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Derive GE without a representative agent:

$$\begin{aligned}
1)\ \frac{dY}{Y} &= \mu_Y dt + \sigma_Y \partial w \\
2)\ \frac{dY^*}{Y^*} &= \mu_Y^* dt + \sigma_Y^* \partial w \\
P(w_*) &\equiv \text{ price of home good in state } -w_* \text{ in term os numerative} \\
P^*(w_*) &\equiv \text{ price of foreign good in state } -w_* \text{ in term os numerative} \\
q(w_*) &\equiv \frac{P(w_*)}{P^*(w_*)}
\end{aligned}$$

Assume complete set of ... securities with dated prices $Q_0(w_*)$
Home:

$$\begin{aligned}
L &= \int_0^T dte^{-\rho*} \int \partial w_* D_0(w_*)\theta_H(w_*) \left\{a_H L C_H(w_*) + (1-a_H)LC_H^*(w_*)\right\} \\
&\quad +\beta_H \left\{w_H(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_H(w_*)\rho(w_*) + C_H^*(w_*)\rho^*(w_*)\right]\right\} \\
\frac{\partial}{\partial C_H(w_*)} = 0 &= e^{-p*} D_0(w_*)\theta_H(w_*) a_H \left(\frac{1}{C_H(w_*)}\right) - \beta_H \theta_0(w_*)\rho(w_*) \forall w_* \\
\frac{\partial}{\partial C_H^*(w_*)} = 0 &= e^{-p*} D_0(w_*)\theta_H(w_*)(1-a_H) \left(\frac{1}{C_H^*(w_*)}\right) - \beta_H \theta_0(w_*)\rho^*(w_*) \forall w_* \\
\frac{\partial}{\partial \beta_H} = 0 &= w_H(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_H(w_*)\rho(w_*) + C_H^*(w_*)\rho^*(w_*)\right] \\
&\Rightarrow \quad C_H(w_*)\rho(w_*) = \left(\frac{1}{\beta_H}\right)\left(\frac{1}{\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_H(w_*) a_H \\
&\Rightarrow \quad C_H^*(w_*)\rho^*(w_*) = \left(\frac{1}{\beta_H}\right)\left(\frac{1}{\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_H(w_*)(1-a_H)
\end{aligned}$$

Foreign:

$$\begin{aligned}
L &= \int_0^T dte^{-\rho*} \int \partial w_* D_0(w_*)\theta_H(w_*) \left\{a_F L C_F(w_*) + (1-a_F)LC_F^*(w_*)\right\} \\
&\quad +\beta_F \left\{w_F(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_F(w_*)\rho(w_*) + C_F^*(w_*)\rho^*(w_*)\right]\right\} \\
\frac{\partial}{\partial C_F(w_*)} = 0 &= e^{-p*} D_0(w_*)\theta_F(w_*) a_F \left(\frac{1}{C_F(w_*)}\right) - \beta_F \theta_0(w_*)\rho(w_*) \forall w_* \\
\frac{\partial}{\partial C_F^*(w_*)} = 0 &= e^{-p*} D_0(w_*)\theta_F(w_*)(1-a_F) \left(\frac{1}{C_F^*(w_*)}\right) - \beta_F \theta_0(w_*)\rho^*(w_*) \forall w_* \\
\frac{\partial}{\partial \beta_F} = 0 &= w_F(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_F(w_*)\rho(w_*) + C_F^*(w_*)\rho^*(w_*)\right] \\
&\Rightarrow \quad C_F(w_*)\rho(w_*) = \left(\frac{1}{\beta_F}\right)\left(\frac{1}{\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_F(w_*) a_F \\
&\Rightarrow \quad C_F^*(w_*)\rho^*(w_*) = \left(\frac{1}{\beta_F}\right)\left(\frac{1}{\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_F(w_*)(1-a_F)
\end{aligned}$$

$$\begin{aligned} W_F(0) &= \frac{1}{\beta_F}\int dt \int \partial w_* e^{-\rho *} D_0(w_*)\theta_F(w_*)\left[a_F + 1 - a_F\right] \\ &= \frac{1}{\beta_F}\int dt \int \partial w_* e^{-\rho *} D_0(w_*)\theta_F(w_*) \\ &= \frac{1}{\rho\beta_F} \text{ using } E\left[\theta_F(w_*)\right] = \theta_F(0) = 1 \\ & \quad \text{vs} \\ W_H(0) &= \frac{1}{\beta_H}\int dt \int \partial w_* e^{-\rho *} D_0(w_*)\theta_H(w_*) \\ &= \frac{1}{\rho\beta_H} \end{aligned}$$

