# 1 Motivation

If $M$ is a smooth closed orientable manifold of dimension $n$, then the de Rham complex of complex-valued differential forms on $M$,

$$0 \to \Omega^0(M) \overset{d_0}{\to} \Omega^1(M) \overset{d_1}{\to} \Omega^2(M) \to \cdots \to \Omega^n(M) \to 0\,,$$

is known to be Fredholm in a suitable $L^2$ completion. This means that the images of $d_k$ are closed and the vector spaces $\ker d_k / \operatorname{im} d_{k-1}$ are finite-dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. The de Rham theorem states that $\ker d_k / \operatorname{im} d_{k-1} \cong H^k(M;\mathbb{C})$, implying that the above index equals $\chi(M)$, the Euler characteristic of $M$.

The goal of the paper is to extend these results to manifolds periodic ends.

# 2 Periodic Manifolds

A manifold $M$ with periodic end $X$ is a Riemann manifold of the form

$$Z_\infty = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots\,,$$

where $W_k$ are isometric copies of the fundamental segment $W$ obtained by cutting $X$ open along an oriented connected submanifold $Y$ and $Z$ is a smooth compact manifold with boundary $Y$.

The completion of the de Rham complex of $M$ in the $L^2$ norm using over the end a Riemann measure $dx$ lifted from that on $X$ is not Fredholm. To rectify this, we will use $L^2_\delta$ norms, which are the $L^2$ norms on $M$ with respect to the measure $e^{\delta f(x)}\,dx$ over the end. Here $\delta$ is a real number and $f : \tilde{X} \to \mathbb{R}$ is a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \to \tilde{X}$. We shall denote the $L^2_\delta$ completion of the de Rham complex on $M$ by $\Omega^*_\delta(M)$.

**Theorem 2.1.** *Let $M$ be a smooth Riemannian manifold manifold with a periodic end modeled on $\tilde{X}$, and suppose that $H_*(M;\mathbb{C})$ is finite-dimensional. Then $\Omega^*_\delta(M)$ is Fredholm for all but finitely many $\delta$ of the form $\delta = \ln|\lambda|$, where $\lambda$ is a root of the characteristic polynomial of $\tau_* : H_*(\tilde{X};\mathbb{C}) \to H_*(\tilde{X};\mathbb{C})$.*

Given a manifold $M$ as in the above theorem, the complex $\Omega^*_\delta(M)$ has a well-defined index $\operatorname{ind}_\delta(M)$. It is known, due to Miller, that $\operatorname{ind}_\delta(M)$ is an even or odd function of $\delta$ according to whether $\dim M = n$ is even or odd, and that $\operatorname{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. The result of following theorem completes the calculation of the function $\operatorname{ind}_\delta(M)$.

**Theorem 2.2.** *Let $M$ be as in Theorem 2.1. Then* $\operatorname{ind}_\delta(M)$ *is a piecewise constant function of $\delta$ whose only jumps occur at $\delta = \ln|\lambda|$, where $\lambda$ is a root of the characteristic polynomial $A_k(t)$ of $\tau_* : H_k(\tilde{X};\mathbb{C}) \to H_k(\tilde{X};\mathbb{C})$ for some $k \in [0 : n-1]$. Every such $\lambda$ contributes $(-1)^{k+1}$ times its multiplicity as a root of $A_k(t)$ to the jump.*

# 3 Finite Dimensionality

Let $M$ be a smooth orientable manifold with a periodic end modeled on $\tilde{X}$. It is stated that vanishing of $\chi(X)$ is a necessary yet not sufficient condition for the vector space $H_*(M;\mathbb{C})$ to be finite dimensional. To obtain a sufficient condition, observe that the derivative *df* defines a closed 1-form on $X$ and let $\xi = [df] \in H^1(X;\mathbb{C})$ be its cohomology class[1]. The cup product with $\xi$ gives rise to the chain complex

$$H^0(X;\mathbb{C}) \xrightarrow{\cup\xi} H^1(X;\mathbb{C}) \xrightarrow{\cup\xi} \cdots \xrightarrow{\cup\xi} H^n(X;\mathbb{C})\,. \tag{1}$$

**Proposition 3.1.** *Suppose the chain complex 1 is exact. Then $H_*(M;\mathbb{C})$ is a finite-dimensional vector space for any smooth orientable manifold with periodic end modeled on $\tilde{X}$.*

[1]Recall that $f : \tilde{X} \to \mathbb{R}$ denotes a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \to \tilde{X}$.