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Exchange economy, with exogenous dividned process

$$d\delta = \mu_\delta dt + \sigma_\delta dw$$

Bond and Stock process :

$$\begin{aligned}\frac{dB}{B} &= rdt \\ \frac{ds+\delta dt}{s} &= \mu dt + \sigma \partial w\end{aligned}$$

(r, μ, σ) to be determined endogenously in equilibrium.

Define:

- $\alpha \equiv$ amount in Bond
- $\theta \equiv$ amount in Stock
- $W \equiv \alpha + \theta =$ Wealth

$$\begin{aligned}dW + cdt &= (\alpha r + \theta\mu)dt + \theta\sigma d\vec{w} \\ U_i(\{c\}) &= E\left[\int_0^T e^{-\rho t} U_i(c_t)dt\right]\end{aligned}$$

Throughout assume $U_2(c_2) = Lc_2$

Endowments:
Agent 1 : Long 1 share stock, short β shares of bond
Agent 2 : Long β shares of bond

Equilibrium:
1) $w_i = \alpha_i + \theta_i = E_t[\int_0^T ds \frac{\zeta^i s}{\zeta^i t} c(s)]$ 2) Market clearing

(i) $c_1 + c_2 = \delta$

(ii) $\alpha_1 + \alpha_2 = 0$

(iii) $\theta_1 = s$

Unrestricted Case: - Dynamically complete markets
Introduce representative agent $U[c, \lambda] = E[\int_0^T e^{-\rho t} u[c_t, \lambda]dt]$ where λ is a constant and where $u[c, \lambda] = \max_{c_1+c_2=c} u_1(c_1) + \lambda u_2(c_2)$

In equal, $(c = \delta)$, so SDF follows

$$\begin{aligned}\zeta(t) &= e^{-\rho t}\frac{U_c\delta_{t,\lambda}}{U_c\delta_{0,\lambda}} \\ \frac{\partial\zeta}{\zeta} &= -rdt - K\partial w \\ K &= \frac{\mu - r}{\sigma} \\ -r = \frac{1}{dt}E\left[\frac{d\zeta}{\zeta}\right] &= \frac{1}{dt}E\left[-\rho dt + \frac{u_{cc}}{u_c}d\delta + \frac{1}{2}\frac{u_{ccc}}{u_c}d\delta^2\right]\end{aligned}$$

$$r = \rho + A\mu_\delta - \frac{1}{2}A\rho\sigma_\delta^2 \tag{12}$$

$$\begin{aligned}
\zeta_t S_t &= E_t\left[\int_t^T \zeta(u)\delta(u)du\right] \\
&\Rightarrow \zeta_t S_t + \int_t^T \zeta(u)\delta(u)du = \text{ P martingale} \\
&\Rightarrow 0 = E\left[\frac{d\zeta}{\zeta} + \frac{\partial s}{s} + \frac{d\zeta}{\zeta}\frac{\partial s}{s} + \frac{\delta}{s}dt\right] \\
&\Rightarrow E\left[\frac{ds + \delta dt}{s}\right] - rdt = -E\left[\frac{\partial\zeta}{\zeta}\frac{\partial s}{s}\right]
\end{aligned}$$

$$\mu - r = K\sigma \tag{13}$$

where, K is defined via, $-K = \frac{d\zeta}{\zeta}|_{stoch} = \frac{U_{cc}}{U_c}\sigma_\delta$

$$K = A\sigma_\delta \tag{14}$$

1

Worth noting that

$$\begin{aligned}
u[c(w_t), \lambda] &= \max_{c_1} u_1[c_1] + \lambda u_2(c_2 = (c - c_1)) \\
FOC: 0 &= u_1'[c_1'(w_t)] - \lambda u_2'[c_2'(w_t) = c'(w_t) - c_1'(w_t)] \\
\lambda &= \frac{u_1'[c_1'(w_t)]}{u_2'[c_2'(w_t)]} \\
&\quad \lambda \text{ is a constant in complete markets case.}
\end{aligned}$$

Now, for a given λ market eventually needs to be determined, we have 2 eqns.

