Bansal Yaron

$$
\begin{aligned}
R_{*,*+1} = P_{*+1} + D_{*+1} &= \left(\frac{P_{*+1} + D_{*+1}}{D_{*+1}}\right)\left(\frac{D_{*+1}}{D_*}\right)\left(\frac{D_*}{P_*}\right) \\
Define, &\quad Z_* = L_r\left(\frac{P_*}{D_*}\right), r \equiv L_r R, g = L\left(\frac{D_{*+1}}{D_*}\right) \\
r &= L_r\left[e^{Z_{*+1}} + 1\right] + g - Z_* \\
&= L_r\left[\frac{e^{Z_{*+1}} + 1}{e^{Z_*} + 1}\right] + L_r\left[e^{\bar{Z}} + 1\right] + g - Z_* \\
&= L_r\left[\frac{e^{\bar{Z}} e^{Z_{*+1} - \bar{Z}} + 1}{e^{\bar{Z}} + 1}\right] + L_r\left[e^{\bar{Z}} + 1\right] + g - Z_* \\
&\approx L_r\left[\frac{e^{\bar{Z}\left(1 + \left(Z_{*+1} - \bar{Z}\right)\right)} + 1}{e^{\bar{Z}} + 1}\right] + L_r\left[e^{\bar{Z}} + 1\right] + g - Z_* \\
&\approx L_r\left[1 + \left(\frac{e^{\bar{Z}}}{e^{\bar{Z}+1}}\right)\left(Z_{*+1} - \bar{Z}\right)\right] + L_r\left[e^{\bar{Z}} + 1\right] + g - Z_* \\
&\approx \left(\frac{e^{\bar{Z}}}{e^{\bar{Z}+1}}\right)\left(Z_{*+1} - \bar{Z}\right) + L_r\left[e^{\bar{Z}} + 1\right] + g - Z_* \\
&= K_0 + K_1 Z_{*+1} - Z_* + g, \text{where,} \\
&\quad K_0 = L\left[e^{\bar{Z}} + 1\right] - \bar{Z}\left(\frac{e^{\bar{Z}}}{e^{\bar{Z}+1}}\right) \\
&\quad K_1 = \left(\frac{e^{\bar{Z}}}{e^{\bar{Z}+1}}\right) \\
\text{For } e^{\bar{Z}} &\approx (25)(12)\,[\text{ the 12 is for monthly }] \\
&= 300 \\
\Rightarrow K_1 &\approx \frac{300}{301} = 0.9966
\end{aligned}
$$

Campbell/ Shdler approximation of an identity:
Since g exogenous, a solution for Z leads to a complete characterization of the return of r

