# Emerging Models and Paradigms in Network Science

## Part #1: Temporal Networks

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# Studying Complex Systems

- Overview of large, integrated systems often requires zooming out from details.
- Systems such as the Internet, metabolism, proteome, and social networks can be represented as graphs.
- Networks often serve as the infrastructure for dynamical systems (e.g., data traffic, disease spread).
- Graph modeling helps estimate network behavior without studying actual dynamics.

# Why graph modeling?

- Advantages of graph modeling:
  - Assessing influence between network parts.
  - Evaluating network optimization.
  - Identifying vertices with similar roles.
- Enhancing models with additional details:
  - Weighted networks include edge weights.
  - Spatial networks consider the position of vertices.

What about modeling a non-static scenario?

# Introducing Temporal Networks

- Temporal networks incorporate the time dimension, showing when edges are active.
- Traditional network studies often aggregate contacts into edges, losing temporal details.
- Segmenting data into time windows still misses aspects of temporal structures.
- Temporal networks show non-transitive edge connections due to time ordering.
- The time ordering of connections affects network behavior beyond static network capabilities.

[References] The content in this presentation is mainly based on Holme, Petter, and Jari Saramäki. "Temporal networks." Physics reports 519.3 (2012): 97-125 [15].

# Why studying temporal networks?

- Traditional network modeling separates static networks from dynamic systems.
- Temporal network approaches integrate timing information into the network structure.
- Suitable systems for temporal network modeling:
  - Information flow (e-mail, phone calls, social media).
  - Spreading dynamics of electronic and biological viruses.
  - Genetic regulation activation sequences.
  - Functional brain networks' time-domain features.
  - Evolving food webs and species networks influenced by environmental conditions.
  - Self-assembled networks of wireless devices and distributed computing systems.

# Example of possible scenarios I

- Person-to-person communication [9, 25, 11]
  - Common examples are emails and mobile phone text messages, as well as instant messages in social networks.
  - Also not instantaneous interactions (e.g., phone calls) can be considered.
- One-to-many information dissemination [17, 16]
  - Contexts in which information is being broadcasted from a source to one or more target
  - Taking the time dimension into account enables a series of interesting analyses
- Proximity
  - Proximity patterns of human: *who is close to whom at what time*
    - Contacts between two patients in a hospital, movement of animals between farms, etc (§ 2.3)
  - Example projects: Reality Mining [8], SocioPatterns [5]

# Example of possible scenarios II

- Cell biology
  - Different systems in cell and microbiology can be modeled as networks and naturally benefit from a temporal modeling
  - Interactome – set of molecular interactions in a cell [23]
    - Proteins or lighter molecules can connect to one another to perform biological functions
  - Metabolism network – set of chemical reactions that occur in a healthy organism [7]
    - Vertices are the molecular species that are connected if they are involved in the same chemical reaction
- Neural and brain networks [4, 10, 24]
  - Several levels of structural and temporal connectivity
    - Spiking patterns of individual neurons, functional connections between brain areas
  - Vertices are associated with time series, and depend on the experimental technique
    - Individual sensors for electroencephalography (EEG) and magnetoencephalography (MEG)
    - Voxels for functional magnetic resonance imaging (fMRI)
  - Edges are assigned if the signals are correlated/in phase

# Example of possible scenarios III

- Artificial neural networks (ANN) [2]
  - The architecture of perceptrons and components of an ANN can be studied via network modeling
  - The time domain is understudied regarding this topic
- Ecological networks
- Distributed computing
- Infrastructural networks

# Definition of temporal networks

- Generally, two definitions of temporal networks are employed in the literature
  - Contact sequence – a model indicating whether two vertices interact with each other at *certain times*
  - Interval graphs – the edges are not active over a set of times but rather over a set of *time intervals*

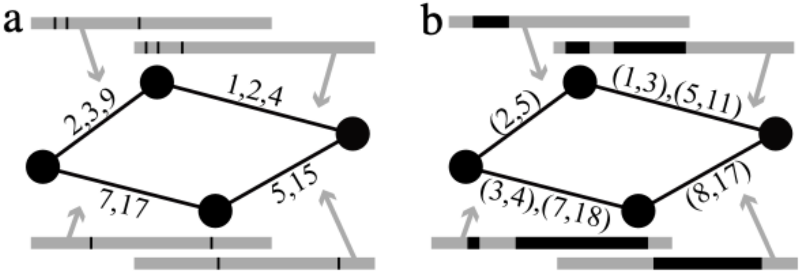


Figure: Contact sequences and interval graphs. Source: [15]