2

$$\begin{aligned}
\delta &= c = c_1'(c, \lambda) + c_2'(c, \lambda) \\
u_1'(c_1'(w_t)) &= \lambda u_2'(c_2'(w_t)) \\
&\quad \text{These jointly determine } c_i'(c, \lambda), i = 1, 2, ... \\
\text{Ex: if } u_1(x) &= u_2(x) = \log x \\
\frac{1}{c_1} &= \frac{\lambda}{c_2} \\
c_2 &= \lambda c_1 \\
c &= c_1 + \lambda c_1 \\
c_1 &= \frac{c}{1+\lambda} \\
c_2 &= \frac{\lambda c}{1+\lambda} \\
&\quad \text{Envelope Condition:} \\
1 \Rightarrow u_c[c(w_t), \lambda] &= u_1'[c_1'(c, \lambda)] \cdot \frac{dc_1}{dc} + \lambda u_2'[c_2']\frac{dc_2}{dc} \\
2 \Rightarrow &= \lambda u_2' \frac{d[c_1 + c_2]}{dc}
\end{aligned}$$

$$u_c = \lambda u_2' = u_1' \tag{16.5}$$

Define

$$f_i(x) \equiv \left([u_i']^{-1}\right)(x)$$

$$c_1'(w_t) = f_1\left[u_c(\delta(w_t),\lambda)\right] \tag{15}$$

$$c_2'(w_t) = f_2\left[\frac{u_c(\delta(w_t),\lambda)}{\lambda}\right] \tag{16}$$

Thus everything is determined if λ is identified. To identify λ, use

$$\begin{aligned}
u_c(w_t) &= \lambda u_2'(c_2') = \lambda \frac{1}{c_2'(w_t)} \\
\zeta & \quad \text{Thus everything is determined if } \lambda \text{ is identified} \\
\zeta(w_t) &= e^{-\rho t}\frac{c_2'(0)}{c_2'(w_t)}, \text{ Note } \zeta(0) = 1 \\
\zeta(0)w_2(0) &= w_2(0) = \beta = E\left[\int_0^T e^{-\rho t}\frac{c_2'(0)}{c_2'(w_t)}c_2'(w_t)dt\right] \\
&= c_2'(0)\frac{1}{\rho}(1-e^{-\rho t})
\end{aligned}$$

$$\beta = \frac{\lambda}{u_c[\delta(0),\lambda]}\frac{1}{\rho}(1-e^{-\rho t}) \tag{17}$$

This is one equation for the one unknown λ. Worth noting that the richest agent-2 can be is that she consumes all output $\Rightarrow c_2$. In that case,

$$\frac{u_c(\delta,\lambda)}{\lambda} = u_2'[c_2 = \delta] = \frac{\partial}{\partial c_2}[c_2]|_{c_2=\delta} = \frac{1}{8}$$

Plugging this into 17 we get $\beta = \frac{\delta}{\rho}(1-e^{-\rho t})$, which is estimation in eq 16.

$$u_1'[c_1'(\delta,\lambda)] = u_c[\delta,\lambda] \tag{A}$$

$$\begin{aligned}
\text{I to LHS } du_i' &= u_1'' dc_1 + \frac{1}{2}u_1''' dc_1^2 \\
\text{I to RHS} & \quad u_{cc}d\delta + \frac{1}{2}r_{ccc}d\delta^2
\end{aligned}$$

$$dc_1 = \frac{1}{u_1''}\left[u_{cc}\left(\mu_\delta dt + \sigma_\delta \partial w\right) + \frac{1}{2}u_{ccc}\sigma_\delta^2 dt - \frac{1}{2}u_1''' dc_1^2\right] \tag{B}$$

