Pavlova Rigabon 2

Derive GE without a representative agent:

$$
\begin{aligned}
1)\ \frac{dY}{Y} &= \mu_Y dt + \sigma_Y \partial w \\
2)\ \frac{dY^*}{Y^*} &= \mu_Y^* dt + \sigma_Y^* \partial w \\
P(w_*) &\equiv \text{ price of home good in state } - w_* \text{ in term os numerative} \\
P^*(w_*) &\equiv \text{ price of foreign good in state } - w_* \text{ in term os numerative} \\
q(w_*) &\equiv \frac{P(w_*)}{P^*(w_*)}
\end{aligned}
$$

Assume complete set of ... securities with dated prices $Q_0(w_*)$
Home:

$$
\begin{aligned}
L &= \int_0^T dt e^{-\rho *} \int \partial w_* D_0(w_*) \theta_H(w_*) \left\{ a_H L C_H(w_*) + (1-a_H) L C_H^*(w_*) \right\} \\
&\quad + \beta_H \left\{ w_H(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_H(w_*)\rho(w_*) + C_H^*(w_*)\rho^*(w_*) \right] \right\} \\
\frac{\partial}{\partial C_H(w_*)} = 0 &= e^{-p*} D_0(w_*) \theta_H(w_*) a_H \left(\frac{1}{C_H(w_*)} \right) - \beta_H \theta_0(w_*) \rho(w_*) \forall w_* \\
\frac{\partial}{\partial C_H^*(w_*)} = 0 &= e^{-p*} D_0(w_*) \theta_H(w_*) (1-a_H) \left(\frac{1}{C_H^*(w_*)} \right) - \beta_H \theta_0(w_*) \rho^*(w_*) \forall w_* \\
\frac{\partial}{\partial \beta_H} = 0 &= w_H(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_H(w_*)\rho(w_*) + C_H^*(w_*)\rho^*(w_*) \right] \\
&\Rightarrow \quad C_H(w_*)\rho(w_*) = \left(\frac{1}{\beta_H} \right) \left(\frac{1}{\theta_0(w_*)} \right) e^{-\rho *} D_0(w_*) \theta_H(w_*) a_H \\
&\Rightarrow \quad C_H^*(w_*)\rho^*(w_*) = \left(\frac{1}{\beta_H} \right) \left(\frac{1}{\theta_0(w_*)} \right) e^{-\rho *} D_0(w_*) \theta_H(w_*) (1-a_H)
\end{aligned}
$$

Foreign:

$$
\begin{aligned}
L &= \int_0^T dt e^{-\rho *} \int \partial w_* D_0(w_*) \theta_H(w_*) \left\{ a_F L C_F(w_*) + (1-a_F) L C_F^*(w_*) \right\} \\
&\quad + \beta_F \left\{ w_F(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_F(w_*)\rho(w_*) + C_F^*(w_*)\rho^*(w_*) \right] \right\} \\
\frac{\partial}{\partial C_F(w_*)} = 0 &= e^{-p*} D_0(w_*) \theta_F(w_*) a_F \left(\frac{1}{C_F(w_*)} \right) - \beta_F \theta_0(w_*) \rho(w_*) \forall w_* \\
\frac{\partial}{\partial C_F^*(w_*)} = 0 &= e^{-p*} D_0(w_*) \theta_F(w_*) (1-a_F) \left(\frac{1}{C_F^*(w_*)} \right) - \beta_F \theta_0(w_*) \rho^*(w_*) \forall w_* \\
\frac{\partial}{\partial \beta_F} = 0 &= w_F(0) - \int dt \int \partial w_* \theta_0(w_*) \left[C_F(w_*)\rho(w_*) + C_F^*(w_*)\rho^*(w_*) \right] \\
&\Rightarrow \quad C_F(w_*)\rho(w_*) = \left(\frac{1}{\beta_F} \right) \left(\frac{1}{\theta_0(w_*)} \right) e^{-\rho *} D_0(w_*) \theta_F(w_*) a_F \\
&\Rightarrow \quad C_F^*(w_*)\rho^*(w_*) = \left(\frac{1}{\beta_F} \right) \left(\frac{1}{\theta_0(w_*)} \right) e^{-\rho *} D_0(w_*) \theta_F(w_*) (1-a_F)
\end{aligned}
$$

