A Comparative Analysis of Value at Risk (VaR) Methodologies: Historical, Parametric, and Monte Carlo Approaches

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Abstract

Value-at-risk (VaR) is a cornerstone metric in financial risk management, used to estimate the potential loss of a portfolio over a given time horizon with a specified confidence level. This paper provides an exposition of three widely used VaR estimation methods: the Historical simulation, Parametric (Variance-Covariance) approach, and Monte Carlo simulation. We compare their mathematical assumptions, computational requirements, and practical advantages and limitations. A demonstration on a five-asset portfolio using Python is shown in Appendix A-C.

1 Introduction

Value-at-risk (VaR) provides a probabilistic measure of potential losses in a portfolio. It plays a critical role in regulatory frameworks (e.g. Basel III) and internal risk management for financial institutions. Basel III frameworks also mandate Conditional Value-at-Risk (CVaR) for market risk capital allocation due to its coherence in capturing tail risks. This paper aims to clarify and compare the three main methods for computing VaR and CVaR.

Value-at-risk is an important metric that is utilised

2 Mathematical Preliminaries

Value-at-Risk can be defined as:

$$\mathrm{VaR}_\alpha(L) = \inf\{x \in \mathbb{R} : F_L(x) \geq 1 - \alpha\}$$

Where L is a random variable for Loss, α is the significance level, $F_L(x)$ is the cumulative density function (CDF) of the loss distribution.

Conditional Value-at-Risk is defined as:

$$\mathrm{CVaR}_\alpha(L) = \mathbb{E}[L | L \geq \mathrm{VaR}_\alpha(L)]$$

This is the average loss value given that the value at risk threshold has been passed. This is useful in determining tail risks

3 Historical Simulation Method

3.1 Methodology

The Historical Simulation Method estimates Value at Risk (VaR) and Conditional Value at Risk (CVaR) by using empirical historical return data without assuming any specific distribution for returns.

The process is as follows:

1. **Data Collection:** Obtain historical daily returns for each asset in the portfolio over a selected window of T trading days (e.g. five years ≈ 1250 days).

2. **Portfolio Returns:** Compute the daily portfolio return series by applying the current portfolio weights $w = (w_1, w_2, \ldots, w_n)$ to the individual asset returns $r_t^{(i)}$:

$$r_t^p = \sum_{i=1}^{n} w_i \cdot r_t^{(i)} \quad \text{for } t = 1, \ldots, T$$

3. **Return based Value at Risk:** Sort the portfolio return series from worst to best and identify the α-quantile (e.g. 5% for 95% confidence). This quantile is the Historical VaR:

$$\text{VaR}_\alpha = \text{Quantile}_\alpha \left(r^p \right)$$

4. **Return based Conditional Value at Risk:** Calculate the average of all losses that fall below the VaR threshold. This is known as Conditional VaR (CVaR) or Expected Shortfall:

$$\text{CVaR}_\alpha = \mathbb{E}[r_t^p \mid r_t^p < \text{VaR}_\alpha]$$

5. **Scaling to Monetary Terms:** Multiply the value of the return-based VaR and CVaR by the total portfolio value V_0 to obtain monetary estimates where we report VaR as a loss value:

$$\text{VaR}_\alpha^\$ = V_0 \cdot \max\left(-\text{Quantile}_\alpha(r^p), 0\right), \quad \text{CVaR}_\alpha^\$ = V_0 \cdot \text{CVaR}_\alpha(r^p)$$

This approach provides a purely data-driven estimate of portfolio downside risk, directly using observed market behaviour.

3.2 Strengths and Weaknesses

Strengths:

- **Non-parametric:** No assumption of a specific distribution (e.g. normality) for asset returns. Captures skewness, fat tails, and asymmetries naturally.

- **Intuitive and transparent:** Directly reflects real historical events, including crises and volatility clustering.

- **Easy to implement:** Requires only historical price data and basic matrix arithmetic.

