Emerging Models and Paradigms in Network Science

Part #2: Multilayer Networks

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Complexity of Real-World Systems

- Real-world systems involve **multiple types of interactions and relationships** that cannot be fully captured by single-layer networks.
  - The human brain is an example:
    - Neurons interact through chemical synapses and electrical gap junctions,
    - Different brain regions have unique connectivity patterns,
    - Different types of synaptic connections and signaling pathways define inter-region interactions.
- Can we represent these systems via traditional networks?
  - Oversimplification – treating all interactions as equivalent often leads to incomplete or inaccurate representations.
  - Lack of context – ignoring possible dimensions (temporal, contextual, etc.) the interactions are based on.

Multilayer Networks

- Multilayer networks are composed of multiple *layers*, each representing a different type of interaction or relationship between the *same* set of nodes.
- Key aspects
  - Nodes and layers – *nodes represent entities of the system*, while *layers represent different types of interactions*
    - A node could be defined for certain layers only!
  - Intralayer and interlayer links – Connections within the same layer and between different layers, respectively

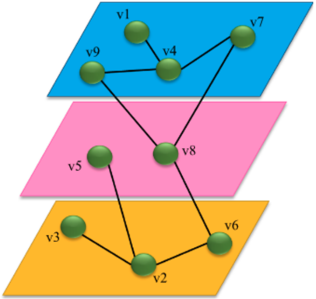

Figure: Source: [14]

Multilayer Networks

- Detailed and comprehensive modeling
  - Traditional single-layer networks often fail to capture the complexity of real-world systems, where multiple types of interactions occur simultaneously
  - Multilayer networks provide a more nuanced and detailed representation
- Understanding of dynamics
  - Dynamics and processes within complex systems often take place among different kinds of interactions

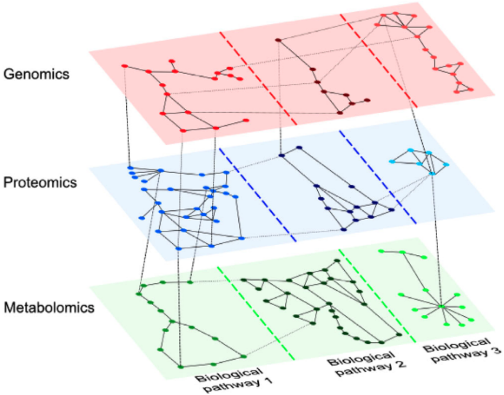

Figure: A illustrative example of multilayer data in the context of biology. Source: [17]

Examples of Application

- Sociology and Social Networks Analysis
  - Friendships on social networks span various contexts: common interests such as sports and cinema, people knowing each other in real life, etc.
  - Nodes are not only users but also the content and topics they discuss.
  - A multilayer model distinguishes these contexts, providing better insights into information spread.

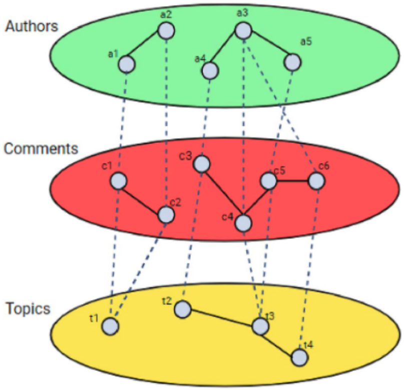

Figure: A multilayer network-based model to represent a social platform. Source: [6]

Examples of Application

- Transportation
  - An air transportation network can be oversimplified via traditional network modeling.
  - However, we can consider each airline as a separate layer.
    - This allows one to model delay propagation and passenger rescheduling more accurately, considering the operational independence of airlines.
- Biology
  - The Caenorhabditis elegans (C. elegans) is a small nematode, one of the first organisms for which the entire genome was sequenced.
    - It consists of 281 neurons and more than 2000 connections.
    - Neurons can be connected either by a chemical link or by an ionic channel; these two connections have completely different dynamics $\rightarrow$ two layers.

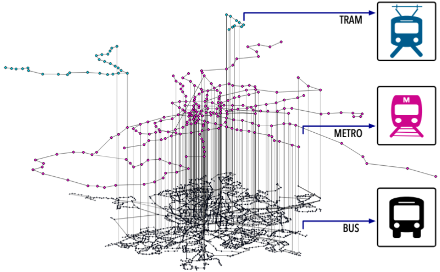

Figure: A multilayer representation of the transport system of Madrid. Source: [1]

Basic Definitions I

Most definitions on this topic are based on [5].