In partial equilibrium with $\{\theta(w_*), D_0(w_*), [\theta_0(w_*), w_H(0), w_F(0), \rho(w_*), \rho^*(w_*)]\}$ exogenous, we would be done, and these last two equations would define $\{\beta_H, \beta_F\}$

From partial equation, I have $4x(*x\omega_*) + 2$ equations

$$C_H(w_*)\rho(w_*) = \left(\frac{1}{\beta_H\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_H(w_*) a_H \forall(*, w_*) \tag{1}$$

$$C_H^*(w_*)\rho^*(w_*) = \left(\frac{1}{\beta_H\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_H(w_*)(1 - a_H) \forall(*, w_*) \tag{2}$$

$$C_F(w_*)\rho(w_*) = \left(\frac{1}{\beta_F\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_F(w_*) a_F \forall(*, w_*) \tag{3}$$

$$C_F^*(w_*)\rho^*(w_*) = \left(\frac{1}{\beta_F\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_F(w_*)(1 - a_F) \forall(*, w_*) \tag{4}$$

$$w_H(0) = \left(\frac{1}{\beta_H}\right)\int dt \int \partial w_* e^{-\rho *} D_0(w_*)\theta_H(w_*) \tag{5}$$

$$w_F(0) = \left(\frac{1}{\beta_H}\right)\int dt \int \partial w_* e^{-\rho *} D_0(w_*)\theta_H(w_*) \tag{6}$$

Here in partial equation, we take $\{\theta(w_*), D_0(w_*), [\theta_0(w_*), w_H(0), w_F(0), \rho(w_*), \rho^*(w_*)]\}$ exogenous

In general equation, these $2 \neq 3x(*x\omega_*)$ are determined via

$$s(0) = w_H(0) = \int dt \int \partial w_* \theta_0(w_*) Y(w_*)\rho(w_*) \text{ leq} \tag{7}$$

$$s^*(0) = w_F(0) = \int dt \int \partial w_* \theta_0(w_*) Y^*(w_*)\rho^*(w_*) \text{ leq} \tag{8}$$

$$Y(w_*) = C_H(w_*) + C_F(w_*) \ (*x\omega_*) \text{ eq} \tag{9}$$

$$Y^*(w_*) = C_H^*(w_*) + C_F^*(w_*) \ (*x\omega_*) \text{ eq} \tag{10}$$

$$1 = \alpha\rho(w_*) + (1-\alpha)\rho^H(w_*) \ (*x\omega_*) \text{ eq} \tag{11}$$

$$(11^*)\ \frac{\theta_0(0)}{D_0(0)} = 1 \Rightarrow \text{ Need this because without, can only determine ratio } \left(\frac{\beta_H}{\beta_F}\right) \tag{11*}$$

$\Rightarrow$ This forces price of consumption unit today equal to 1.

To solve, I will first determine $q(w_*) \equiv \frac{\rho(w_*)}{\rho^H(w_*)}$ and then use equation 11 to solve for $\rho(w_*), \rho^*(w_*)$. I will then solve for $\theta_0(w_*)$ and finally solve for $w_H(0), w_F(0)$.