Weaknesses:

- **Dependence on historical window:** Results are sensitive to the chosen sample period. Too short a window may omit rare events; too long a window may overweight outdated dynamics.
- **Assumes history repeats:** Implies that the empirical distribution of past returns is a good proxy for future outcomes, which may not hold in changing market conditions.
- **Lack of extrapolation:** Cannot estimate risk beyond the range of observed outcomes (i.e. no interpolation or tail modelling).

3.3 Conclusion

The Historical Simulation Method is a powerful non-parametric alternative to parametric VaR. It is particularly useful when return distributions deviate significantly from normality or when risk managers prefer data-driven insights. However, its reliability hinges on the quality and relevance of historical data, and it may fail to anticipate extreme losses not present in the sample window.

4 Parametric (Variance-Covariance) Method

4.1 Assumptions

The parametric Value at Risk (VaR) method, also known as the variance-covariance approach, is a model-based technique that relies on several simplifying assumptions:

- **Linearity:** The portfolio is assumed to consist of only linear financial instruments, such as stocks and bonds. Derivatives and other non-linear instruments (e.g. options) are excluded unless linearised using sensitivity measures (such as delta).
- **Normality:** Asset returns are assumed to be independently and identically distributed (i.i.d.) and follow a multivariate normal distribution with mean μ and standard deviation σ.
- **Stationarity:** Historical return statistics (mean and covariance) are assumed to remain constant over the risk horizon.

These assumptions allow for a closed-form solution of VaR, making the method computationally efficient, but potentially less robust to real-world phenomena such as fat tails and volatility clustering.

4.2 Mathematical Formulation

Let the portfolio consist of n assets with returns $r_1, r_2, \ldots, r_n$, and weights $w_1, w_2, \ldots, w_n$. Let the vector of portfolio weights be denoted by:

$$w = \begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix}$$

Let C denote the $n \times n$ variance-covariance matrix of asset returns:

$$C = \begin{bmatrix} \mathrm{Var}(r_1) & \mathrm{Cov}(r_1, r_2) & \dots & \mathrm{Cov}(r_1, r_n) \\ \mathrm{Cov}(r_2, r_1) & \mathrm{Var}(r_2) & \dots & \mathrm{Cov}(r_2, r_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(r_n, r_1) & \mathrm{Cov}(r_n, r_2) & \dots & \mathrm{Var}(r_n) \end{bmatrix}$$

Then, the variance of the portfolio return is computed as:

$$\sigma_p^2 = w^T C w$$

and the standard deviation is:

$$\sigma_p = \sqrt{w^T C w}$$

4.3 Value at Risk Calculation

Assuming the portfolio return is normally distributed, the one-period (e.g. daily) return based VaR at a confidence level $1 - \alpha$ is given by:

$$\mathrm{VaR}_\alpha^\$ = V_0 \cdot \left(-\mu_t + z_\alpha \cdot \sigma_p \cdot \sqrt{t} \right)$$

where:

- z_α is the critical value (usually 1% or 5%) of the standard normal distribution, such that $\mathbb{P}(Z < -z_\alpha) = \alpha$.
- σ_p is the standard deviation of the portfolio's return.
- t is the time horizon (e.g. $t = 1$ for daily VaR, $t = 10$ for 10-day VaR).
- μ_t is the mean return over the time horizon.

This gives the maximum expected loss over the time horizon at the α significance level, assuming normality.

CVaR is then computed by integrating the left tail of the normal distribution beyond the VaR cut-off:

$$\mathrm{CVaR}_\alpha^\$ = V_0 \cdot \left(-\mu t + \sigma \sqrt{t} \cdot \frac{\phi(z_\alpha)}{\alpha} \right)$$

where $\phi(z)$ is the probability density function (PDF) of the standard normal distribution evaluated at z_α.

