HWS 2021
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Stochastic Calculus
Problem Sheet 1

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Exercise 1.1

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion (BM). Show that

(i) $W^1 := -W$ is a BM.

(ii) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(iii) $W^3 := \alpha B + \sqrt{1-\alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0,1]$.

(iv) Show that the independence of B and B' in c) cannot be omitted, i.e., if B and B' are not independent, then W^3 need not be a BM. Give two examples.

Exercise 1.2

Let $(X_t)_{t\in[0,\infty)}$ be adapted to a right-continuous filtration $(\mathcal{F}_{t\in[0,T]})$, then the hitting times $\tau_A := \inf\{t \geq 0 : X_t \in A\}$ are stopping times if

(i) A is open, X is right-continuous or

(ii) A is closed, X is continuous.

Exercise 1.3

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion and its complete natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Show that the following processes are $(\mathcal{F}_t)_{t\in[0,T]}$-martingales:

(i) $X_t = B_t$, $t \geq 0$,

(ii) $X_t = B_t^2 - t$, $t \geq 0$,

(iii) $X_t = \exp(\lambda B_t - \frac{\lambda^2}{2} t)$,$t \geq 0$, for all $\lambda \in \mathbb{R}$.

Programming exercise

Doing this exercise is optional! Don't submit your solution, unless you find it nicer than the sample solution.

(i) Install Python and Visual Code, and then the packages *numpy* and *matplotlib.pyplot* as it is explained on the website.

(ii) Let N denote your discretization number per time unit, and T your time horizon. Use the function *random.normal* from *numpy* to generate $N * T$ normal distributed random numbers.

(iii) Generate an approximation of a realization of a Brownian Motion by starting at 0 and then summing up your normally distributed random variables. Make sure to choose the correct standard deviation in (ii).

(iv) Generate approximations of the martingales in Lemma 1.4 of the lecture notes, by using your approximated BM.

(v) Plot your approximated processes by using the functions *plot* and *show* from *matplotlib.pyplot.*

Please submit your solutions by Tuesday, the 14th of September, at noon (12 pm).