Solve for $q(w_*)$:

$$\begin{aligned}(1,3,9): \rho(w_*)Y(w_*) &= \rho(w_*)C_H(w_*) + \rho(w_*)C_F(w_*) \\ &= \left(\frac{1}{\beta_H\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_H(w_*)a_H + \\ &\quad \left(\frac{1}{\beta_F\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_F(w_*)a_F \end{aligned} \tag{12}$$

$$\begin{aligned}(2,4,10): \theta_0(w_*)\rho^*(w_*)Y^*(w_*) &= \theta_0(w_*)\rho^*(w_*)C_H^*(w_*) + \theta_0(w_*)\rho^*(w_*)C_F^*(w_*) \\ &= e^{-\rho*}D_0(w_*)\left\{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)(1-\theta_H)+\right. \\ &\quad \left.\left(\frac{1}{\beta_F}\right)\theta_F(w_*)(1-\theta_F)\right\} \end{aligned} \tag{13}$$

$$\frac{12}{13}: q(w_*)\frac{Y(w_*)}{Y^*(w_*)} = \frac{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)a_H + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F}{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)(1-a_H) + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)(1-a_F)} \tag{13b}$$

This agrees with equation in P/R if $\frac{1}{\beta_H} \sim \lambda_H, \frac{1}{\beta_F} \sim \lambda_F$ which is the typical asset that lagrange multiplier of agent reciprocal of lagrange multiplier of agents weight in social planner problem.

Eq 13b gives $q(w_*)$ as a fact of $\{\beta_H, \beta_F\}$, and exogenous variables -

$$(11) \Rightarrow \frac{1}{\rho(w_*)} = \alpha + (1-\alpha)\frac{1}{q(w_*)}$$

$$\Rightarrow \rho(w_*) = \frac{q(w_*)}{(1-\alpha)+\alpha q(w_*)} \tag{13c}$$

$$(11) \Rightarrow \frac{1}{\rho^*(w_*)} = \alpha q(w_*) + (1-\alpha)$$

$$\Rightarrow \rho(w_*) = \frac{1}{(1-\alpha)+\alpha q(w_*)} \tag{13d}$$

Since $q(w_*)$ written in exogenous terms, so are $\{\rho(w_*), \rho^*(w_*)\}$

To solve for $\theta_0(w_*)$, use 12, 13 and 11

$$12: \alpha\rho\theta_0(w_*) = \alpha e^{-\rho*}D_0(w_*)\left(\frac{1}{Y(w_*)}\right)\left\{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)a_H + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F\right\} \tag{14}$$

$$\begin{aligned}13: (1-\alpha)\rho^*\theta_0(w_*) &= (1-\alpha)e^{-\rho*}D_0(w_*)\left(\frac{1}{Y^*(w_*)}\right)\left\{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)a_H\right. \\ &\quad \left. + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F\right\} \end{aligned} \tag{15}$$

Combine 14, 15, and use in 11:

$$\begin{aligned}\theta_0(w_*) = e^{-\rho*}D_0(w_*)&\left\{\left(\frac{\alpha}{Y(w_*)}\right)\left[\frac{1}{\beta_H}\theta_H(w_*)a_H + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F\right] + \left(\frac{1-\alpha}{Y^*(w_*)}\right)\right. \\ &\left.\left[\frac{1}{\beta_H}\theta_H(w_*)(1-a_H) + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)(1-a_F)\right]\right\} \end{aligned} \tag{15*}$$

This is consistent with $q(w_*) \equiv \frac{\theta_0(w_*)}{D_0(w_*)}$ if $\frac{1}{\beta_H} = \lambda_H, \frac{1}{\beta_F} = \lambda_F$

Now, solve for $\{w_H(0), w_F(0)\}$. For $w_H(0)$, use (7), (12)

$$
\begin{aligned}
w_H(0) &= \int dt \int \partial w_* e^{-\rho *}(w_*) \left\{ \frac{1}{\beta_H}\theta_H(w_*)a_H + \frac{1}{\beta_F}\theta_F(w_*)a_F \right\} \\
&\quad \text{use fact that } \theta_H, \theta_F \text{ are marginals: } \int \partial w_* D_0(w_*) = \theta_H(0) = 1, \forall * \\
&= \left(\frac{1}{\rho}\right)\left(\frac{a_H}{\beta_H} + \frac{a_F}{\beta_F}\right) \qquad (16) \\
&\quad \text{Similarly, for } w_F(0) \text{, use (8) and (13)} \\
w_F(0) &= \int dt \int \partial w_* e^{-\rho *} D_0(w_*) \left\{ \frac{1}{\beta_H}\theta_H(w_*)(1-a_H) + \frac{1}{\beta_F}\theta_F(w_*)(1-a_F) \right\} \\
&= \left(\frac{1}{\rho}\right)\left(\frac{1-a_H}{\beta_H} + \frac{1-a_F}{\beta_F}\right) \qquad (17) \\
&\quad \text{But from partial eq, we have } w_H(0) = \tfrac{1}{\rho\beta_H}, w_F(0) = \tfrac{1}{\rho\beta_F} \\
&\quad \text{Combining,} \\
1 &= a_H + a_F\frac{\beta_H}{\beta_F}, 1 = (1-a_H)\frac{\beta_F}{\beta_H} + (1-a_F) \\
\frac{\beta_H}{\beta_F} &= \frac{1-a_H}{a_F} \qquad (18) \\
&\quad \text{Plugging back we get,}
\end{aligned}
$$