$$
\begin{aligned}
W_F(0) &= \frac{1}{\beta_F}\int dt\int \partial w_* e^{-\rho *} D_0(w_*)\theta_F(w_*)\left[a_F+1-a_F\right] \\
&= \frac{1}{\beta_F}\int dt\int \partial w_* e^{-\rho *} D_0(w_*)\theta_F(w_*) \\
&= \frac{1}{\rho\beta_F} \text{ using } E\left[\theta_F(w_*)\right]=\theta_F(0)=1 \\
& \quad \text{vs} \\
W_H(0) &= \frac{1}{\beta_H}\int dt\int \partial w_* e^{-\rho *} D_0(w_*)\theta_H(w_*) \\
&= \frac{1}{\rho\beta_H}
\end{aligned}
$$

In partial equilibrium with $\{\theta(w_*), D_0(w_*), [\theta_0(w_*), w_H(0), w_F(0), \rho(w_*), \rho^*(w_*)]\}$ exogenous, we would be done, and these last two equations would define $\{\beta_H, \beta_F\}$

From partial equation, I have $4x(*x\omega_*)+2$ equations

$$
\begin{aligned}
C_H(w_*)\rho(w_*) &= \left(\frac{1}{\beta_H\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_H(w_*)a_H\forall(*, w_*) && (1)\\
C_H^*(w_*)\rho^*(w_*) &= \left(\frac{1}{\beta_H\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_H(w_*)(1-a_H)\forall(*, w_*) && (2)\\
C_F(w_*)\rho(w_*) &= \left(\frac{1}{\beta_F\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_F(w_*)a_F\forall(*, w_*) && (3)\\
C_F^*(w_*)\rho^*(w_*) &= \left(\frac{1}{\beta_F\theta_0(w_*)}\right) e^{-\rho *} D_0(w_*)\theta_F(w_*)(1-a_F)\forall(*, w_*) && (4)\\
w_H(0) &= \left(\frac{1}{\beta_H}\right)\int dt\int \partial w_* e^{-\rho *} D_0(w_*)\theta_H(w_*) && (5)\\
w_F(0) &= \left(\frac{1}{\beta_H}\right)\int dt\int \partial w_* e^{-\rho *} D_0(w_*)\theta_H(w_*) && (6)
\end{aligned}
$$

Here in partial equation, we take $\{\theta(w_*), D_0(w_*), [\theta_0(w_*), w_H(0), w_F(0), \rho(w_*), \rho^*(w_*)]\}$ exogenous

In general equation, these $2 \neq 3x(*x\omega_*)$ are determined via

$$
\begin{aligned}
s(0)=w_H(0) &= \int dt\int \partial w_* \theta_0(w_*)Y(w_*)\rho(w_*) \text{ leq} && (7)\\
s^*(0)=w_F(0) &= \int dt\int \partial w_* \theta_0(w_*)Y^*(w_*)\rho^*(w_*) \text{ leq} && (8)\\
Y(w_*) &= C_H(w_*)+C_F(w_*)\ (*x\omega_*) \text{ eq} && (9)\\
Y^*(w_*) &= C_H^*(w_*)+C_F^*(w_*)\ (*x\omega_*) \text{ eq} && (10)\\
1 &= \alpha\rho(w_*)+(1-\alpha)\rho^H(w_*)\ (*x\omega_*) \text{ eq} && (11)
\end{aligned}
$$

$$
(11^*)\ \frac{\theta_0(0)}{D_0(0)}=1 \Rightarrow \text{ Need this because without, can only determine ratio } \left(\frac{\beta_H}{\beta_F}\right) \qquad (11^*)
$$

$\Rightarrow$ This forces price of consumption unit today equal to 1.

To solve, I will first determine $q(w_*) \equiv \frac{\rho(w_*)}{\rho^H(w_*)}$ and then use equation 11 to solve for $\rho(w_*), \rho^*(w_*)$. I will then solve for $\theta_0(w_*)$ and finally solve for $w_H(0), w_F(0)$.

