HWS 2021
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Stochastic Calculus
Problem Sheet 2

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Exercise 2.1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume that $X = (X_t)_{t\in[0,T]}$, $Y = (Y_t)_{t\in[0,T]}$ are two stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. For two processes Z and Z' on $(\Omega, \mathcal{F}, \mathbb{P})$ one says,

- Z and Z' are *modifications* of each other if $\mathbb{P}(Z_t = Z'_t) = 1 \ \forall\ t \in [0,T]$
- Z and Z' are *indistinguishable* if $\mathbb{P}(Z_t = Z'_t \ \forall\ t \in [0,T]) = 1$.

(i) Assume that X and Y are both right-continuous or both left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

(ii) Give an example showing that one of the implications of part **(i)** does not hold for general processes X, Y.

Exercise 2.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, $(\mathcal{F}_t)_{t\in[0,T]}$ be a right-continuous filtration and $(M_t)_{t\in[0,T]}$ be a continuous martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ which has paths of locally bounded variation. Prove that

$$\mathbb{P}\big(\{\omega \in \Omega : \ M_t(\omega) = M_0(\omega) \,\forall t \in [0,T]\}\big) = 1.$$

Proceed as follows:

(i) Assume w.l.o.g. that $M_0 = 0$.

(ii) Assuming that M and its variation process $|M|_\cdot$ are bounded, show that

$$\mathbb{E}[M_t^2] = \mathbb{E}\Big[\sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2\Big] \qquad \text{for all } 0 = t_0 \le \dots \le t_n = t,\ t \in [0,T].$$

(iii) Deduce under the assumption from (ii) that $\mathbb{E}[M_t^2] = 0$ for $t \in [0,T]$ and conclude the assertion in this case.

(iv) Use stopping times $\tau_n := \inf\{t \ge 0 \ :\ |M_t| \ge n \text{ or } |M|_t \ge n\}$, $n \in \mathbb{N}$, to obtain the same result for the general unbounded case.

[Hint: You may use without a proof that $(|M|_t)_{t\in[0,T]}$ is a continuous and adapted process.]

Exercise 2.3

Let $(M_t)_{t\in[0,T]}$ be a continuous local martingale with $M_0 \in \mathbb{R}$.

(i) Show that $(\tau_n)_{n\in\mathbb{N}}$, given by

$$\tau_n := \inf\{t \in [0,T] \ :\ |M_t| \ge n\} \wedge T,\ n \in \mathbb{N},$$

is a localizing sequence for $(M_t)_{t\in[0,T]}$.

(ii) If $(M_t)_{t\in[0,T]}$ is bounded from below, $(M_t)_{t\in[0,T]}$ is a super-martingale.

Programming exercise 2

Doing this exercise is optional! Do not submit your solution for correction. If you found an elegant solution, please do submit it so that we can improve our sample solution and thus help all students.

Write a function *pth_variation* in Python which takes the variable $p > 0$, the discretization per time step number N and the time horizon T as arguments. The function should

- simulate one realization of a standard Brownian motion $(B_t)_{t\in[0,T]}$,
- then approximate the pth *variation* of $(B_t)_{t\in[0,T]}$,

$$|B|_{p,t} := \lim_{n\to\infty} \sum_{J\in\Pi_n} |\Delta_{J\cap[0,t]}B|^p, \quad t \in [0,T],$$

  where Π_n and $\Delta_{J\cap[0,t]}B$ are defined as in the lecture, by using the equal width N-discretization,
- and then return the approximated realization of $(|B|_{p,t})_{t\in[0,T]}$.

Use the function *pth_variation* to show the 1*th variation*, *quadratic variation* and 3*rd variation* of a standard Brownian motion with $T = 5$ and $N = 10^5$ in one plot.

Remark 1. *Note that the pth variation from Programming exercise 2 does for $p > 1$ **not** coincide with the p-variation*

$$|B|_t^p := \sup_{\Pi} \sum_{J\in\Pi} |\Delta_{J\cap[0,t]}B|^p, \quad t \in [0,T].$$

Indeed, $|B|_{p,t} = |B|_t^p$ does only hold for $p = 1$.

Please submit your solutions by Tuesday, the 21st of September, at noon (12 pm).