HWS 2021
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Stochastic Calculus
Problem Sheet 1

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Exercise 1.1

Let $(B_t)_{t\in[0,T]}$ and $(B'_t)_{t\in[0,T]}$ be two independent Brownian motions, $s \in (0,T)$ and $\alpha \in [0,1]$.

(i) Setting $W^1_t := -B_t$ for $t \in [0,T]$, show that $(W^1_t)_{t\in[0,T]}$ is a Brownian motion.

(ii) Setting $W^2_t := B_{s+t} - B_s$ for $t \in [0, T-s]$, show that $(W^1_t)_{t\in[0,T-s]}$ is a Brownian motion.

(iii) Setting $W^3_t := \alpha B_t + \sqrt{1-\alpha^2} B'_t$ for $t \in [0,T]$, show that $(W^3_t)_{t\in[0,T]}$ is a Brownian motion.

(iv) Show that the independence of B and B' in (iii) cannot be omitted, i.e., if B and B' are not independent, then $(W^3_t)_{t\in[0,T]}$ need not be a Brownian motion. Give an example.

Exercise 1.2

Let $(X_t)_{t\in[0,\infty)}$ be a real-valued, right-continuous stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and let $A \subset \mathbb{R}$. Prove that the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A\}$$

is a stopping time if

(i) A is open and $(\mathcal{F}_t)_{t\in[0,\infty)}$ is right-continuous or

(ii) A is closed and $(X_t)_{t\in[0,\infty)}$ is continuous.

Exercise 1.3

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion and $(\mathcal{F}_t)_{t\in[0,T]}$ be its completed natural filtration. Show that the following processes are $(\mathcal{F}_t)$-martingales:

(i) $X^1_t := B_t$ for $t \in [0,T]$;

(ii) $X^2_t := B^2_t - t$ for $t \in [0,T]$;

(iii) $X^3_t := \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ for $t \in [0,T]$ and $\sigma > 0$.

Programming exercise 1

Doing this exercise is optional! Do not submit your solution for correction. If you found an elegant solution, please do submit it so that we can improve our sample solution and thus help all students.

(i) Install Python and Visual Code, and then the packages *numpy* and *matplotlib.pyplot* as it is explained on the website.

(ii) Let N denote your discretization number per time unit, and T your time horizon. Use the function *random.normal* from *numpy* to generate $N \cdot T$ normal distributed random numbers.

(iii) Generate an approximation of a realization of a Brownian motion by starting at 0 and then summing up your normally distributed random variables. Make sure to choose the correct standard deviation in (ii).

(iv) Generate approximations of the martingales in Exercise 1.3 by using your approximated Brownian motion from (iii).

(v) Plot your approximated processes by using the functions *plot* and *show* from *matplotlib.pyplot*.

Please submit your solutions by Tuesday, the 14th of September, at noon (12 pm).