Solve for $q(w_*)$:

$$\begin{aligned}(1,3,9): \rho(w_*)Y(w_*) &= \rho(w_*)C_H(w_*) + \rho(w_*)C_F(w_*) \\ &= \left(\frac{1}{\beta_H\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_H(w_*)a_H + \\ &\quad \left(\frac{1}{\beta_F\theta_0(w_*)}\right) e^{-\rho*} D_0(w_*)\theta_F(w_*)a_F \end{aligned} \tag{12}$$

$$\begin{aligned}(2,4,10): \theta_0(w_*)\rho^*(w_*)Y^*(w_*) &= \theta_0(w_*)\rho^*(w_*)C_H^*(w_*) + \theta_0(w_*)\rho^*(w_*)C_F^*(w_*) \\ &= e^{-\rho*}D_0(w_*)\left\{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)(1-\theta_H) + \right. \\ &\quad \left. \left(\frac{1}{\beta_F}\right)\theta_F(w_*)(1-\theta_F)\right\} \end{aligned} \tag{13}$$

$$\frac{12}{13}: q(w_*)\frac{Y(w_*)}{Y^*(w_*)} = \frac{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)a_H + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F}{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)(1-a_H) + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)(1-a_F)} \tag{13b}$$

This agrees with equation in P/R if $\frac{1}{\beta_H} \sim \lambda_H, \frac{1}{\beta_F} \sim \lambda_F$ which is the typical asset that lagrange multiplier of agent reciprocal of lagrange multiplier of agents weight in social planner problem.

Eq 13b gives $q(w_*)$ as a fact of $\{\beta_H, \beta_F\}$, and exogenous variables -

$$(11) \Rightarrow \frac{1}{\rho(w_*)} = \alpha + (1-\alpha)\frac{1}{q(w_*)}$$

$$\Rightarrow \rho(w_*) = \frac{q(w_*)}{(1-\alpha) + \alpha q(w_*)} \tag{13c}$$

$$(11) \Rightarrow \frac{1}{\rho^*(w_*)} = \alpha q(w_*) + (1-\alpha)$$

$$\Rightarrow \rho(w_*) = \frac{1}{(1-\alpha) + \alpha q(w_*)} \tag{13d}$$

Since $q(w_*)$ written in exogenous terms, so are $\{\rho(w_*), \rho^*(w_*)\}$

To solve for $\theta_0(w_*)$, use 12, 13 and 11

$$12: \alpha\rho\theta_0(w_*) = \alpha e^{-\rho*}D_0(w_*)\left(\frac{1}{Y(w_*)}\right)\left\{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)a_H + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F\right\} \tag{14}$$

$$\begin{aligned}13: (1-\alpha)\rho^*\theta_0(w_*) &= (1-\alpha)e^{-\rho*}D_0(w_*)\left(\frac{1}{Y^*(w_*)}\right)\left\{\left(\frac{1}{\beta_H}\right)\theta_H(w_*)a_H \right. \\ &\quad \left. + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F\right\} \end{aligned} \tag{15}$$

Combine 14, 15, and use in 11:

$$\begin{aligned}\theta_0(w_*) = e^{-\rho*}D_0(w_*)&\left\{\left(\frac{\alpha}{Y(w_*)}\right)\left[\frac{1}{\beta_H}\theta_H(w_*)a_H + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)a_F\right] + \left(\frac{1-\alpha}{Y^*(w_*)}\right)\right. \\ &\left.\left[\frac{1}{\beta_H}\theta_H(w_*)(1-a_H) + \left(\frac{1}{\beta_F}\right)\theta_F(w_*)(1-a_F)\right]\right\}\end{aligned} \tag{15*}$$

This is consistent with $q(w_*) \equiv \frac{\theta_0(w_*)}{D_0(w_*)}$ if $\frac{1}{\beta_H} = \lambda_H, \frac{1}{\beta_F} = \lambda_F$