4.4 Advantages and Disadvantages

Advantages:

- **Analytical simplicity:** The closed-form expression makes this method fast and computationally efficient.
- **Low data requirements:** Only the mean vector and covariance matrix are needed, which can be estimated from historical data.
- **Scalability:** Easily applied to large portfolios using matrix operations.

Disadvantages:

- **Normality assumption:** Real financial returns often exhibit fat tails and skewness, leading to underestimation of extreme losses.
- **Linearity assumption:** This method is not valid for portfolios containing options or other instruments with non-linear payoffs.
- **Static covariance:** The method assumes a constant covariance structure, ignoring dynamic market volatility.
- **Sensitivity to estimation error:** The quality of the VaR estimate heavily depends on the stability and accuracy of historical covariance estimates.

4.5 Conclusion

The parametric approach provides a fast and transparent method for computing VaR under idealized assumptions. However, its limitations must be recognised especially in markets characterized by non-Gaussian behaviour, structural breaks, or portfolios with derivative exposure. In such cases, non-parametric approaches such as the historical or Monte Carlo methods may be more appropriate.

5 Monte Carlo Simulation Method for VaR and CVaR

5.1 Overview

The Monte Carlo (MC) simulation method provides a flexible and powerful approach to estimating the value at risk (VaR) and the conditional value at risk (CVaR). Unlike historical or parametric methods, it uses simulated return paths based on statistical properties of the portfolio to estimate potential losses.

5.2 Theoretical Foundation

Let $\mathbf{R} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ represent the multivariate normal distribution of asset returns with the expected return vector $\boldsymbol{\mu}$ and the covariance matrix Σ. The Monte Carlo process involves the following steps:

1. Estimate the historical means $\boldsymbol{\mu}$ and the covariance matrix Σ of the asset returns.
2. Simulate a large number of return paths using:
$$\mathbf{R}_t^{(m)} = \boldsymbol{\mu} + L\mathbf{z}_t^{(m)}, \quad \mathbf{z}_t^{(m)} \sim \mathcal{N}(0, I)$$
where L is the Cholesky decomposition of Σ, $m = 1, \ldots, M$ (number of simulations) and $t = 1, \ldots, T$ (time steps).
3. Compute the cumulative portfolio values over time for each simulation:
$$V^{(m)} = V_0 \cdot (1 + \mathbf{w}^T \mathbf{R}^{(m)})$$
4. Use the distribution of simulated final values V_T to estimate:
   - **VaR:** The α-percentile of the profit/loss distribution.
   - **CVaR:** The expected loss given that the loss exceeds the VaR threshold.

5.3 Advantages and Limitations

- **Advantages:** While this example assumes normality, Monte Carlo simulations can model non-normal distributions such as the Student-t.
- **Limitations:** Computationally intensive and dependent on input assumptions and historical data quality.

Appendix A: Empirical Estimation of Historical VaR and CVaR Using Python

To illustrate the application of Value at Risk (VaR) and Conditional Value at Risk (CVaR), we implement a historical simulation approach using real financial market data. This empirical analysis is conducted in Python and uses the `yfinance`, `NumPy`, `Pandas`, and `SciPy` libraries.

Portfolio Construction

A portfolio of five financial assets was selected: TLT (bonds), SPY (U.S. equities), QQQ (tech stocks), GLD (gold), and CL=F (crude oil futures). The value of the initial investment is $100,000. Equal portfolio weights were assigned to each asset:

$$w = [0.2, 0.2, 0.2, 0.2, 0.2]$$

Daily historical price data for each asset over approximately five years (1250 trading days) was downloaded. Returns were computed as daily percentage changes.

Portfolio Returns and Risk

The portfolio return series was calculated as the weighted sum of the individual asset returns:

$$r_{\text{portfolio}}(t) = \sum_{i=1}^{n} w_i \cdot r_i(t)$$

We then compute the expected daily return μ_p and standard deviation σ_p of the portfolio, which are used for scaling and context in risk metrics.

