Programming exercise 1

(i) Install Python and Visual Code, and then the packages *numpy* and *matplotlib.pyplot*.

(ii) Let N denote your discretization number per time unit, and T your time horizon. Use the function *random.normal* from *numpy* to generate $N \cdot T$ normal distributed random numbers.

(iii) Generate an approximation of a realization of a Brownian motion by starting at 0 and then summing up your normally distributed random variables. Make sure to choose the correct standard deviation in (ii).

(iv) Generate approximations of the martingales in Exercise 1.3 by using your approximated Brownian motion from (iii).

(v) Plot your approximated processes by using the functions *plot* and *show* from *matplotlib.pyplot*.

Programming exercise 2

Write a function *pth_variation* in Python which takes the variable $p > 0$, the discretization per time step number N and the time horizon T as arguments. The function should

- simulate one realization of a standard Brownian motion $(B_t)_{t\in[0,T]}$,
- then approximate the pth *variation* $(|B|_{p,t})_{t\in[0,T]}$ of $(B_t)_{t\in[0,T]}$, that is,

$$|B|_{p,t} := \lim_{n\to\infty} \sum_{J\in\Pi_n} |\Delta_{J\cap[0,t]} B|^p, \quad t \in [0,T],$$

  where Π_n and $\Delta_{J\cap[0,t]}B$ are defined as in the lecture, by using an equidistance discretization with N partition points on each interval of length 1.
- and then return the approximated realization of $(|B|_{p,t})_{t\in[0,T]}$.

Use the function *pth_variation* to plot the 1*th variation*, 2*th variation* (also called *quadratic variation*) and 3*rd variation* of a standard Brownian motion with $T = 5$ and $N = 10^5$ in one plot.

Remark. *You might have seen the definition of p-variation, as defined by*

$$\|B\|_{p\text{-}var,t} := \sup_{\Pi} \sum_{J\in\Pi} |\Delta_{J\cap[0,t]} B|^p, \quad t \in [0,T],$$

where the supermum is taken over all partitions Π *of the interval* $[0,T]$*, somewhere already.*

Note that $|B|_{1,t} = |B|_{1\text{-}var,t}$ *does only hold for* $p = 1$*. Indeed, for* $p > 1$ *the pth variation* $|\cdot|_{p,t}$ *(from Programming exercise* 2*) does* ***not*** *coincide with the p-variation* $\|\cdot\|_{p\text{-}var,t}$ *in general.*

Programming exercise 3

In this exercise, your task is to approximate pathwise Lebesque–Stieltjes integrals of the form

$$X_t := \int_0^t f(s, B_s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{1}$$

where $B = (B_t)_{t\in[0,T]}$ is a Brownian motion, and stochastic Itô integrals of the form

$$Y_t := \int_0^t g(s, B_s)\,\mathrm{d}B_s, \quad t \in [0,T]. \tag{2}$$

To do that, use an equal width discretization of $[0,T]$ with N discretization steps per time unit, and the approximation formulas

$$\hat{X}_{t_{n+1}} = \hat{X}_{t_n} + f(t_n, B_{t_n})\Delta_n, \quad n = 0, \dots, TN-1, \tag{3}$$

and

$$\hat{Y}_{t_{n+1}} = \hat{Y}_{t_n} + g(t_n, B_{t_n})\Delta B_n, \quad n = 0, \dots, TN-1, \tag{4}$$

where $\Delta_n := t_{n+1} - t_n$ and $\Delta B_n := B_{t_{n+1}} - B_{t_n}$. Use (3) and (4) to visualize that the following equalities hold:

(i)
$$\int_0^t B_s \,\mathrm{d}B_s = \frac{1}{2}(B_t^2 - t),$$

(ii)
$$\int_0^t B_s^2 \,\mathrm{d}B_s = \frac{1}{3}B_t^3 - \int_0^t B_s \,\mathrm{d}s,$$

(iii)
$$\int_0^t s^2 \,\mathrm{d}B_s = t^2 B_t - 2\int_0^t sB_s \,\mathrm{d}s.$$

Plot your approximations of the left- and right-hand sides of (i), (ii) and (iii) with $T = 5$ and $N = 5$, 100, 10000 to visualize that the approximations (3) and (4) converge against the integrals (1) and (2).