Now, solve for $\{w_H(0), w_F(0)\}$. For $w_H(0)$, use (7), (12)

$$\begin{aligned}
w_H(0) &= \int dt \int \partial w_* e^{-\rho *}(w_*) \left\{ \frac{1}{\beta_H} \theta_H(w_*) a_H + \frac{1}{\beta_F} \theta_F(w_*) a_F \right\} \\
&\quad \text{use fact that } \theta_H, \theta_F \text{ are marginals: } \int \partial w_* D_0(w_*) = \theta_H(0) = 1, \forall * \\
&= \left(\frac{1}{\rho}\right) \left(\frac{a_H}{\beta_H} + \frac{a_F}{\beta_F}\right) \qquad (16) \\
&\quad \text{Similarly, for } w_F(0), \text{ use (8) and (13)} \\
w_F(0) &= \int dt \int \partial w_* e^{-\rho *} D_0(w_*) \left\{ \frac{1}{\beta_H} \theta_H(w_*)(1-a_H) + \frac{1}{\beta_F} \theta_F(w_*)(1-a_F) \right\} \\
&= \left(\frac{1}{\rho}\right) \left(\frac{1-a_H}{\beta_H} + \frac{1-a_F}{\beta_F}\right) \qquad (17) \\
&\quad \text{But from partial eq, we have } w_H(0) = \tfrac{1}{\rho\beta_H}, w_F(0) = \tfrac{1}{\rho\beta_F} \\
&\quad \text{Combining,} \\
1 &= a_H + a_F \frac{\beta_H}{\beta_F}, 1 = (1-a_H)\frac{\beta_F}{\beta_H} + (1-a_F) \\
\frac{\beta_H}{\beta_F} &= \frac{1-a_H}{a_F} \qquad (18) \\
&\quad \text{Plugging back we get,}
\end{aligned}$$

$$\frac{w_H(0)}{w_F(0)} = \frac{\beta_F}{\beta_H} = \frac{a_F}{1-a_H} \qquad (19)$$

$$\begin{aligned}
&\quad \text{Also from 14, 18 we have:} \\
q(0) &= \frac{Y^*(0)}{Y(0)} \left[\frac{\frac{a_H}{\beta_H} + \frac{a_F}{\beta_F}}{\frac{1-a_H}{\beta_H} + \frac{1-a_F}{\beta_F}} \right] \\
&= \frac{Y^*(0)}{Y(0)} \left[\frac{\frac{a_F a_H}{1-a_H} + a_F}{a_F + (1-a_F)} \right] \\
&= \frac{Y^*(0)}{Y(0)} \frac{a_F}{a-a_H} \qquad (20) \\
&\quad \text{use equation 11* and 15*: } \theta_H(0) = 1, \theta_F(0) = 1 \\
1 &= \frac{\theta_0(0)}{D_0(0)} = \frac{\alpha}{Y(0)} \left[\frac{a_H}{\beta_H} + \frac{a_F}{\beta_F}\right] + \frac{1-\alpha}{Y^*(0)} \left[\frac{1-a_H}{\beta_H} + \frac{1-a_F}{\beta_F}\right] \\
&\quad \text{Use equation 19 : } \tfrac{\beta_F}{\beta_H} = \tfrac{a_F}{a-a_H} \\
\beta_F &= \frac{\alpha}{Y(0)} \left[a_H \frac{a_F}{1-a_H} + a_F\right] + \frac{1-\alpha}{Y^*(0)} \left[(1-a_H)\frac{a_F}{1-a_H} + (1-a_F)\right] \\
&= \frac{1}{Y(0)} \frac{\alpha}{1-a_H} [a_H a_F + a_F(1-a_H)] + (1-\alpha)\frac{1}{Y^*(0)} \\
&= \frac{\alpha}{Y(0)} \frac{a_F}{1-a_H} + \frac{1-\alpha}{Y^*(0)} \\
&= \alpha \left(\frac{a_F}{a-a_H}\right) \left(\frac{q(0)}{Y^*(0)}\right) \frac{\frac{1}{\beta_H}(1-a_H) + \frac{1}{\beta_F}(1-a_F)}{\frac{1}{\beta_H} a_H + \frac{1}{\beta_F} a_F} + \frac{1-\alpha}{Y^*(0)} \\
&= \frac{1}{Y^*(0)} \left\{ \alpha \left(\frac{a_F}{1-a_H}\right) q(0) \left[\frac{\left(\frac{a_F}{1-a_H}\right)(1-a_H) + (1-a_F)}{\left(\frac{a_F}{1-a_H}\right) a_H + a_F} \right] + (1-\alpha) \right\} \\
&= \frac{1}{Y^*(0)} \left\{ \alpha \left(\frac{a_F}{1-a_H}\right) q(0) \left[\frac{1-a_H}{a_F a_H + a_F(1-a_H)} \right] + (1-\alpha) \right\} \\
&= \frac{1}{Y^*(0)} \{\alpha q(0) + (1-\alpha)\} \\
\beta_F &= \frac{1}{Y^*(0)} \frac{1}{\rho^*(0)} \qquad (21)
\end{aligned}$$

