HWS 2021
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Stochastic Calculus
Problem Sheet 1

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Exercise 1.1

Let $X = (X_t, t \geq 0)$ be adapted to a right-continuous filtration , then the entry times $\tau_A := \inf\{t \geq 0 : X_t \in A\}$ are stopping times if

1. A is open, X is right-continuous or
2. A is closed, X is continuous.

Exercise 1.1

Let X and Y be integrable random variables with

$$\mathbb{E}[X|Y] = Y \quad \text{and} \quad \mathbb{E}[Y|X] = X, \quad \mathbb{P}\text{-a.s.}$$

Prove that $X = Y$ $\mathbb{P}$-a.s.

Exercise 1.2

Let $x\colon [0,1] \to \mathbb{R}$ be a continuous function, and let $(\pi_n)_{n\in\mathbb{N}}$ be the sequence of partitions given by the dyadic rational in $[0,1]$, that is, π_n is given by the set $\{\frac{k}{2^n} : k = 0, \dots, 2^n\}$, and thus $\lim_{n\to\infty} |\pi^n| = 0$. Show that, if the sum

$$\sum_{t_k^n \in \pi^n} y_{t_k^n} \big(x_{t_{k+1}^n} - x_{t_k^n}\big)$$

converges to a finite limit for every continuous function $y\colon [0,1] \to \mathbb{R}$, then x is of finite variation. *[Hint: Use the Banach-Steinhaus theorem.]*

Exercise 1.3

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion (BM). Show that

(a) $W^1 := -W$ is a BM.

(b) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha B + \sqrt{1-\alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0,1]$.

(d) Show that the independence of B and B' in c) cannot be omitted, i.e., if B and B' are not independent, then W^3 need not be a BM. Give two examples.

Please submit your solutions by Tuesday, the 14th of September, at noon (12 pm).