1 Motivation

If M is a smooth closed orientable manifold of dimension n, then the de Rham complex of complex-valued differential forms on M,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0\,,$$

is known to be Fredholm in a suitable L^2 completion. This means that the images of d_k are closed and the vector spaces $\ker d_k / \operatorname{im} d_{k-1}$ are finite-dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. The de Rham theorem states that $\ker d_k / \operatorname{im} d_{k-1} \cong H^k(M;\mathbb{C})$, implying that the above index equals $\chi(M)$, the Euler characteristic of M.

The goal of the paper is to extend these results to manifolds periodic ends.

2 Periodic Manifolds

A manifold M with periodic end modeled on an infinite cyclic cover $\tilde{X}$ of a compact manifold X associated with a primitive cohomology class $\gamma \in H^1(X;\mathbb{Z})$ is a Riemann manifold of the form

$$Z_\infty = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots\,,$$

where W_k are isometric copies of the fundamental segment W obtained by cutting X open along an oriented connected submanifold Y and Z is a smooth compact manifold with boundary Y.

The completion of the de Rham complex of M in the L^2 norm using over the end a Riemann measure dx lifted from that on X is not Fredholm. To rectify this, we will use L^2_δ norms, which are the L^2 norms on M with respect to the measure $e^{\delta f(x)}\,dx$ over the end. Here δ is a real number and $f : \tilde{X} \to \mathbb{R}$ is a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \to \tilde{X}$. We shall denote the L^2_δ completion of the de Rham complex on M by $\Omega^*_\delta(M)$.

Theorem 2.1. *Let M be a smooth Riemannian manifold manifold with a periodic end modeled on $\tilde{X}$, and suppose that $H_*(M;\mathbb{C})$ is finite-dimensional. Then $\Omega^*_\delta(M)$ is Fredholm for all but finitely many δ of the form $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial of $\tau_* : H_*(\tilde{X};\mathbb{C}) \to H_*(\tilde{X};\mathbb{C})$.*

Given a manifold M as in the above theorem, the complex $\Omega^*_\delta(M)$ has a well-defined index $\operatorname{ind}_\delta(M)$. It is known, due to Miller, that $\operatorname{ind}_\delta(M)$ is an even or odd function of δ according to whether $\dim M = n$ is even or odd, and that $\operatorname{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. The result of following theorem completes the calculation of the function $\operatorname{ind}_\delta(M)$.

Theorem 2.2. *Let M be as in Theorem 2.1. Then $\mathrm{ind}_\delta(M)$ is a piecewise constant function of δ whose only jumps occur at $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial $A_k(t)$ of $\tau_* : H_k(\tilde{X};\mathbb{C}) \to H_k(\tilde{X};\mathbb{C})$ for some $k \in [0 : n-1]$. Every such λ contributes $(-1)^{k+1}$ times its multiplicity as a root of $A_k(t)$ to the jump.*

To be precise, we have a formula

$$\mathrm{ind}_\delta(M) = (-1)^n \chi(M) + \sum (-1)^k \; \#\{\lambda \mid A_k(\lambda) = 0\,, |\lambda| > e^\delta\} \; .$$

3 Finite Dimensionality

Let M be a smooth orientable manifold with a periodic end modeled on $\tilde{X}$. It is stated that vanishing of $\chi(X)$ is a necessary yet not sufficient condition for the vector space $H_*(M;\mathbb{C})$ to be finite dimensional. To obtain a sufficient condition, observe that the derivative df defines a closed 1-form on X and let $\xi = [df] \in H^1(X;\mathbb{C})$ be its cohomology class[1]. The cup product with ξ gives rise to the chain complex

$$H^0(X;\mathbb{C}) \xrightarrow{\cup\xi} H^1(X;\mathbb{C}) \xrightarrow{\cup\xi} \cdots \xrightarrow{\cup\xi} H^n(X;\mathbb{C})\,. \tag{1}$$

Proposition 3.1. *Suppose the chain complex (1) is exact. Then $H_*(M;\mathbb{C})$ is a finite-dimensional vector space for any smooth orientable manifold with periodic end modeled on $\tilde{X}$.*

4 Examples

Example 4.1. A manifold with product end is a smooth Riemannian manifold whose end is modeled on $\tilde{X} = \mathbb{R} \times Y$, where Y is a closed Riemannian manifold. The metric on $\mathbb{R} \times Y$ is presumed to be the product metric. The covering translation induces an identity map τ_* on the homology of $\mathbb{R} \times Y$. Since $\lambda = 1$ is the only root of the characteristic polynomial of τ_*, the complex $\Omega^*_\delta(M)$ is Fredholm for all $\delta \neq 0$. Its index $\mathrm{ind}_\delta(M)$ equals $\chi(M)$ if the dimension of M is even, and $-\operatorname{sgn}\delta \cdot \chi(M)$ if the dimension of M is odd. Note that the same is true for any manifold whose periodic end is modeled on $\tilde{X}$ such that the characteristic polynomial of $\tau_* : H_*(\tilde{X};\mathbb{C}) \to H_*(\tilde{X};\mathbb{C})$ only has unitary roots.

[1]Recall that $f : \tilde{X} \to \mathbb{R}$ denotes a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \to \tilde{X}$.