- A multilayer network is a pair $\mathcal{M} = (\mathcal{G}, \mathcal{C})$, where
- $\mathcal{G} = \{G_\alpha : \alpha \in \{1, \ldots, M\}\}$ is a family of (directed/undirected, weighted/unweighted) graphs $G_\alpha = (X_\alpha, E_\alpha)$ called *layers* of $\mathcal{M}$
- $\mathcal{C} = \{E_{\alpha\beta} \subseteq X_\alpha \times X_\beta : \alpha, \beta \in \{1, \ldots, M\}, \alpha \neq \beta\}$ is the set of interconnections between nodes of different layers G_α and G_β with $\alpha \neq \beta$
    - The elements of $\mathcal{C}$ are called *crossed layers*,
    - The elements of each E_α are called *intralayer connections* of $\mathcal{M}$,
    - The elements of each $E_{\alpha\beta}$ $(\alpha \neq \beta)$ are called interlayer connections.
- Exercise: `exercises/multilayer-network-0.md`

Basic Definitions II

- Usually, the set of nodes of the layer G_α is denoted by $X_\alpha = \{x_1^\alpha, \ldots, x_{N_\alpha}^\alpha\}$
- The *adjacency matrix* of each layer G_α is denoted by $A^{[\alpha]} = (a_{ij}^\alpha) \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, where

$$a_{ij}^\alpha = \begin{cases} 1 & \text{if } (x_i^\alpha, x_j^\alpha) \in E_\alpha \\ 0 & \text{otherwise} \end{cases}$$

  for $1 \leqslant i, j \leqslant N_\alpha$ and $1 \leqslant \alpha \leqslant M$.
- The *interlayer adjacency matrix* corresponding to $E_{\alpha\beta}$ is the matrix $A^{[\alpha,\beta]} = (a_{ij}^{\alpha\beta}) \in \mathbb{R}^{N_\alpha \times N_\alpha}$ given by

$$a_{ij}^{\alpha\beta} = \begin{cases} 1 & \text{if } (x_i^\alpha, x_j^\beta) \in E_{\alpha\beta} \\ 0 & \text{otherwise} \end{cases}$$

Projection Network

- Given a multilayer network $\mathcal{M}$, the projection network of $\mathcal{M}$ is the (single layer) network $proj(\mathcal{M}) = (X_{\mathcal{M}}, E_{\mathcal{M}})$ where

$$X_{\mathcal{M}} = \bigcup_{\alpha=1}^{M} X_{\alpha}$$

  and

$$E_{\mathcal{M}} = \left(\bigcup_{\alpha=1}^{M} E_{\alpha} \right) \bigcup \left(\bigcup_{\alpha,\beta=1,\alpha\neq\beta}^{M} E_{\alpha\beta} \right)$$

- $proj(\mathcal{M})$ is a network that merges all the layers of $\mathcal{M}$ into a single layer, representing the overall connectivity of the nodes across all layers
  - The set of nodes is the union of the nodes from all layers,
  - The set of edges includes *(i)* all intralayer edges, and *(ii)* all interlayer edges.

Multiplex Networks

- A multiplex network [22] is a special type of multilayer network where
  - $X_1 = X_2 = \cdots = X_M = X$
  - $E_{\alpha\beta} = \{(x, x) : x \in X\}$
- All layers have the same exact nodes, and the only possible type of interlayer connections are those in which a given node x is only connected to its counterpart nodes in the rest of layers
  - Pragmatically, multiplex networks consist of a fixed set of nodes connected by different types of links.
  - One of the common paradigms of multiple networks is social systems, which can be seen as a superposition of a multitude of complex social networks.

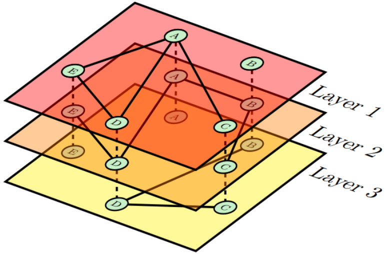

Figure: A multiplex network

Temporal Networks

- A temporal network $(G(t))_{t=1}^{T}$ can be represented as a multilayer network with a set of layers $\{G_1, \ldots, G_T\}$ where
  - $G_t = G(t)$,
  - $E_{\alpha\beta} = \emptyset$ if $\beta \neq \alpha + 1$, and
  - $E_{\alpha,\alpha+1} = \{(x, x) : x \in X_\alpha \cap X_{\alpha+1}\}$
- Note how, in this case, the temporal network is represented as a sequence of graphs.