$$\begin{aligned}
\text{A + RHS1:} &= \frac{u_{cc}}{u_c}\frac{u_1'}{u_1''}\left(\mu_\delta dt + \sigma_\delta \partial w\right) = \frac{A}{A_1}\left(\mu_\delta dt + \sigma_\delta \partial w\right) \\
\text{A + RHS2:} &= \frac{1}{2}\frac{u_{ccc}}{u_c}\frac{u_1'}{u_1''}\sigma_\delta^2 dt = -\frac{1}{2}A\rho\frac{1}{A_1}\sigma_\delta^2 dt \\
\text{RHS3:} & \quad -\frac{1}{2}\frac{u_1'''}{u_1''}dc_1^2 \\
& \quad \text{But from B, we have, } dc_1^2 = \left(\frac{u_{cc}}{u_1''}\right)^2\sigma_\delta^2 \\
&= \frac{1}{2}\rho_1\left(\frac{u_{cc}/u_c}{u_1''/u_1'}\right)^2\sigma_\delta^2 \\
&= \frac{1}{2}\rho_1\left(\frac{A}{A_1}\right)^2\sigma_\delta^2 \\
\sigma_{c'} &= \frac{u_{cc}}{u_1''}\sigma_\delta = \frac{A}{A_1}\sigma_\delta \\
\mu_{c'} &= \frac{A}{A_1}\mu_\delta - \frac{1}{2}\frac{A}{A_1}\rho\sigma_\delta^2 + \frac{1}{2}\left(\frac{A}{A_1}\right)^2\rho_1\sigma_\delta^2
\end{aligned}$$

$$
\begin{aligned}
w_2(t) &= E_t\left[\int_t^T e^{-\rho t}\frac{u_c[\delta_s,\lambda]}{u_c[\delta_t,\lambda]}c_2'(s)ds\right] \\
&= \frac{1}{u_c(\delta_t,\lambda)}E_t\left[\int_t^T e^{-\rho t}\frac{\lambda}{c_2'(s)}c_2'(s)ds\right] \\
&= \frac{\lambda}{u_c(\delta_t,\lambda)}\frac{1}{\rho}\left(1-e^{-\rho(T-t)}\right)\ \text{, using 17, } \frac{\lambda}{\rho}=\frac{\beta u_c(\delta_0,\lambda)}{1-e^{-\rho t}} \\
&= \beta\frac{u_c(\delta_0,\lambda)}{u_c(\delta_t,\lambda)}\frac{1-e^{-\rho(T-t)}}{1-e^{-\rho t}} \\
&= \beta\left[\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho t}}\right]\frac{1}{\zeta(t)} \\
dw_2|_{stoch} &= w_2\left(\frac{-ds}{s}\right)|_{stoch} \\
&= w_2K,\ \text{vs } dw_2|_{stoch}=\theta_2\sigma \text{ from wealth dynamics} \\
\theta_2 &= \frac{K}{\sigma}w_2=\frac{\mu-r}{\sigma^2}w_2 \\
\alpha_2 &= w_2-\theta_2=w_2\left[1-\frac{\mu-r}{\sigma^2}\right] \\
\theta_1 &= s-\theta_2 \\
\alpha_1 &= -\alpha_2
\end{aligned}
$$

Restricted Case :
When agent-2 restricted from participating in markets, equlibrium not pareto efficient, so solution is not same as social planner/representative agent with a constant weight λ. However, can still introduce representation agent with stochastic weight $\lambda(w_t) \Rightarrow$ This reduces search for an equilibrium to the specification of the weighting process $\alpha\lambda$ [and $\lambda(0)$].