$$
\begin{aligned}
& \text{Back to 13b} \\
q(0)\left(\frac{Y(0)}{Y^*(0)}\right) &= \frac{\frac{a_H}{\beta_H}+\frac{a_F}{\beta_F}}{\frac{1-a_F}{\beta_H}+\frac{1-a_F}{\beta_F}} \\
&= \frac{a_H\frac{a_F}{1-a_H}+a_F}{(1-a_H)\frac{a_F}{1-a_H}+(1-a_F)} \\
&= \frac{a_F}{1-a_H} \\
&= \frac{\beta_F}{\beta_H} \\
\text{Thus, } \beta_H &= \beta_F\left(\frac{Y^*(0)}{Y(0)}\right)\left(\frac{\rho^*(0)}{\rho(0)}\right) \\
&= \frac{1}{\rho(0)Y(0)} \qquad (22) \\
& \text{From 5, 6:} \\
s(0) &= w_H(0)=\frac{1}{\rho\beta_H}=\frac{\rho(0)Y(0)}{\rho} \qquad (23) \\
s^*(0) &= w_F(0)=\frac{1}{\rho\beta_F}=\frac{\rho^*(0)Y(0)}{\rho} \qquad (24) \\
\text{1x3:}\frac{C_H(w_*)}{C_F(w_*)} &= \frac{\beta_F}{\beta_H}\frac{\theta_H(w_*)}{\theta_F(w_*)}\frac{a_H}{a_F} \\
Y(w_*) &= C_H(w_*)\left[1+\frac{\beta_H a_F\theta_F}{\beta_F\theta_H a_H}\right] \\
\Rightarrow C_H(w_*) &= Y(w_*)\frac{\beta_F a_H\theta_H}{\beta_F a_H\theta_H+\beta_H a_F\theta_F} \\
&= Y(w_*)\frac{\frac{1}{\beta_H}a_H\theta_H}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F} \qquad (25) \\
\Rightarrow C_F(w_*) &= Y(w_*)-C_H(w_*) \\
&= Y(w_*)\left(\frac{\frac{1}{\beta_F}a_F\theta_F}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F}\right) \qquad (26) \\
\Rightarrow C_H^*(w_*) &= Y^*(w_*)\frac{\beta_F a_H\theta_H}{\beta_F a_H\theta_H+\beta_H a_F\theta_F} \\
&= Y(w_*)\frac{\frac{1}{\beta_H}a_H\theta_H}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F} \\
\Rightarrow C_F^*(w_*) &= Y^*(w_*)-C_H^*(w_*) \\
&= Y^*(w_*)\left(\frac{\frac{1}{\beta_F}a_F\theta_F}{\frac{1}{\beta_H}a_H\theta_H+\frac{1}{\beta_F}a_F\theta_F}\right) \\
& \text{From here I go back to P/R's notation,} \\
