

1 Motivation

If M is a smooth closed orientable manifold of dimension n , then the de Rham complex of complex-valued differential forms on M ,

$$0 \rightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \rightarrow \cdots \rightarrow \Omega^n(M) \rightarrow 0,$$

is known to be Fredholm in a suitable L^2 completion. This means that the images of d_k are closed and the vector spaces $\ker d_k / \operatorname{im} d_{k-1}$ are finite-dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. The de Rham theorem states that $\ker d_k / \operatorname{im} d_{k-1} \cong H^k(M; \mathbb{C})$, implying that the above index equals $\chi(M)$, the Euler characteristic of M .

The goal of the paper is to extend these results to manifolds periodic ends.

2 Periodic Manifolds

A manifold M with periodic end X is a Riemann manifold of the form

$$Z_\infty = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots,$$

where W_k are isometric copies of the fundamental segment W obtained by cutting X open along an oriented connected submanifold Y and Z is a smooth compact manifold with boundary Y .

The completion of the de Rham complex of M in the L^2 norm using over the end a Riemann measure dx lifted from that on X is not Fredholm. To rectify this, we will use L_δ^2 norms, which are the L^2 norms on M with respect to the measure $e^{\delta f(x)} dx$ over the end. Here δ is a real number and $f : \tilde{X} \rightarrow \mathbb{R}$ is a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \rightarrow \tilde{X}$. We shall denote the L_δ^2 completion of the de Rham complex on M by $\Omega_\delta^*(M)$.

Theorem 2.1. *Let M be a smooth Riemannian manifold with a periodic end modeled on \tilde{X} , and suppose that $H_*(M; \mathbb{C})$ is finite-dimensional. Then $\Omega_\delta^*(M)$ is Fredholm for all but finitely many δ of the form $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial of $\tau_* : H_*(\tilde{X}; \mathbb{C}) \rightarrow H_*(\tilde{X}; \mathbb{C})$.*

Given a manifold M as in the above theorem, the complex $\Omega_\delta^*(M)$ has a well-defined index $\operatorname{ind}_\delta(M)$. It is known, due to Miller, that $\operatorname{ind}_\delta(M)$ is an even or odd function of δ according to whether $\dim M = n$ is even or odd, and that $\operatorname{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. The result of following theorem completes the calculation of the function $\operatorname{ind}_\delta(M)$.

Theorem 2.2. *Let M be as in Theorem 2.1. Then $\text{ind}_\delta(M)$ is a piecewise constant function of δ whose only jumps occur at $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial $A_k(t)$ of $\tau_* : H_k(\tilde{X}; \mathbb{C}) \rightarrow H_k(\tilde{X}; \mathbb{C})$ for some $k \in [0 : n - 1]$. Every such λ contributes $(-1)^{k+1}$ times its multiplicity as a root of $A_k(t)$ to the jump.*

To be precise, we have the formula

$$\text{ind}_\delta(M) = (-1)^n \chi(M) + \sum (-1)^k \# \{ \lambda \mid A_k(\lambda) = 0, |\lambda| > e^\delta \}.$$

3 Finite Dimensionality

Let M be a smooth orientable manifold with a periodic end modeled on \tilde{X} . It is stated that vanishing of $\chi(X)$ is a necessary yet not sufficient condition for the vector space $H_*(M; \mathbb{C})$ to be finite dimensional. To obtain a sufficient condition, observe that the derivative df defines a closed 1-form on X and let $\xi = [df] \in H^1(X; \mathbb{C})$ be its cohomology class¹. The cup product with ξ gives rise to the chain complex

$$H^0(X; \mathbb{C}) \xrightarrow{\cup \xi} H^1(X; \mathbb{C}) \xrightarrow{\cup \xi} \dots \xrightarrow{\cup \xi} H^n(X; \mathbb{C}). \quad (1)$$

Proposition 3.1. *Suppose the chain complex (1) is exact. Then $H_*(M; \mathbb{C})$ is a finite-dimensional vector space for any smooth orientable manifold with periodic end modeled on \tilde{X} .*

4 Examples

Example 4.1. A manifold with product end is a smooth Riemannian manifold whose end is modeled on $\tilde{X} = \mathbb{R} \times Y$, where Y is a closed Riemannian manifold. The metric on $\mathbb{R} \times Y$ is presumed to be the product metric. The covering translation induces an identity map τ_* on the homology of $\mathbb{R} \times Y$. Since $\lambda = 1$ is the only root of the characteristic polynomial of τ_* , the complex $\Omega_\delta^*(M)$ is Fredholm for all $\delta \neq 0$. Its index $\text{ind}_\delta(M)$ equals $\chi(M)$ if the dimension of M is even, and $-\text{sgn } \delta \cdot \chi(M)$ if the dimension of M is odd. Note that the same is true for any manifold whose periodic end is modeled on \tilde{X} such that the characteristic polynomial of $\tau_* : H_*(\tilde{X}; \mathbb{C}) \rightarrow H_*(\tilde{X}; \mathbb{C})$ only has unitary roots.

¹Recall that $f : \tilde{X} \rightarrow \mathbb{R}$ denotes a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \rightarrow \tilde{X}$.