$$
\frac{w_H(0)}{w_F(0)} = \frac{\beta_F}{\beta_H} = \frac{a_F}{1-a_H} \qquad (19)
$$

$$
\begin{aligned}
&\quad \text{Also from 14, 18 we have:} \\
q(0) &= \frac{Y^*(0)}{Y(0)}\left[\frac{\frac{a_H}{\beta_H} + \frac{a_F}{\beta_F}}{\frac{1-a_H}{\beta_H} + \frac{1-a_F}{\beta_F}}\right] \\
&= \frac{Y^*(0)}{Y(0)}\left[\frac{\frac{a_F a_H}{1-a_H} + a_F}{a_F + (1-a_F)}\right] \\
&= \frac{Y^*(0)}{Y(0)}\frac{a_F}{a-a_H} \qquad (20) \\
&\quad \text{use equation 11* and 15*: } \theta_H(0) = 1, \theta_F(0) = 1 \\
1 &= \frac{\theta_0(0)}{D_0(0)} = \frac{\alpha}{Y(0)}\left[\frac{a_H}{\beta_H} + \frac{a_F}{\beta_F}\right] + \frac{1-\alpha}{Y^*(0)}\left[\frac{1-a_H}{\beta_H} + \frac{1-a_F}{\beta_F}\right] \\
&\quad \text{Use equation 19 : } \tfrac{\beta_F}{\beta_H} = \tfrac{a_F}{a-a_H} \\
\beta_F &= \frac{\alpha}{Y(0)}\left[a_H\frac{a_F}{1-a_H} + a_F\right] + \frac{1-\alpha}{Y^*(0)}\left[(1-a_H)\frac{a_F}{1-a_H} + (1-a_F)\right] \\
&= \frac{1}{Y(0)}\frac{\alpha}{1-a_H}\left[a_H a_F + a_F(1-a_H)\right] + (1-\alpha)\frac{1}{Y^*(0)} \\
&= \frac{\alpha}{Y(0)}\frac{a_F}{1-a_H} + \frac{1-\alpha}{Y^*(0)} \\
&= \alpha\left(\frac{a_F}{a-a_H}\right)\left(\frac{q(0)}{Y^*(0)}\right)\frac{\frac{1}{\beta_H}(1-a_H) + \frac{1}{\beta_F}(1-a_F)}{\frac{1}{\beta_H}a_H + \frac{1}{\beta_F}a_F} + \frac{1-\alpha}{Y^*(0)} \\
&= \frac{1}{Y^*(0)}\left\{\alpha\left(\frac{a_F}{1-a_H}\right)q(0)\left[\frac{\left(\frac{a_F}{1-a_H}\right)(1-a_H) + (1-a_F)}{\left(\frac{a_F}{1-a_H}\right)a_H + a_F}\right] + (1-\alpha)\right\} \\
&= \frac{1}{Y^*(0)}\left\{\alpha\left(\frac{a_F}{1-a_H}\right)q(0)\left[\frac{1-a_H}{a_F a_H + a_F(1-a_H)}\right] + (1-\alpha)\right\} \\
&= \frac{1}{Y^*(0)}\left\{\alpha q(0) + (1-\alpha)\right\} \\
\beta_F &= \frac{1}{Y^*(0)}\frac{1}{\rho^*(0)} \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
& \text{Back to 13b} \\
q(0)\left(\frac{Y(0)}{Y^*(0)}\right) &= \frac{\frac{a_H}{\beta_H}+\frac{a_F}{\beta_F}}{\frac{1-a_F}{\beta_H}+\frac{1-a_F}{\beta_F}} \\
&= \frac{a_H\frac{a_F}{1-a_H}+a_F}{(1-a_H)\frac{a_F}{1-a_H}+(1-a_F)} \\
&= \frac{a_F}{1-a_H} \\
&= \frac{\beta_F}{\beta_H} \\