# Contact sequence

- Defined over a set $V$ of $N$ vertices
- Interactions are represented via a *contact sequence*
  - A contact sequence $C$ is a set of contacts,
  - each contact is a triple $(i, j, t)$ where $i, j \in V$ and $t$ denotes time
- Equivalently, we can represent the whole system by
  - the set $V$ of vertices,
  - a set $E$ of $M$ edges,
  - for $e \in E$, a non-empty set of times of contacts $T_e = \{t_1, \dots, t_n\}$.
- Typical systems suitable to be represented as a contact sequence:
  - Communication data,
  - Proximity data.
- Here, the duration of a contact is not considered.

# Interval graphs

- Edges are active over a set of intervals $T_e = \{(t_1, t_1'), \ldots, (t_n, t_n')\}$
  - $t_k$ indicates the beginning of the interval, while $t_k'$ indicates the end, $k = 1..n$
- Typical systems suitable to be represented as an interval graph:
  - Proximity networks (a contact represents two individuals that have been close for some extent of time)
  - Ecological networks (e.g., seasonal food webs)
  - Infrastructural systems (e.g., data network)

# Restrictions

- Network theory poses some restrictions on what we deal with
  - E.g., in simple graphs an edge never occurs twice between the same nodes
- Assumption #1 – (contact sequence) A triple of a contact sequence never occurs twice
- Assumption #2 – (interval graphs) Consider two intervals $(t_i, t_i'), (t_j, t_j') \in T_e$, then the following must hold
  - $t_i < t_i'$,
  - $t_j < t_j'$,
  - $t_i < t_j$ iff $t_i' < t_j$

# Lossless and Lossy Representations I

- Holme [13] proposes two representations: lossless and lossy.
- Lossless representations – these carry all information about a temporal network
  - Contact sequences,
  - Graph sequences – aka multilayer networks,
  - Dynamic networks – static networks evolving over time with respect to an underlying mechanism (e.g., preferential attachment [3])
  - Time-node graphs – each node is represented for every timestep
  - Time series of contacts on a static graph
  - Time-lines of contacts
  - Adjacency tensors

# Lossless and Lossy Representations II

- Lossy representations – representations where some information of the original temporal network is lost
  - Weighted graphs – edge weights count the number of contacts between two nodes
  - Reachability and influence graphs
  - Time-window graphs – a static network including links present in the temporal network within a time window
  - Concurrency graphs – two nodes are linked if they had contacts both before a certain $t_{start}$ and after a certain $t_{stop}$
  - Difference graphs – a static network highlighting the change over time (i.e., the difference between two consecutive timesteps)

# Examples

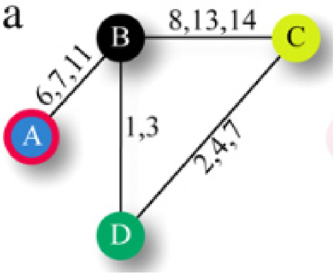


Figure: A graph visualization of a contact sequence. Source: [15]

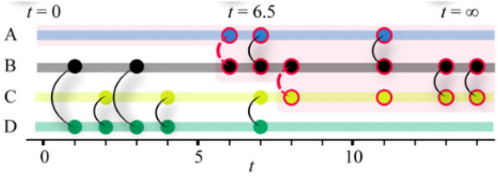


Figure: A timeline visualization of a contact sequence. Source: [15]

# Measures

- The topological structure of static networks can be characterized by a plethora of measures
- When the additional degree of freedom of time is include, many of them need rethinking or revising
- We discuss the following aspects for measures
  - Time-respecting paths and reachability,
  - Distances, latencies, and fastest paths,
  - Connectivity and components,
  - Diameter and network efficiency,
  - Centrality measures,
  - Patterns and motifs,
  - Burstiness