Remark. *(a) The approximation in* (3) *is a simple Riemann sum, and we could therefore also evaluate f on the right interval bound $(t_{n+1}, B_{t_{n+1}})$ here. The approximation in* (4) *though is a simple form of the so-called Euler-scheme. Evaluating g on $(t_{n+1}, B_{t_{n+1}})$ would yield a wrong convergence here.*

(b) It is easy to prove the equalities (i), (ii) and (iii) by using Itô's formula, which is the equivalent of the fundamental theorem of calculus for the stochastic integration theory, and will be content of the lecture soon.

Programming exercise 4

In this exercise, we will show that stochastic integration (in contrast to Riemann integration) is not unique in terms of where to evaluate the integrand. Define therefore for general $(A_t)_{t\in[0,T]}$ and $f\colon [0,T] \to \mathbb{R}$, and for some constant $\alpha \in [0,1]$, the α-integrals

$$\int_0^t f(s)\,\mathrm{d}^\alpha A_s := \lim_{n\to\infty} \sum_{t_i^n \in \Pi^n} \Big(f(t_i^n) + \alpha\Big(f(t_{i+1}^n) - f(t_i^n)\Big)\Big) \cdot \Big(A_{t_{i+1}^n \wedge t} - A_{t_i^n \wedge t}\Big), \tag{5}$$

where $(\Pi^n)_{n\in\mathbb{N}}$ is any zero-sequence of partitions of $[0,T]$. Show numerically (by approximating both sides of the equations in one plot respectively), that:

(i) For $A_t := t$, one has

$$\int_0^t B_s \,\mathrm{d}^0 s = \int_0^t B_s \,\mathrm{d}^{\frac{1}{2}} s = \int_0^t B_s \,\mathrm{d}^1 s = 2tB_t - \int_0^t s \,\mathrm{d}B_s, \quad \forall t \in [0,T].$$

(ii) For $A_t := B_t$ $\forall t \in [0,T]$ for some Brownian motion $(B_t)_{t\in[0,T]}$, one has

(a)
$$\int_0^t B_s \,\mathrm{d}^0 B_s = \frac{1}{2}(B_t^2 - t),$$

(b)

$$\int_0^t B_s \, \mathrm{d}^{\frac{1}{2}} B_s = \frac{1}{2} B_t^2,$$

(c)

$$\int_0^t B_s \, \mathrm{d}^1 B_s = \frac{1}{2}(B_t^2 + t).$$

for $t \in [0, T]$.

Remark. *In (ii), the stochastic integral in (a) is the well-known Itô integral, while (b) and (c) are the so-called 'Stratonovich' and 'backward-Itô' integrals. If f is a stochastic process, then the integrals in (b) and (c) are not adapted anymore, in contrast to the Itô integral. This is due to the fact that you need to 'look a bit into the future' of the process f to evaluate the integrals in (b) and (c). On the other hand, calculating with the Stratonovich integral has its advantages since the chain rule of ordinary calculus holds here (instead of Itô's formula).*

Programming exercise 5

From Exercise 5.3 (i) we know that the quadratic covariation of two Itô processes $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ can be written as

$$\langle X, Y\rangle_t = \lim_{n\to\infty} \sum_{J\in\Pi_n} (\Delta_{J\cap[0,t]} X)(\Delta_{J\cap[0,t]} Y) \qquad \text{in probability,}$$

for any zero-sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ and all $t \in [0, T]$.