Historical Value at Risk (VaR)

The Historical VaR at confidence level $\alpha = 5\%$ was computed non-parametrically as the 5th percentile of the empirical distribution of historical portfolio returns:

$$\text{VaR}_\alpha = \text{Quantile}_\alpha(r_{\text{portfolio}})$$

This represents the maximum expected loss under normal market conditions with 95% confidence.

Conditional Value at Risk (CVaR)

CVaR, or Expected Shortfall, is defined as the average loss given that the loss exceeds the VaR threshold:

$$\text{CVaR}_\alpha = \mathbb{E}[r \mid r < \text{VaR}_\alpha]$$

This provides a more conservative risk measure by accounting for tail losses.

Results

The Python output prints the following:

- **Expected Portfolio Return:** Total expected return over one day (scaled by investment).
- **VaR (95% confidence):** Worst-case daily loss not exceeded 95% of the time.
- **CVaR (95% confidence):** Average loss conditional on exceeding the VaR.

Visualization

To visualize the distribution of historical returns:

- A histogram of portfolio returns was plotted.
- A kernel density estimate (KDE) was superimposed to smooth the distribution.
- Markers for the expected return, VaR, and CVaR were added to the plot.

The KDE estimate provides a continuous approximation of the portfolio return distribution, highlighting the fat-tailed nature of empirical financial returns and supporting the interpretation of VaR and CVaR in a realistic context.

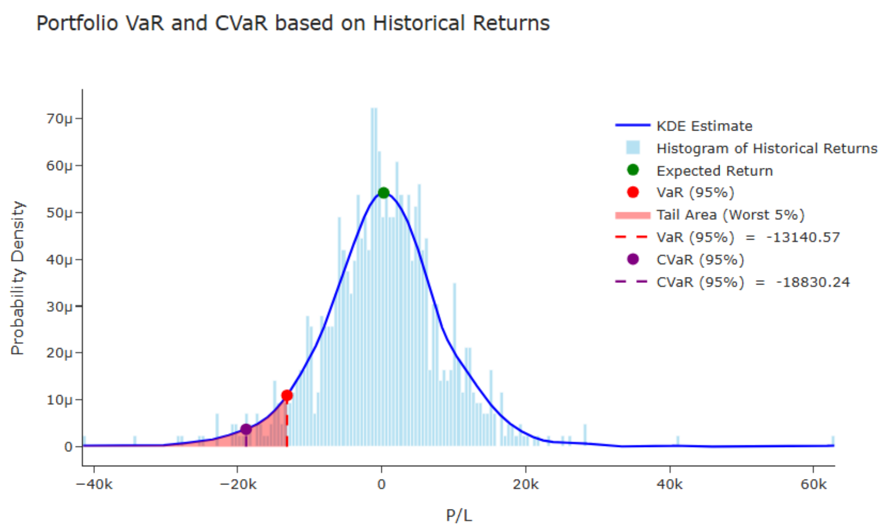

Figure 1: Histogram and KDE of Historical Portfolio Returns with VaR and CVaR at 95% Confidence

Conclusion

This simple empirical framework demonstrates the utility and limitations of historical simulation for estimating risk measures. While easy to implement and assumption-free, it relies heavily on the stability of past return distributions and may understate risk during regime shifts or crisis periods.

Appendix B: Parametric (Variance-Covariance) Method: Empirical Implementation in Python

This appendix provides a numerical implementation of the variance-covariance approach to Value at Risk (VaR) and Conditional Value at Risk (CVaR) using historical market data and Python.

1. Data Retrieval: We collect historical daily closing prices for a portfolio of five equally weighted assets: TLT, SPY, QQQ, GLD, and crude oil futures (CL=F). Data is obtained from Yahoo Finance for a period of approximately 1250 trading days.

2. Portfolio Return and Risk: Daily returns are calculated and used to compute the portfolio's expected return and standard deviation:

$$r^p = \sum_{i=1}^{n} w_i \mu_i \cdot, \quad \sigma^p = \sqrt{w^T \Sigma w}$$

where Σ is the sample covariance matrix of asset returns, w_i are the portfolio weights, and 250 adjusts the returns from annual to daily.