(2):q &= \frac{\lambda_H\theta a_H+\lambda_F a_F}{\lambda_H\theta(1-a_H)+\lambda_F(1-a_F)}\frac{Y^*}{Y},\theta\equiv\frac{\theta_H}{\theta_F} \\
Lq &= L\left[\lambda_H a_H\theta+\lambda_F a_F\right]-L\left[\lambda_H(1-a_H)\theta+\lambda_F(1-a_F)\right]+LY^*-LY \\
dLq &= \left(\frac{\lambda_H a_H}{\lambda_H a_H\theta+\lambda_F a_F}\right)d\theta-\frac{\lambda_H(1-a_H)}{\lambda_H(1-a_H)\theta+\lambda_F(1-a_F)}d\theta+\frac{dY^*}{Y^*}-\frac{dY}{Y}+\theta(d*) \\
&\equiv Ad\theta+\frac{dY^*}{Y^*}-\frac{dY}{Y}+\theta(d*) \\
& \text{where, } A\equiv\left(\frac{\lambda_H a_H}{\lambda_H a_H\theta+\lambda_F a_F}\right)-\left(\frac{\lambda_H(1-a_H)}{\lambda_H(1-a_H)\theta+\lambda_F(1-a_F)}\right) \\
\frac{dq}{q} &= ()d*+[Ad\theta-\sigma_Y dw+\sigma_Y^*\partial w^*] \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
S(*) &= \left(\frac{1}{\rho}\right)\left(\frac{q(*)}{\alpha q(*)+(1-\alpha)}\right)Y \\
LS &= Lq - L\left[(*)+(1-\alpha)\right] + LY \\
dLS &= \frac{dq}{q} - \frac{\alpha dq}{\alpha q(*)+(1-\alpha)} + \frac{dY}{Y} + ()dt \\
&= \left[1-\frac{\alpha q}{\alpha q+(1-\alpha)}\right]\frac{dq}{q}+\frac{dY}{Y} \\
&= \left[\frac{1-\alpha}{\alpha q+(1-\alpha)}\right]\frac{dq}{q}+\frac{dY}{Y} \\
&= \left[\frac{1-\alpha}{\alpha q+(1-\alpha)}\right]\left[Ad\theta-\sigma_Y\partial w+\sigma_Y^*\partial w^*\right]-\sigma_Y\partial w \qquad (28) \\
S^* &= \frac{1}{\rho}\left(\frac{1}{\alpha q+(1-\alpha)}\right)Y^* \\
dLS^* &= (\cdot)d*-\left(\frac{\alpha q}{\alpha q+(1-\alpha)}\right)\frac{dq}{q}+\frac{dY^*}{Y^*} \\
&= -\left[\frac{\alpha q}{\alpha q+(1-\alpha)}\right]\left[Ad\theta-\sigma_Y\partial w+\sigma_Y^*\partial w^*\right]-\sigma_Y^*\partial w^* \qquad (29)
\end{aligned}
$$