# Time-respecting paths I

- In a static graph, a path is a sequence of edges such that one edge ends at the node where the next edge of the path begins
- In temporal networks, paths are defined as sequences of contacts with non-decreasing times that connect sets of vertices
  - Such paths are called time-respecting paths (TRP) [14]
  - For instance, in Figure 2, there are TRPs from $A$ to $C$ (e.g., $(A, B, 7)$, $(B, C, 8)$).
- TRPs are not transitive
  - A TRP from $i$ to $j$ via $k$ does not imply that there is a path from $i$ to $k$
    - A path from $k$ to $k$ via $j$ exists only if the first contact on the $j - k$ path takes place *after* the last contact on the $i - j$ path.

# Time-respecting paths II

- TRPs define which nodes can be reached from which other nodes within some **observation window** $t \in [t_0, T]$
- The set of nodes that can be reached by TRPs from $i$ is called the **set of influence of** $i$.
  - The **reachability ratio** is the average fraction of nodes in the sets of influence of all nodes.
- The **source set** of $i$ is the set of nodes that can reach $i$ through TRPs within a given observation window $t$
  - In spreading dynamics, this set consists of all nodes that can have been the source of an infection infecting $i$.
- Both sets are time-dependent, therefore an often useful analysis is to monitor them as a function of time
  - Study how many nodes may reach vertex $i$ by TRPs by time $t'$, when the paths begin no earlier than $t < t'$

# Distances, latencies, and fastest paths I

- *How quickly vertices can reach each other through TRPs?*
- **Duration of a TRP** – the difference between the last and first contacts on the path
  - Also called temporal path length
- **Fastest TRP between two nodes** – the shortest TRP between two nodes
  - The shortest time within which $i$ can reach $j$ is called their **latency**
- In literature, the term distance is used to measure the number of links, while duration/latency is for measuring times
- Latency is a useful measure in several scenarios
  - Distributed computing – keeping track of the age of information that a node has about other nodes
  - Information spreading – the fastest possible trajectories of information between nodes

# Distances, latencies, and fastest paths II

- Consider a node $i$ at time $t$ in a temporal network
- We denote with $\phi_{i,t}(j)$ the latest time before $t$ such that information from $j$ can have reached $i$ by time $t$
  - $\phi_{i,t}(j)$ is called quantity $i$'s *view* of $j$'s information at time $t$
- Also, $\lambda_{i,t}(j) = t - \phi_{i,t}(j)$ is called $j$'s *latency* with respect to $i$ at time $t$
  - It is a measure of how old $i$'s information coming from $j$ is at time $t$
- The vector $[\phi_{i,t}(1), \ldots, \phi_{i,t}(N)]$ is called $i$'s *vector clock*
  - It captures the partial ordering of events in a distributed system for the node $i$
  - Example: `examples/temporal-network-0.md`
- Usually, these measures are used to characterize the overall "velocity" of the network
  - Measuring how quickly vertices can, on average, transmit something.
  - How can we do it?

# Distances, latencies, and fastest paths III

- An approach is to enumerate all fastest TRPs between vertices and then compute the average duration
  - However, this does not take into account the frequency of the paths
- Computing average latency is also difficult
  - Close to the end of the observation window, latency becomes infinite (paths no longer have enough time to be completed)
  - Some posed constraints on paths or just repeated values considered in previous steps [19]
- In conclusion, there is no de facto approach for averaging such quantities.

# Diameter and Network Efficiency I

- For a static network $G(V, E)$
  - Diameter: $D = \max_{u,v \in V} d_{u,v}$, i.e., the largest distance between any pairs of vertices,
  - (Global) Efficiency: $E = \frac{1}{|V|(|V|-1)} \sum_{u \neq v \in V} \frac{1}{d_{u,v}}$ the avg over inverse path lengths of all paths
- No general definitions in temporal networks
  - The diameter could be defined as the longest average latency,
  - [6] requires that the diameter should be a number as small as possible such that increasing it would not make you find more pairs of vertices connected by a TRP that is shorter than the diameter
  - [20] defines network efficiency as the harmonic average of the latency
    - It combines the average latency and the reachability ratio defined in [12]