3. Parametric VaR: Under the assumption of normally distributed returns, we compute VaR at a significance level α using the inverse cumulative distribution function:

$$\text{VaR}_\alpha = V_0 \cdot \left(-\mu t + z_\alpha \cdot \sigma \sqrt{t}\right)$$

where z_α is the quantile of the standard normal distribution corresponding to $\alpha\%$ and t is the time horizon on risk. In the python example, only daily VaR is calculated where $t = 1$. However we can adjust based on t, if $t = 2$, the VaR would reflect the potential loss in value of the portfolio in 2 trading days and so on.

4. Conditional VaR (CVaR): CVaR is computed by integrating the left tail of the normal distribution beyond the VaR cut-off:

$$\text{CVaR}_\alpha^\$ = V_0 \cdot \left(-\mu + \sigma \cdot \frac{\phi(z_\alpha)}{\alpha}\right)$$

where $\phi(z)$ is the probability density function (PDF) of the standard normal distribution evaluated at z_α.

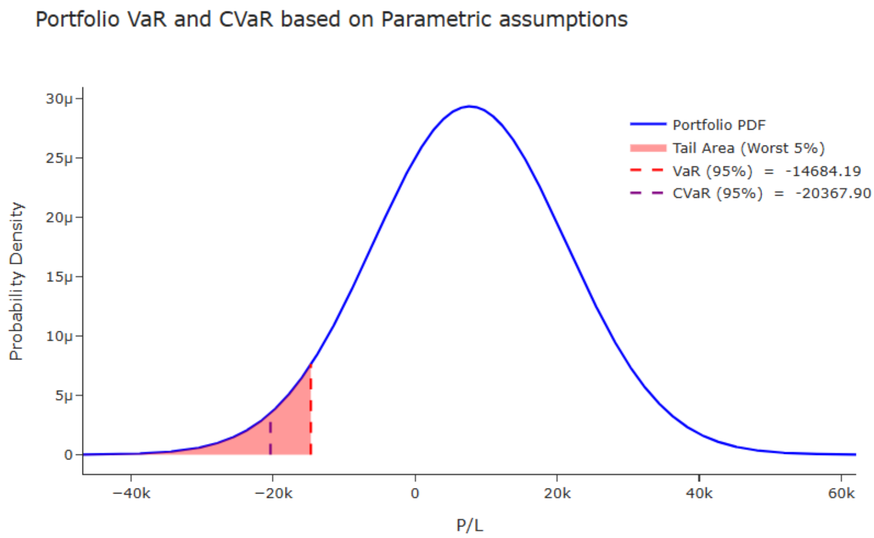

Figure 2: Simulated profit P/L with parametric assumptions

5. Visualization: We construct a probability density function (PDF) of the portfolio returns under the normal assumption. The figure below illustrates:

- The **Portfolio PDF** in blue.
- The **VaR threshold** as a red dashed line.
- The **CVaR** as a purple dashed line.
- The **Tail Area** (left of VaR) shaded in red, representing the worst α% losses.

Key Outputs:

- **Expected Portfolio Return:** for the given time horizon and portfolio allocation.
- **Value at Risk (95%):** Maximum expected portfolio loss over 250 days at 95% confidence.
- **Conditional Value at Risk (95%):** Expected loss given that the VaR threshold is breached.

Appendix C: Monte Carlo Simulation Method: Empirical Implementation in Python

This appendix describes the implementation of the Monte Carlo simulation method to estimate Value at Risk (VaR) and Conditional Value at Risk (CVaR) using simulated return paths. This approach is more flexible than parametric methods and can incorporate complex distributional characteristics and non-linear instruments.