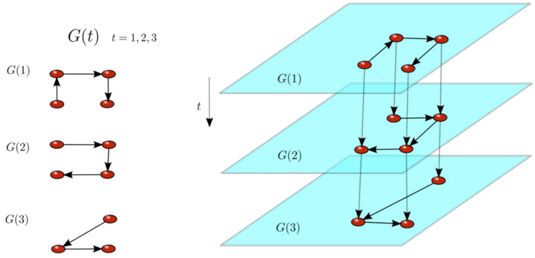

Figure: Mapping a temporal network to a multilayer one. Source: [5]

Multidimensional Networks

- Formally, a multidimensional network (also called edge-labeled multigraph) is a triple $G = (V, E, D)$ where
  - V is a set of nodes,
  - D is a set of labels representing the dimensions,
  - E is a set of labeled edges, i.e., $E = \{(u, v, d) : u, v \in V, d \in D\}$
    - We assume that given a pair of nodes $u, v \in V$ and a label $d \in D$, there may exist only one edge (u, v, d)
- $G = (V, E, D)$ can be modeled as a multiplex network by mapping each label to a layer
  - We associate G to a multiplex network of layers $\{G_1, \ldots, G_{|D|}\}$ where, for every $\alpha \in D$, we have
    - $G_\alpha = (X_\alpha, E_\alpha), X_\alpha = V$,
    - $E_\alpha = \{(u, v) \in V \times V : (u, v, d) \in E, d = \alpha\}$, and
    - $E_{\alpha\beta} = \{(x, x) : x \in V\}$, for $1 \leqslant \alpha \neq \beta \leqslant |D|$.

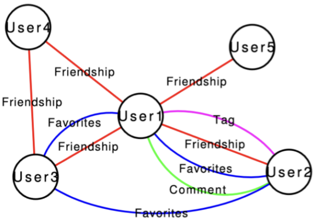

Figure: A multidimensional network. Source: [10]

Interdependent Networks

- An interdependent (or layered) network is a collection of different networks (the layers) whose nodes are interdependent on each other [8, 11]
  - Nodes from one layer of the network depend on control nodes in a different layer
  - The dependencies are additional edges connecting the different layers.
  - Such a structure is often called a *mesostructure*.

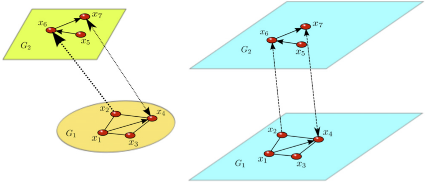

Figure: An illustration of two interdependent networks mapped to a multilayer one. Source: [5]

Multilevel Networks

- Let $G = (X, E)$ be a network.
- A multilevel network is a triple $\mathbb{M} = (X, E, \mathcal{S})$, where $\mathcal{S} = \{S_1, \ldots, S_p\}$ is a family of subgraphs $S_j = (X_j, E_j), j = 1, \ldots, p$ of G such that

$$X = \bigcup_{j=1}^{p} X_j \qquad E = \bigcup_{j=1}^{p} E_j$$

- We note how $G = \mathit{proj}(\mathbb{M})$ and each subgraph $S_j \in \mathcal{S}$ is called a *slice* of the multilevel network $\mathbb{M}$.
- $\mathbb{M}$ can be seen as a multilayer network with layers $\mathcal{S}$ and crossed layers $E_{\alpha\beta} = \{(x, x) : x \in X_\alpha \cap X_\beta\}$ for every $1 \leqslant \alpha \neq \beta \leqslant p$.
- Also, every multiplex network is a multilevel network, and a multilevel network $\mathbb{M}$ is a multiplex network if and only if $X_\alpha = X_\beta$, for all $1 \leqslant \alpha, \beta \leqslant p$.

Hypernetworks

- Hypernetworks are networks where edges (called hyperedges) connect more than two nodes.
- Mathematically, they are represented by hypergraphs:
  - A hypergraph is a pair $\mathcal{H} = (X, H)$ where
    - X is a non-empty set of nodes, and
    - $H = \{H_1, \ldots, H_p\}$ is a family of non-empty subsets of X, called hyperedges (or hyperlinks) of $\mathcal{H}$.
- Given a hypergraph $\mathcal{H} = (X, H)$, it can be modeled as a multilayer network, such that for every hyperedge $H_j = (x_1, \ldots, x_k) \in H$ we have
  - a layer G_h which is a complete graph of nodes $x_1, \ldots, x_k$,
  - and the crossed layers are $E_{\alpha\beta} = \{(x, x) : x \in X_\alpha \cap X_\beta\}$

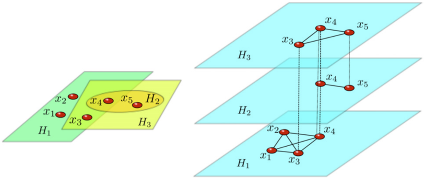

Figure: An illustration of a hypernetwork mapped to a multilayer one. Source: [5]

Centrality and Ranking I

- Well-known measures for the structural relevance of each node can be extended to multilayer networks.
- Let us make an example with *degree centrality*
  - The more links a node has, the more relevant it is.
  - The degree of a node $i \in X$ of a multiplex network $\mathcal{M}$ is the vector

$$\daleth_i = (k_i^{[1]}, \ldots, k_i^{[M]})$$

where $k_i^{[\alpha]}$ is the degree of the node i in the layer α, that is

$$k_i^{[\alpha]} = \sum_j a_{ij}^{[\alpha]}$$

  - Such a vector is called the vector-type node degree, and is the natural extension of the established definition of node degree in a monolayer network.