(i) Let $(B_t^{(1)})_{t\in[0,T]}$ and $(B_t^{(2)})_{t\in[0,T]}$ be two independent Brownian motions, and define for $\rho \in [0, 1]$:

$$\tilde{B}_t := \rho B_t^{(1)} + \sqrt{1-\rho^2} B_t^{(2)}, \quad t \in [0, T].$$

Show graphically that

$$\langle B^{(1)}, \tilde{B}\rangle_t = \rho t, \quad t \in [0, T],$$

and

$$\langle B^{(2)}, \tilde{B}\rangle_t = \sqrt{1-\rho^2}\, t, \quad t \in [0, T].$$

(ii) Consider the Itô processes

$$X_t = X_0 + \int_0^t a(\cdot, s) ds + \int_0^t b(\cdot, s) \, \mathrm{d}B_s, \quad t \in [0, T],$$

and

$$\tilde{X}_t = X_0 + \int_0^t a(\cdot, s) ds + \int_0^t \tilde{b}(\cdot, s) \, \mathrm{d}B_s, \quad t \in [0, T],$$

where $X_0 = 0$, $a(\cdot, t) = t$, $b(\cdot, t) = \sqrt{2t}$ and $\tilde{b}(\cdot, t) = \frac{3}{\sqrt{2}} t^2$, for $t \in [0, T]$.

Show graphically that

$$\langle X\rangle_t = \int_0^t b^2(\cdot, s) \, \mathrm{d}s, \quad t \in [0, T],$$

and

$$\langle X, \tilde{X}\rangle_t = \int_0^t b(\cdot, s)\tilde{b}(\cdot, s) \, \mathrm{d}s, \quad t \in [0, T].$$

Programming exercise 6

Let $X = (X_t)_{t\in[0,T]}$ be a solution of the SDE

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t, \quad t \in [0, T], \quad X_0 = x_0$$

where $(B_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion. For the equidistant grid given by $t_i = \frac{i}{n}T$, $i = 0, \dots, n$, the *Euler-Maruyama approximation* $Y = (Y_i, i \in \{0, \dots, n\})$ of X is given by $Y_0 = x_0$ and

$$Y_{i+1} = Y_i + \mu(t_i, Y_i)\frac{T}{n} + \sigma(t_i, Y_i)(B_{t_{i+1}} - B_{t_i}), \qquad i = 0, \dots, n-1,$$

Implement this approximation scheme for

(i) $\mathrm{d}X_t = X_t\,\mathrm{d}t + X_t\,\mathrm{d}B_t$, $t \in [0, T]$ with $X_0 = 1$ and

(ii) $\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t$, $t \in [0, T]$ with $X_0 = 1$.

Choose an appropriate time horizon T and step size n. For (i), also visualize the exact solution that you know from the lecture in the plot and compare it to the *Euler-Maruyama approximation*.

Programming exercise 7

Consider a European call option with strike price $K = 4$ and time horizon $T = 10$, on a risky asset $(S_t^1)_{t\in[0,T]}$ with $S_0^1 = 5$ and volatility $\sigma = 1$ in the Bachelier model, such that the payoff function is

$$g(S_T^1) = (S_T^1 - 4)^+.$$

Approximately calculate the arbitrage-free price of the option by

(i) (approximately) evaluating

$$\pi(g(S_T^1)) = \int_{\mathbb{R}} g(y)\frac{1}{\sigma\sqrt{2\pi T}}exp\Big(-\frac{(S_0^1 - y)^2}{2\sigma^2 T}\Big)\,\mathrm{d}y.$$

Hint: you can use the package scipy.stats to get the density function of the normal distribution by the function scipy.stats.norm.pdf. You can also use the function scipy.integrate.quad from the package scipy.integrate to (approximately) calculate integrals.

(ii) implementing a Monte-Carlo-scheme to evaluate

$$\pi(g(S_T^1)) = \mathbb{E}^{\mathbb{Q}}[g(S_T^1)],$$

for an appropriate sample size N.