Representative agent:

$$
\begin{aligned}
U[c,\lambda] &= E_0\left[\int_0^T e^{-\rho t}u[c,\lambda]dt\right] \\
&\quad \text{where, } u[c,\lambda]=\max_{c_1}\{u_1[c_1]+\lambda(w_t)u_2[c_2=c-c_1]\} \\
&\quad \text{FOC : } u_1'[c_1']=\lambda(w_t)u_2'[c_2'']\\
&\quad \text{A1 optimum, must be indifferent to small, affordable, consumption changes}
\end{aligned}
$$

$$
u[c_0',\lambda_0]+D_0(w_t)e^{-\rho t}u[c(w_t),\lambda(w_t)] \quad = \quad u[c_0'-\Delta c_0]+D_0(w_t)e^{-\rho t}u(c'+\Delta c(w_t),\lambda(w_t))
$$

$$
\begin{aligned}
0 &= -\Delta c_0 u'[c_0', \lambda_0] + \Delta c(w_t) u'[c'(w_t), \lambda(w_t)] D_0(w_t) e^{-\rho t} \\
&\quad \text{where, } [\Delta c(w_t), -\Delta c(0)] \text{ have zero cost via A/D prices:} \\
&\quad 0 = -\Delta c(0) + \Delta c(w_t).AD_0(w_t) \\
AD_0(w_t) u'[c_0', \lambda_0] &= D_0(w_t) e^{-\rho t} u'[c'(w_t), \lambda(w_t)] \\
\zeta_0(w_t) &\equiv \frac{AD_0(w_t)}{D_0(w_t)} = e^{-\rho t} \frac{u'[c'(w_t), \lambda(w_t)]}{u'[c'(0), \lambda(0)]} = e^{-\rho t} \frac{u'[\delta(w_t), \lambda(w_t)]}{u'[\delta(0), \lambda(0)]} \\
&\quad \text{Also, from bottom of page 6,} \\
u_c(c, \lambda) &= u_1'[c_1'] \frac{\partial c_1'}{\partial c} + \lambda u_2'(c_1') \frac{\partial c_1'}{\partial c}, \text{ use, } \mu_1' = \lambda u_2' \\
&= u_1'(c_1') \frac{d(c_1' + c_2')}{dc} = u_1'(c_1'), \text{ since, } c_1' + c_2' = c = \delta \\
u_c(c, \lambda) &= u_1'(c_1') = \lambda \mu_2'(c_2')
\end{aligned}
$$

Implications

i

$$
c_1'[\delta(w_t), \lambda(w_t)] = [u_1']^{-1} \left(u_c(\delta(w_t), \lambda(w_t)) \right) \equiv f_1[u_c(\delta(w_t), \lambda(w_t))] \tag{18}
$$

ii :

$$
c_2'[\delta(w_t), \lambda(w_t)] = [u_2']^{-1} \frac{(u_c(\delta(w_t), \lambda(w_t)))}{\lambda} \equiv f_2 \left[\frac{u_c(\delta(w_t), \lambda(w_t))}{\lambda} \right] \tag{19}
$$

iii :

$$
\lambda(w_t) = \frac{u_1'[]c_1'(w_t)}{u_2'[c_2'(w_t)]} \tag{20}
$$

Important!!! When they wrote $u[c] = u_1[c_1] + \lambda u_2[c]_2]$, they already assumed agent-1 was unrestricted.

More generally, need to write $u[c] \equiv \max_{c_1+c_2=c} \lambda_1(w_t) u_1[c_1] + \lambda_2(w_t) u_2[c_2]$ and will need to identify $(\lambda_1(w_t), \lambda_2(w_t))$ separately.

Have shown above that,

$$
\begin{aligned}
u_1'[c_1'(\delta(w_t), \lambda(w_t))] &= u_c[\delta(w_t), \lambda(w_t)] = e^{-\rho t} \zeta(w_t) u_c[\delta_0, \lambda] \\
\Rightarrow e^{-\rho t} u_1'[c_1'] &\propto \zeta(w_t)
\end{aligned}
$$

B/C interpret,"Since agent-1 facing a dynamically complete market, optimality of c_1', equvivalent to $e^{-\rho t} u_1'(c_1')$ being proportional to $\zeta(t)$"

Also shown above that,