Simulation Setup

We use the same five-asset portfolio as in previous appendices: TLT, SPY, QQQ, GLD, and CL=F, with equal weights assigned to each asset:

$$w = [0.2, 0.2, 0.2, 0.2, 0.2]$$

Daily returns are computed from historical price data spanning approximately five years. The mean return vector $\boldsymbol{\mu}$ and the covariance matrix Σ are estimated from this data set.

Monte Carlo Process

Using the estimated $\boldsymbol{\mu}$ and Σ, we simulate the $M = 10{,}000$ paths of one-day returns assuming a multivariate normal distribution. The Cholesky decomposition of the covariance matrix is used to introduce correlation among asset returns.

$$\mathbf{R}^{(m)} = \boldsymbol{\mu} + L \cdot \mathbf{z}^{(m)}, \quad \text{where } \mathbf{z}^{(m)} \sim \mathcal{N}(0, I)$$

Each simulated return vector is used to compute a portfolio return:

$$r_p^{(m)} = \mathbf{w}^T \cdot \mathbf{R}^{(m)}$$

Risk Measures

From the simulated portfolio return distribution $\{r_p^{(1)}, r_p^{(2)}, \ldots, r_p^{(M)}\}$, the VaR and CVaR at a 95% confidence level ($\alpha = 5\%$) are estimated:

- **Monte Carlo VaR:**
$$\text{VaR}_\alpha = \text{Quantile}_\alpha(r_p)$$

- **Monte Carlo CVaR:**
$$\text{CVaR}_\alpha = \mathbb{E}[r_p \mid r_p < \text{VaR}_\alpha]$$

The results are scaled to monetary losses using the total portfolio value V_0.

Results

Python outputs the following statistics:

- **Expected Return:** Mean of the simulated portfolio returns.
- **VaR (95%):** The 5th percentile of simulated losses.
- **CVaR (95%):** The mean loss beyond the 5th percentile.

Visualization

The distribution of the simulated portfolio returns is visualised using a histogram and kernel density estimate (KDE), with annotations for expected return, VaR, and CVaR.

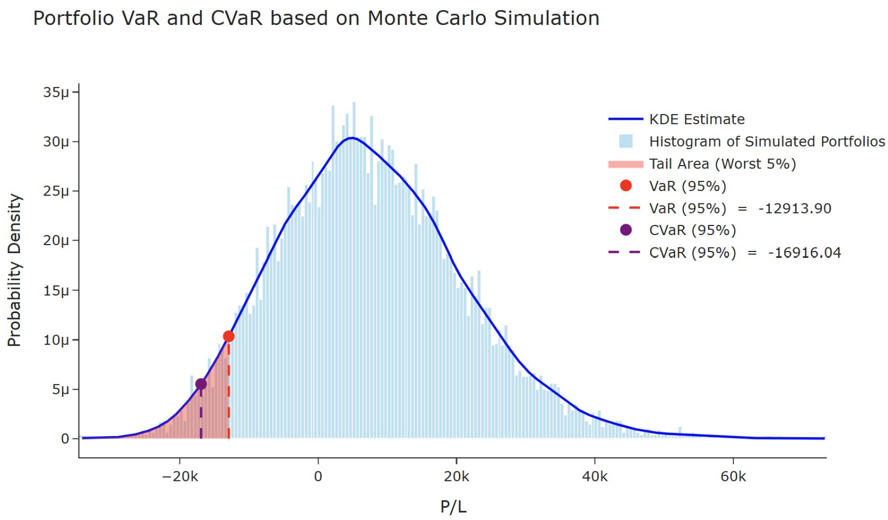

Figure 3: Simulated Portfolio Return Distribution with VaR and CVaR (95% Confidence)

Conclusion

The Monte Carlo approach offers a highly flexible risk estimation framework that is particularly useful for capturing portfolio behaviour under complex return dynamics. It allows for the incorporation of fat tails, skewness, and dynamic volatility, but at the cost of increased computational effort and reliance on the accuracy of model inputs.