# Centrality measures I

- Used for identifying important vertices w.r.t. some properties,
  - Degree centrality, Betweenness centrality, PageRank centrality, etc
- There is usually no unique candidate for the temporal version of a static centrality measure
- Simple approach: replace the role of paths by TRP
  - Closeness centrality $C_C$ in temporal networks [22]: $C_C(i,t) = \frac{N-1}{\sum_{j\neq i}\lambda_{i,t}(j)}$, where $\lambda_{i,t}(j)$ is the latency between $i$ and $j$.
  - Efficiency-based closeness centrality $C_E(i,t) = \frac{1}{N-1}\sum_{j\neq i}\frac{1}{\lambda_{i,t}(j)}$.
  - Betweenness centrality $C_B$ measures the fraction of shortest paths passing through a given node
    - It is defined in static networks as $C_B(i) = \frac{\sum_{i\neq j\neq k}\nu_i(j,k)}{\sum_{i\neq j\neq k}\nu(j,k)}$, where $\nu_i(j,k)$ is the number of shortest paths between $j$ and $k$ that pass $i$, and $\nu(j,k)$ is the total number of shortest paths between $j$ and $k$.
    - It is generalizable to temporal networks by adding a dependence on time $t$ and counting the fraction of fastest TRPs that pass through the given node [21]

# Centrality measures II

- Matrix-based centrality measures like *eigenvector centrality* and *PageRank* are less straightforward to be generalized to temporal networks
  - Any generalization would have to use 3D tensors representing the temporal networks
- A simple algorithms to generalize the eigenvector centrality
  1. Start with a centrality value 1 at each node,
  2. At every contact between nodes $i$ and $j$, let $C_E$ values after the contact at timestep $t$ be

$$C_E^{t+1}(i) = \rho C_E^t(i) + (1 - \rho) C_E^t(j)$$

     and

$$C_E^{t+1}(j) = \rho C_E^t(j) + (1 - \rho) C_E^t(i)$$

     where the parameter $\rho \in \mathbb{R}_{>0}$ sets the rate of centrality transmitted at a contact (if $\rho$ is large it puts a bigger emphasis on recent contacts.

# Patterns and Motifs

- Also in this case, there is usually no unique candidate for the definition of patterns and motifs in temporal networks
- Data mining-inspired approaches: *subgraph mining* [18]
  - The support set $S(G')$ of a subgraph $G'$ is the set of timesteps when $G' \subseteq G_t$
    - $G_t$ is the graph of all edges active at time $t$ in a temporal network
  - Persistent subgraphs are graphs having support larger than a certain threshold.

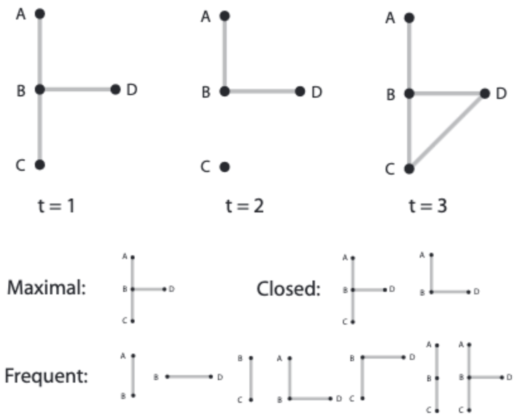


Figure: A temporal network with three timesteps and various types of frequent subgraphs at minimum support 2. Source: [18]

# Patterns and Motifs

- A network motif [1] is an equivalence class of subgraphs that is overrepresented in terms of its cardinality w.r.t. some null model in a network
  - i.e., a larger number of such subgraphs can be found than in a randomized reference system.
- Motifs are generally defined a priori or selected from snapshots of the network
  - A simple approach is to aggregate edges over a period of time, and count the different subgraphs in the snapshots
  - Also, motifs can be defined on the basis of static subgraphs, and their presence then assessed on the temporal version
  - For instance, [26] define motifs such as *chains*, *stars*, and *"ping pong"* subgraphs.
- Several aspects are still understudied
  - What reference or null models should be used?
  - Recurrence of temporal subgraphs – exactly the same or similar?

