c r dt + e^{-\rho t}\left[u_{cc} + \frac{u_{c\lambda}}{u_c}\lambda\left(\frac{u_c u_{cc}}{u_c - \lambda u_{c\lambda}}\right)\right]\sigma_\delta\partial w \\
&= e^{-\rho t}u_c r dt + \left(\frac{e^{-\rho t}}{u_c - \lambda u_{c\lambda}}\right)[u_{cc}u_c - \lambda u_{cc}u_{c\lambda} + \lambda u_{cc}u_{c\lambda}]\sigma_\delta\partial w
\end{aligned}
$$

use 33 again,

$$
d(e^{-\rho t}u_c) = e^{-\rho t}u_c r dt + e^{-\rho t}u_1''\sigma_\delta\partial w \tag{34}
$$

$$
\begin{aligned}
d\left(\frac{e^{-\rho t}u_c}{\lambda}\right) &= \frac{1}{\lambda}d(e^{-\rho t u_c}) - \frac{e^{-\rho t}u_c}{\lambda}\frac{d\lambda}{\lambda} - \frac{e^{-\rho t}}{\lambda}du_c\frac{d\lambda}{\lambda} + \frac{1}{2}2e^{-\rho t}u_c\frac{1}{\lambda}\frac{\partial\lambda}{\lambda^2} \\
&= \frac{1}{\lambda}\left\{-e^{-\rho t}u_c r dt + e^{-\rho t}u_1''\sigma_\delta\partial w\right\} - \frac{e^{-\rho t}u_c}{\lambda}\frac{u_1''}{u_c}\sigma_\delta\partial w - \frac{e^{-\rho t}}{\lambda}u_1''\sigma_\delta\frac{u_1}{u_c} + \frac{e^{-\rho t}}{\lambda}u_c\left(\frac{u_1''}{u_c}\right)^2\sigma_\delta^2 dt
\end{aligned}
$$

$$
d\left(\frac{e^{-\rho t}u_c}{\lambda}\right) = -\left(\frac{e^{-\rho t}u_c}{\lambda}\right) r dt \tag{35}
$$

Compare with $\frac{dB}{B} = rdt \Rightarrow d(\frac{1}{B}) = -\frac{1}{B^2}rBdt = -(\frac{1}{B})rdt$
$\Rightarrow \frac{d(1/B)}{1/B} = -rdt = \frac{d\left(\frac{e^{-\rho t}u_c}{\lambda}\right)}{\left(\frac{e^{-\rho t}u_c}{\lambda}\right)}$
$\Rightarrow B(t) - \Psi\frac{e^{\rho t}\lambda(t)}{u_c(t)}$ for some $\Psi \Rightarrow B(0) = 1 = \Psi\frac{\lambda(0)}{u_c(0)}$

$$B(t) = e^{\rho t}\frac{\lambda(t)}{\lambda(0)}\frac{u_c(0)}{u_c(t)} \tag{36}$$

$$\begin{aligned}
c_2'(w_t) &= f_2\left[\frac{u_c(\delta_t,\lambda_t)}{\lambda_t}\right] = \frac{\lambda(t)}{u_c(\delta_t,\lambda_t)} \text{ due to log utility} \\
&= e^{-\rho t}B(t)\frac{\lambda(0)}{u_c(0)} = e^{-\rho t}B(t)\frac{\beta\rho}{1-e^{-\rho t}} \\
& \quad \text{We also have,} \\
w_2(w_t) &= E_t\left[\int_t^T dse^{-\rho(s-t)}\frac{u_2'(c_s)}{u_2'(c_t)}c(w_s)\right], u_2'(x) = \frac{1}{x} \\
&= c(w_t)\frac{1}{\rho}(1-e^{-\rho(T-t)}) \\
&= \beta\frac{\rho e^{-\rho t}}{1-e^{-\rho t}}B(t)\frac{1}{\rho}(1-e^{-\rho(T-t)}) \\
&= \beta\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}B(t) \\
dw_2 &= d\left[\beta\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}B(t)\right] \\
&= \beta\left(\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}\right)rBdt - \rho\frac{\beta B(t)}{1-e^{-\rho T}}e^{-\rho t}dt \\
&\equiv (\alpha_2 r - c_2)dt \\
&= \left[\alpha_2 r - \beta\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}B(t)\right] \\
\alpha_2 &= \beta\left(\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}\right)B
\end{aligned}$$

Step 2:

$$\begin{aligned}
w_1(t) &= s(t) - w_2(t) \\
&= E_t\left[\int_t^T dve^{-\rho(v-t)}\frac{u_c(v)}{u_c(t)}f_1(f_c(v))\right] \\
u_c(t)e^{-\rho t}s(t) - u_c(t)e^{-\rho t}\left[w_2(t) = \frac{1-e^{-\rho(T-t)}}{\rho}\frac{\lambda(t)}{u_c(t)}\right] &= E_t\left[\int_t^T dve^{-\rho v}u_c(v)f_1(u_c(v))\right] \\
E_t\left[\int_t^T dve^{-\rho v}u_c(v)f_1(u_c(v))\right] &= u_c(t)e^{-\rho t}s(t) - \frac{e^{-\rho t}-e^{-\rho T}}{\rho}\lambda(t) \\
&\quad + \int_0^T dve^{-\rho v}u_c(v)f_c(u_c(v)) \\
&= \text{p-martingale}
\end{aligned}$$