Home bond pays out $(1-C_H d*)$ units of home goods, in terms of numeraire:

$$
\begin{aligned}
\text{1 unit of home good} &= \rho(*) \text{ at date } * \\
a-C_H d* \text{ units of home good} &= (1+C_H d*)\rho(*_\Delta *) \text{ at date } *... \\
\Rightarrow &= (1+C_H d*)(\rho+d\rho)-\rho \\
&= \rho+d\rho+C_H\rho d* \\
\Rightarrow \frac{\Delta Price}{\rho} &= \frac{d\rho}{\rho}+C_H d* \\
\Rightarrow \frac{\partial B}{B} &= C_H d*+\frac{\partial\rho}{\rho} \\
\rho &= \frac{q}{(1-\alpha)j+\alpha q} \\
dL\rho &= ()d*+\frac{dq}{q}-\left(\frac{\alpha q}{(1-\alpha)+\alpha q}\right)\frac{dq}{q} \\
&= ()d*+\left[\frac{1-\alpha}{(1-\alpha)+\alpha q}\right]\left[Ad\theta-\sigma_Y\partial w+^*_Y\partial w^*\right] \qquad (30)
\end{aligned}
$$

Foreign Bond:

$$\begin{aligned}
\Delta\rho &= (1 + C_F d*)(\rho^* + d\rho^*) - \rho^* \\
\frac{\partial B^*}{B^*} &= \frac{\partial \rho^*}{\rho^*} + C_F d* \\
dL\rho^* &= -\left[\frac{\alpha\rho}{(1-\alpha)+\alpha q}\right]\frac{dq}{q} + ()d* \\
\frac{\partial B^*}{B^*} &= -\left[\frac{\alpha\rho}{(1-\alpha)+\alpha q}\right][Ad\theta - \sigma_Y \partial w + \sigma_Y^* \partial w^*] \qquad (31) \\
&\quad \text{all agree with property 1, equation 17.} \\
&\quad \text{I do not want to do all consumption dynamics. I will do just one:} \\
LC_H &= L\theta - L[\lambda_H a_H \theta + \lambda_F a_F] + LY + ... \\
\frac{dC_H}{C_H} &= ()d* + \frac{d\theta}{\theta} - \left(\frac{\lambda_H a_H \theta}{\lambda_H a_H \theta + \lambda_F a_F}\right)\frac{d\theta}{\theta} + \frac{dY}{Y} \\
&= \left(\frac{\lambda_F a_F}{\lambda_F a_F + \lambda_H a_\theta \theta}\right)\frac{d\theta}{\theta} + \frac{dY}{Y} \qquad (32) \\
&\quad \text{agrees with signsin equaiton 19.}
\end{aligned}$$

13a: if $Y_H = 0, Y_F = 0$ then I see only 2 B/M's $\frac{\partial Y}{Y} = \mu d* + \sigma\partial w$, $\frac{\partial Y^*}{Y^*} = \mu_* d* + \sigma^* \partial w^*$ So the fact that $S + S^*$ are perfectly correlated, to me, still seems like complete markets.

$$\begin{aligned}
13b: \frac{d\theta_H}{\theta_H} &= Y_H \partial w_\theta \\
\frac{d\theta_F}{\theta_F} &= Y_F \partial w_\theta \\
\theta &= \frac{\theta_H}{\theta_H}, \theta_1 = \frac{1}{\theta_F}, \theta_2 = \frac{1}{\theta_F^2}, \theta_{11} = 0, \theta_{12} = -\frac{1}{\theta_F^2}, \theta_{22} = \frac{2\theta_H}{\theta_F^3} \\
\frac{d\theta}{\theta} &= Y_H \partial w_\theta - Y_F \partial w_\theta + Y_F^2 \partial * - Y_H Y_F \partial *
\end{aligned}$$

$$= (Y_F^2 - Y_H Y_F)d* + (Y_H - Y_F)\partial w_\theta \qquad (33)$$

Proposition 2: First, introduce "world bond" : If you want a portfolio that pays α units of home good and $(1-\alpha)$ units of foreign good at date $(* + d*)$, its date $(* + d*)$ value is,

$$\begin{aligned}
V(* + d*) &= \alpha\rho(* + d*) + (1-\alpha)\rho^*(* + d*) \equiv 1 \\
&\quad \text{at date *, this will cost:} \\
V(*) &= \left(\frac{\alpha}{1 + C_H d*}\right)\rho(*) + \left(\frac{1-\alpha}{1 + C_F d*}\right)\rho^*(*) \\
&= (\alpha\rho(*) + (1-\alpha)\rho^*(*)) - (\alpha\rho(*)C_H + (1-\alpha)\rho^*(*)C_F)d* \\
&= 1 - C_w d*
\end{aligned}$$

$$\Rightarrow C_w \equiv \alpha c_F(*)\rho(*) + (1-\alpha)C_F(*)\rho^*(*) \qquad (34)$$

Interpretations:
B: 4 assets : 3 risky $[S, S^*, B]$ and one riskless B^w
C: 3 assets : 2 risky $[S, S^*]$ and one riskless B^w
D:
Going back to P/R notation via equation A4:

$$
\begin{aligned}
\zeta(w_*) &\quad \alpha \quad e^{-\rho *}\left\{\left(\frac{\alpha}{Y(w_*)}\right)(\lambda_H\theta_H(w_*)a_H+\lambda_F\theta_F(w_*)a_F)\right. \\
&\qquad \left.+\left(\frac{1-\alpha}{Y^*(w_*)}\right)(\lambda_H\theta_H(w_*)a_H+\lambda_F\theta_F(w_*)a_F)\right\} \\
d\zeta &= -\rho q d*+e^{-\rho *}\left(\frac{\alpha}{Y(w_*)}\right)\left[\lambda_H a_H Y_H^T\theta_H\partial w+\lambda_F a_F Y_F^T\theta_F\partial w\right] \\
&\quad -e^{-\rho *}\left(\frac{\alpha}{Y}\right)\left[\lambda_H a_H\theta_H+\lambda_F a_F\theta_F\right]\left[\mu_Y d*+\sigma_Y\partial w\right] \\
&\quad -e^{-\rho *}\left(\frac{\alpha}{Y}\right)\left[\lambda_H a_H\theta_H\sigma_Y Y_{H,1}+\lambda_F a_F\theta_F\sigma_Y Y_{F,1}\right]dt \\
&\quad +e^{-\rho *}\left(\frac{1-\alpha}{Y}\right)\left[\lambda_H(1-a_H)\theta_H\sigma_Y Y_H\partial w+\lambda_F(1-a_F)\theta_F\sigma_Y Y_F\partial w\right] \\
&\quad -e^{-\rho *}\left(\frac{1-\alpha}{Y}\right)\left[\lambda_H(1-a_H)\theta_H+\lambda_F(1-a_F)\theta_F\right]\left[\mu_Y^* d*+\sigma_Y^*+\sigma_Y^*\partial w^*\right] \\
&\quad -e^{-\rho *}\left(\frac{1-\alpha}{Y}\right)\left[\lambda_H(1-a_H)\theta_H\sigma_Y^* Y_{H,2}+\lambda_F(1-a_F)\theta_F\sigma_Y^* K_{F,2}\right]dt \\
&\quad +e^{-\rho *}\left(\frac{\alpha}{Y}\right)\left[\lambda_H\theta_H a_H+\lambda_F\theta_F a_F\right]\sigma_Y^* dt+e^{-\rho *}\left(\frac{1-\alpha}{Y^*}\right)\left[\lambda_H\theta_H(1-a_H)+\lambda_F\theta_F(1-a_F)\right](\sigma_Y^*)^2 dt \\
-\gamma &= -\rho-\frac{e^{-\rho *}}{\zeta}\left(\frac{\alpha}{Y}\right)\left[\lambda_H a_H\theta_H+\lambda_F a_F\theta_F\right]\mu_Y \\
&\quad -\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{\alpha}{Y}\right)\left[\lambda_H a_H\theta_H\sigma_Y Y_{H,1}+\lambda_F a_F\theta_F\sigma_Y K_{F,1}\right] \\
&\quad -\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{1-\alpha}{Y^*}\right)\left[\lambda_H(1-a_H)\theta_H+\lambda_F(1-a_F)\theta_F\right]\mu_Y^* \\
&\quad -\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{1-\alpha}{Y^*}\right)\left[\lambda_H(1-a_H)\theta_H\sigma_Y^* Y_{H,2}+\lambda_F(1-a_F)\theta_F\sigma_Y^* Y_{F,2}\right] \\
&\quad +\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{\alpha}{Y}\right)\left[\lambda_H\theta_H a_H+\lambda_F\theta_F a_F\right]\sigma_Y^2+\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{1-\alpha}{Y^*}\right)\left[\lambda_H\theta_H(1-a_H)+\lambda_F\theta_F(1-a_F)\right] \\
&\quad \text{where wer are using } \frac{d\zeta}{\zeta}\equiv-\gamma d*-m^T\partial\vec{w} \\
-\vec{m} &= \left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{\alpha}{Y}\right)\left(\lambda_H a_H Y_H^T\theta_H+\lambda_F a_F Y_F^T\theta_F\right) \\
&\quad -\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{\alpha}{Y}\right)(\lambda_H a_H\theta_H+\lambda_F a_F\theta_F)(\sigma_Y,0,0) \\
&\quad +\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{1-\alpha}{Y^*}\right)\left(\lambda_H(1-a_H)Y_H^T\theta_H+\lambda_F(1-a_F)Y_F^T\theta_F\right) \\
&\quad -\left(\frac{e^{-\rho *}}{\zeta}\right)\left(\frac{1-\alpha}{Y^*}\right)(\lambda_H(1-a_H)\theta_H+\lambda_F(1-a_F)\theta_F)(0,\sigma_Y^*,0)
\end{aligned}
$$