# Representing Temporal Data as Static Graph

- The literature on static networks is abundantly larger than that on temporal ones
  - Analyzing static graphs is easier than analyzing temporal ones
- One approach to analyzing temporal networks is to derive static networks that capture both temporal and topological properties of the system
- The straightforward way: accumulate the contacts over some time to form edges
  - This creates a “projection” of the temporal network
  - While it is useful whenever the topological aspects are considered, it can discard information on the temporal ones
- There are different ways of encoding the temporal network structure into a static graph

# Reachability Graphs

- Aka *path graphs* or *associated influence digraph*
- A static network having the same nodes as the temporal one is created
- Then, a directed edge from node $i$ to $j$ is created if there is a TRP from $i$ to $j$ in the temporal network
- This is very useful when dynamics on the network are considered
  - For instance, the average degree $k$ of a reachability graph is the average worst-case outbreak size, which effectively tells how vertices can possibly affect which others

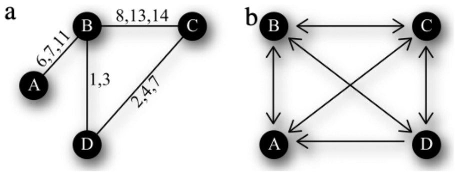


Figure: (a) shows a contact sequence, and (b) shows its reachability graph. Source: [15]

# Line Graphs

- A line graph of a static graph $G$ is a graph whose nodes are the edges of $G$ that are connected if they share a vertex in $G$
  - Aka *interchange graph* or *dual graph*
- It finds several uses in epidemiology
  - The line graph is closely related to the structure of concurrent partnerships in the original graph
  - Also used extensively in the analysis of higher-order interactions

# Transmission Graphs I

- A version of line graphs with temporal information [92]
- Built over an interval graph where an edge $e$ is active over an interval $[t_{start}(e), t_{end}(e)]$
- The transmission graph has a directed edge from $e$ to $e'$ if $e$ and $e'$ share a node in the interval graph,

$$t_{start}(e) < t_{start}(e') + \delta$$

  and

$$t_{start}(e') < t_{start}(e)$$

  - where $\delta$ is used to identify the last possible time a member of an edge can transmit information.
- A transmission graph encodes the directionality arising from the order of non-concurrent relationships.

# Transmission Graphs II

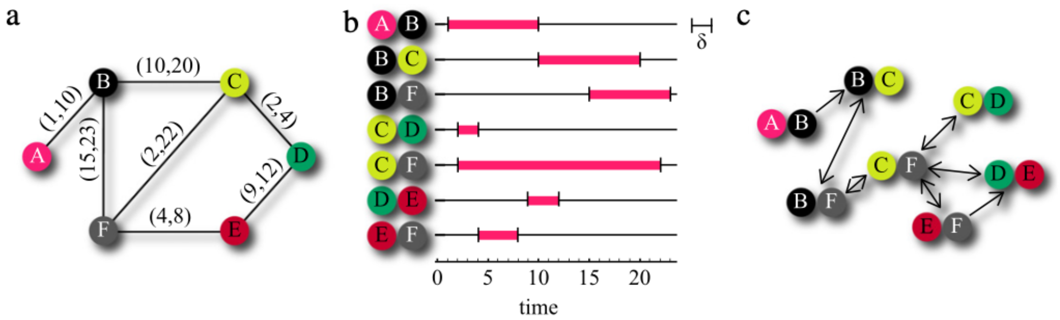


Figure: Transmission graphs. (a) shows an interval graph representation of a temporal network (where each edge has only one interval). (b) gives slightly reduced picture of the system, where a line corresponds to an edge in (a) and the active interval is indicated. The derived transmission graph is illustrated in (c). Source: [15]

# Exercises

- Let's try playing with temporal networks
  - The exercises must be done in Python using iPython Notebook.
- Two steps
  1. Define a `TemporalNetwork` class
     - It represents a temporal network using the contact sequence representation.
     - It should at least provide methods to *(i)* add and remove contacts, *(ii)* compute and return all TRPs between two nodes $i$ and $j$
  2. Use it to compute the *reachability ratio* and the *average latency*.
- A notebook to start with is available at `exercises/temporal-networks-0.ipynb`

Thanks for your attention!

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