$$
\begin{aligned}
r_{*,*+1} &\cong K_0 + K_1 Z_{*+1} - Z_* + g_{*+1} \\
&\quad \epsilon/Z \text{ Pacing Kernel} \\
M_{*,*+1} &= \theta L_r \delta - \frac{\theta}{\psi} g_{*+1} + (\theta - 1)\, r_{a,*+1} \\
&\quad \text{Note : when } \theta = 1 \text{ which imples } \frac{1}{\psi} = r \\
&\quad \Rightarrow m = L_r \delta - r g_{*+1} = CRRA \\
&\quad \text{ exogenously specify Lucas consumption dynamics,} \\
g_{*+1} &= \mu + X_* + \sigma n_{*+1} \\
g^d_{*+1} &= \mu_d + Q X_* + Q_d \sigma n_{*+1} \\
X_{*+1} &= \rho X_* + Q_e \sigma e_{*+1} \\
1 &= E\left[e^{m+r}\right] \\
&= E\left[e^{\left(\left[\theta L_r \delta - \frac{\theta}{\psi} g_{*+1} + (\theta - 1)\right] + r_{a,*+1}\right)}\right] \\
&= E\left[e^{\left[\theta L_r \delta - \frac{\theta}{\psi} g_{*+1} + \theta\left[K_0 + K_1 Z_{*+1} - Z_* + g_{*+1}\right]\right]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E\left[e^{\left[\theta L_r\delta+\theta\left(1-\frac{1}{\psi}\right)g_{*+1}+\theta K_0+\theta K_1 Z_{*+1}-\theta Z_*\right]}\right] \\
1 &= e^{\theta L_r\delta+\theta K_0-\theta Z_*}E_* e^{\theta\left(1-\frac{1}{\psi}\right)(\mu+X_*+\sigma n_{*+1})+\theta K_1 Z_{*+1}} \\
\text{Guess, } Z_* &= Z_0+A_1A_*, Z_{*+1}=A_0+A_1X_{*+1} \\
1 &= e^{\theta L_r\delta+\theta K_0-\theta(A_0+A_1X_*)+\theta\left(1-\frac{1}{\psi}\right)(\mu+X_*)+\theta K_1 A_0}Ee^{\theta\left(1-\frac{1}{\psi}\right)\sigma n_{*+1}+\theta K_1A_1(\rho X_*+Q_e\sigma e_{*+1})} \\
0 &= \theta L_r\delta+\theta K_0-\theta\left(A_0+A_1X_*\right)+\theta\left(a-\frac{1}{\psi}\right)(\mu+X_*)+\theta K_1A_0+\theta K_1A_1\rho X_*+\frac{1}{2}\theta^2\left(1-\frac{1}{\psi}\right)^2\sigma^2 \\
&\quad +\frac{1}{2}\theta^2K_1^2A_1^2Q_e^2\sigma^2 \\
X':0 &= -\theta A_1+\theta\left(1-\frac{1}{\psi}\right)+\theta K_1A_1\rho \\
X^0:0 &= \theta L_r\delta+\theta K_0-\theta A_0+\theta\left(1-\frac{1}{\psi}\right)\mu+\theta K_1A_0+\frac{\sigma^2}{2}\theta^2\left(1-\frac{1}{psi}\right)^2+\frac{\sigma^2}{2}\theta^2K_1^2A_1^2 \\
X':0 &= A_1\left(K_1\rho-1\right)+\left(1-\frac{1}{psi}\right) \\
\Rightarrow A_1 &= \frac{1-\frac{1}{\psi}}{1-K_1\rho} \\
M_{*,*+1} &= \theta L_r\delta-\frac{\theta}{\psi}g_{*+1}+(\theta-1)\left(K_0+K_1Z_{*+1}-Z_*+g_{*+1}\right) \\
&= \theta L_r\delta+(\theta-1)\left(L_0-Z_*\right)+(\theta-1)K_1\left(A_0+A_1X_{*+1}\right)+\left(\theta-1-\frac{\theta}{\psi}\right)\left(\mu+X_*+\sigma n_{*+1}\right) \\
&= \theta L_r\delta+(\theta-1)\left(L_0-Z_*\right)+(\theta-1)K_1A_0+\left(\theta-1-\frac{\theta}{\psi}\right)\left(\mu+X_*\right) \\
&\quad +(\theta-1)K_1A_1\left(\rho X_*+Q_e\sigma e_{*+1}\right)+\left(\theta-1-\frac{\theta}{\psi}\right)\sigma n_{*+1} \\
&= \theta L_r\delta+(\theta-1)\left(K_0-\left(A_0+A_1X_*\right)\right)+(\theta-1)K_1A_0+(\theta-1)K_1A_1\rho X_* \\
&\quad (\theta-1)K_1A_1Q_e\sigma e_{*+1}+\left(\theta-1-\frac{\theta}{\psi}\right)\sigma n_{*+1} \\
&= E_*\left(m_{*+1}\right)-(1-\theta)\left(K_1\frac{1-\frac{1}{\psi}}{1-K_1\rho}Q_e\right)\sigma e_{*+1}+\left(\theta-1-\frac{\theta}{\psi}\right)\sigma n_{*+1} \\
&= E_*\left(m_{*+1}\right)-\lambda_{m,e}\sigma e_{*+1}+\lambda_{m,n}\sigma n_{*+1} \\
&\quad \text{where, } \lambda_{m,e}\equiv(1-\theta)\left(K_1\left(\frac{1-\frac{1}{\psi}}{1-K_1\rho}\right)Q_e\right) \\
&\quad \lambda_{m,n}\equiv-\frac{\theta}{\psi}+\theta-1 \\
e^{-r_F}\equiv\frac{1}{R_F} &= E_*M_{*,*+1}=E_*e^{M_{*,*+1}}=e^{E(m)+\frac{1}{2}Var(m)} \\
r_F &= -E\left(m\right)-\frac{1}{2}Var\left(m\right) \\
&= -\left(\theta L_r\delta+(\theta-1)\left(K_0-A_0+K_1A_0\right)+\left(\theta-1-\frac{\theta}{\psi}\right)\mu\right) \\
&\quad -X_*\left(-(\theta-1)A_1+\left(\theta-1-\frac{\theta}{\psi}\right)+(\theta-1)K_*-\frac{1}{2}Var\left(m\right)\right)
\end{aligned}
$$

$$
\text{Term Linear in X} = (\theta-1)A_1\left(1-K_1\rho\right)-\left(\theta-1-\frac{\theta}{\psi}\right)
$$

$$
\begin{aligned}
&= (\theta-1)\left(1-\frac{1}{\psi}\right)-\left(\theta-1-\frac{\theta}{\psi}\right) \\
&= -\left(\frac{1}{\psi}\right)(\theta-1-\theta) \\
&= \frac{1}{\psi} \\
e^0=1 &= E\left(e^{m+r_i}\right)e^{E(m)+E(r_i)+\frac{1}{2}Var(m)+\frac{1}{2}Var(i)+cov(m,r_i)} \\
0 &= \left(-r_F-\frac{1}{2}Var\left(m\right)\right)+E\left(r_i\right)+\frac{1}{2}Var\left(m\right)+\frac{1}{2}Var\left(i\right)+cov\left(m,r_i\right) \\
\Rightarrow E\left(r_i\right)-r_F &= -cov(m,r_i)-\frac{1}{2}Var\left(r_i\right)
\end{aligned}
$$

For Market portfolio, where,

$$
\begin{aligned}
r_m &= K_{0,m}+K_{1,m}Z_m(*+1)-Z_m(*)+g_{d,*+1} \\
&= const+K_{1,m}A_{1,m}Q_e\sigma e_{*+1}+Q_d\sigma u_{*+1} \\
-cov\left(m,r\right) &= -\left(K_{1,m}A_{1,m}Q_e\sigma\left(-\lambda_{m,e}\sigma\right)\right) \\
&= K_{1,m}\frac{\theta 0\frac{1}{\psi}}{1-K_{1,m}\rho}Q_e\lambda_{m,e}\sigma^2 \\
&= \beta_{m,e}\lambda_{m,e}\sigma^2 \\
&\quad \text{where, } \beta_{m,e}\equiv K_{1,m}\left(\frac{Q-\frac{1}{\psi}}{1-K_{1,m}\rho}\right)Q_e
\end{aligned}
$$