HWS 2021
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Stochastic Calculus
Problem Sheet 7 (Bonus)

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Exercise 7.1

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion, $n \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_n = T$ and $u_1, \ldots, u_n \in \mathbb{R}$. Prove that there is some $\varphi \in \mathscr{H}^2$ such that

$$\prod_{j=1}^{n} \exp\Big(iu_j(B_{t_j} - B_{t_{j-1}}) + \frac{u_j^2}{2}(t_j - t_{j-1})\Big) = 1 + \int_0^T \varphi(s)\,\mathrm{d}B_s.$$

Hint: Let X, Y be bounded random variables with $X = \mathbb{E}[X] + \int_0^T \varphi(s)\,\mathrm{d}B_s$ and $Y = \mathbb{E}[Y] + \int_0^T \psi(s)\,\mathrm{d}B_s$ for some $\varphi, \psi \in \mathscr{H}^2$ satisfying $\int_0^T \varphi(s)\psi(s)\,\mathrm{d}s = 0$. Then you can deduce with the product formula that $X \cdot Y = \mathbb{E}[X]\mathbb{E}[Y] + \int_0^T (X_s\varphi(s) + Y_s\psi(s))\,\mathrm{d}B_s$, with $X_t := \mathbb{E}[X] + \int_0^t \varphi(s)\,\mathrm{d}B_s$, $t \in [0,T]$, and an analogous definition of Y_t.

Exercise 7.2

Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a complete and right-continuous filtration and let $(\tau_t)_{t\in[0,\infty)}$ be a strictly increasing sequence of $(\mathcal{F}_t)$-stopping times such that $t \mapsto \tau_t$ is continuous. Prove that if $X = (X_t)_{t\in[0,\infty)}$ is a continuous local martingale with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$, then $(X_{\tau_t})_{t\in[0,\infty)}$ is a continuous local martingale with respect to $(\mathcal{F}_{\tau_t})_{t\in[0,\infty)}$.

Hint: Consider $A_t := \tau_t^{-1} = \inf\{s \geq 0 : \tau_s \geq t\}$, $t \in [0,\infty)$. Show that A_t is a $(\mathcal{F}_{\tau_u})_{u\in[0,\infty)}$-stopping time for $t \in [0,\infty)$. Secondly, deduce that A_{ν_n} is a $(\mathcal{F}_{\tau_u})_{u\in[0,\infty)}$-stopping time for $n \in \mathbb{N}$, where $(\nu_n)_{n\in\mathbb{N}}$ is a localizing sequence for X. Finally, prove that $(A_{\nu_n})_{n\in\mathbb{N}}$ is a localizing sequence for $(X_{\tau_t})_{t\in[0,\infty)}$, which then gives the claim.

Exercise 7.3

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\in[0,T]}$ be the associated Brownian standard filtration. Suppose that $S_0, \sigma > 0$ and $\mu \in \mathbb{R}$.
Define the probability measure Q by

$$\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}} := L_T \quad \text{with} \quad L_t := \exp\left(-\frac{\mu}{\sigma}B_t - \frac{1}{2}\Big(\frac{\mu}{\sigma}\Big)^2 t\right), \quad t \in [0,T].$$

(i) Show that $(S_t)_{t\in[0,T]}$ is a Q-martingale, where

$$S_t := S_0 + \sigma B_t + \mu t, \quad t \in [0,T].$$

(ii) Show that $(S_t)_{t\in[0,T]}$ is a Q-martingale, where

$$S_t := S_0 \exp(\mu - \frac{1}{2}\sigma^2 t + \sigma B_t), \quad t \in [0,T].$$

Please submit your solutions by Tuesday, the 26th of October, at noon (12 pm).

Programming exercise 7

Consider a European call option with strike price $K = 4$ and time horizon $T = 10$, on a risky asset $(S_t^1)_{t\in[0,T]}$ with $S_0^1 = 5$ and volatility $\sigma = 1$ in the Bachelier model, such that the payoff function is

$$g(S_T^1) = (S_T^1 - 4)^+.$$

Approximately calculate the arbitrage-free price of the option by

(i) (approximately) evaluating

$$\pi(g(S_T^1)) = \int_{\mathbb{R}} g(y) \frac{1}{\sigma\sqrt{2\pi T}} exp\Big(- \frac{(S_0^1 - y)^2}{2\sigma^2 T} \Big) \, \mathrm{d}y.$$

Hint: you can use the package scipy.stats to get the density function of the normal distribution by the function scipy.stats.norm.pdf. You can also use the function scipy.integrate.quad from the package scipy.integrate to (approximately) calculate integrals.

(ii) implementing a Monte-Carlo-scheme to evaluate

$$\pi(g(S_T^1)) = \mathbb{E}^{\mathbb{Q}}[g(S_T^1)],$$

for an appropriate sample size N.