\text{Thus, } \beta_H &= \beta_F\left(\frac{Y^*(0)}{Y(0)}\right)\left(\frac{\rho^*(0)}{\rho(0)}\right) \\
&= \frac{1}{\rho(0)Y(0)} \qquad (22) \\
& \text{From 5, 6:} \\
s(0) &= w_H(0)=\frac{1}{\rho\beta_H}=\frac{\rho(0)Y(0)}{\rho} \qquad (23) \\
s^*(0) &= w_F(0)=\frac{1}{\rho\beta_F}=\frac{\rho^*(0)Y(0)}{\rho} \qquad (24) \\
\text{1x3:}\frac{C_H(w_*)}{C_F(w_*)} &= \frac{\beta_F}{\beta_H}\frac{\theta_H(w_*)}{\theta_F(w_*)}\frac{a_H}{a_F} \\
Y(w_*) &= C_H(w_*)\left[1+\frac{\beta_H a_F\theta_F}{\beta_F\theta_H a_H}\right] \\
\Rightarrow C_H(w_*) &= Y(w_*)\frac{\beta_F a_H\theta_H}{\beta_F a_H\theta_H+\beta_H a_F\theta_F} \\
&= Y(w_*)\frac{\frac{1}{\beta_H}a_H\theta_H}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F} \qquad (25) \\
\Rightarrow C_F(w_*) &= Y(w_*)-C_H(w_*) \\
&= Y(w_*)\left(\frac{\frac{1}{\beta_F}a_F\theta_F}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F}\right) \qquad (26) \\
\Rightarrow C_H^*(w_*) &= Y^*(w_*)\frac{\beta_F a_H\theta_H}{\beta_F a_H\theta_H+\beta_H a_F\theta_F} \\
&= Y(w_*)\frac{\frac{1}{\beta_H}a_H\theta_H}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F} \\
\Rightarrow C_F^*(w_*) &= Y^*(w_*)-C_H^*(w_*) \\
&= Y^*(w_*)\left(\frac{\frac{1}{\beta_F}a_F\theta_F}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F}\right) \\
& \text{From here I go back to P/R's notation,} \\
(2):q &= \frac{\lambda_H\theta a_H+\lambda_F a_F}{\lambda_H\theta(1-a_H)+\lambda_F(1-a_F)}\frac{Y^*}{Y},\theta\equiv\frac{\theta_H}{\theta_F} \\
Lq &= L\left[\lambda_H a_H\theta+\lambda_F a_F\right]-L\left[\lambda_H(1-a_H)\theta+\lambda_F(1-a_F)\right]+LY^*-LY \\
dLq &= \left(\frac{\lambda_H a_H}{\lambda_H a_H\theta+\lambda_F a_F}\right)d\theta-\frac{\lambda_H(1-a_H)}{\lambda_H(1-a_H)\theta+\lambda_F(1-a_F)}d\theta+\frac{dY^*}{Y^*}-\frac{dY}{Y}+\theta(d*) \\
&\equiv Ad\theta+\frac{dY^*}{Y^*}-\frac{dY}{Y}+\theta(d*) \\
& \text{where, } A\equiv\left(\frac{\lambda_H a_H}{\lambda_H a_H\theta+\lambda_F a_F}\right)-\left(\frac{\lambda_H(1-a_H)}{\lambda_H(1-a_H)\theta+\lambda_F(1-a_F)}\right) \\
\frac{dq}{q} &= ()d*+\left[Ad\theta-\sigma_Y dw+\sigma_Y^*\partial w^*\right] \qquad (27)
\end{aligned}
$$