Centrality and Ranking II

- Centrality measures are often used to rank the nodes according to their relevance in the structure.
  - Since the node degree in a multiplex network is a vector, there is not a clear ordering in $\mathbb{R}^M$ that could produce such ranking.
  - We can aggregate the vector-type node degree and define the overlapping degree [23] of the node $i \in X$ as

$$o_i = \sum_{\alpha=1}^{M} k_i^{[\alpha]}$$

  - Many other aggregation measures $f(\neg_i)$ can be used to compute the degree centrality in a multiplex network.
- Measures based on the metric structure of the network, such as closeness and betweenness centrality, can be easily extended to multilayer networks.
  - To do so, we need to define paths and distances.

Shortest Paths and Distances

- The metric structure of a complex network is related to the topological distance between nodes, considered in terms of walks and paths.
- We need to establish these notions (path, walks, length) in the context of multilayer networks.

Walk

- Given a multilayer network $\mathcal{M} = (\mathcal{G}, \mathcal{C})$, we consider the set

$$E(\mathcal{M}) = E_1, \ldots, E_M \bigcup \mathcal{C}$$

- A walk of length $q - 1$ in $\mathcal{M}$ is a non-empty alternating sequence

$$\{x^{\alpha_1}1, l_1, x^{\alpha_2}2, l_2, \ldots, l_{q-1}, x_q^{\alpha_q}\}$$

  of nodes and edges with $\alpha_1, \alpha_2, \ldots, \alpha_q \in 1, \ldots, M$, such that for all $r < q$ there exists an $\mathcal{E} \in E(\mathcal{M})$ with $l_r = (x^{\alpha_r}r, x^{\alpha r+1}r + 1) \in \mathcal{E}$.
  - If the edges l_1, l_2, l_{q-1} are weighted, the length of the walk can be defined as the sum of the inverse of the corresponding weights.
  - If $x_1^{\alpha_1} = x_q^{\alpha_q}$, the walk is said to be *closed*.

- Example: `examples/multilayer-network-0.md`

Paths and Cycles

- A path $\omega = \{x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_1^{\alpha_1}\}$ between two nodes $x_1^{\alpha_1}$ and $x_q^{\alpha_q}$ in $\mathcal{M}$ is a walk through the nodes of $\mathcal{M}$ in which each node is visited only once.
- A cycle is a closed path starting and ending at the same node.
- Connectedness – if it is possible to find a path between any pair of its nodes, $\mathcal{M}$ is referred to as *connected*; otherwise, it is called *disconnected*.
  - Note that this depends on the definition of reachability used!
  - For instance, we could consider only the edges included in certain layers.

Length

- The length of a path ω is the number of edges in it.
- This definition changes depending on whether we consider interlayer and intralayer edges to be equivalent.

Metric Definitions

- A *geodesic* between two nodes u and v is one of the shortest paths that connects u and v.
  - Here, the length of the path is the discriminant value.
- The distance d_{uv} between u and v is the length of any geodesic between u and v.
- The diameter of $\mathcal{M}$ is the maximum distance $D(\mathcal{M})$ between any two vertices in $\mathcal{M}$.
  - The number of different geodesics that join u and v is often denoted as n_{uv}.
  - If x is a node and l is a link, then $n_{uv}(x)$ and $n_{uv}(l)$ are the number of geodesics that join the nodes u and v passing through x and l respectively.

Connected Components

- A multilayer network $\mathcal{N} = (\mathcal{G}', \mathcal{C}')$ is a subnetwork of $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ if for every γ, δ with $\gamma \neq \delta$, there exist α, β with $\alpha \neq \beta$ such that $X'_\gamma \subseteq X_\alpha$, $E'_\gamma \subset E_\alpha$, and $E'_{\gamma\delta} \subseteq E_{\alpha\beta}$.
- A connected component of $\mathcal{M}$ is a maximal connected subnetwork of $\mathcal{M}$.
- A k-component is a maximal subset of the vertices such that every vertex in the subset is connected to every other by k independent paths.
  - When $k = 2, 3$, the components are called *bi-components* and *three-components* respectively.
  - For any given network, the k-components are nested (e.g., every three-component is a subset of a bi-component, etc)

Characteristic Path Length and Efficiency

- The characteristic path length is a way of measuring the "performance" of a graph.
- In a multilayer network $\mathcal{M}$, it is defined as

$$L(\mathcal{M}) = \frac{1}{N(N-1)} \sum_{u,v \in X_{\mathcal{M}}, u \neq v} d_{uv}$$

  where $|X_{\mathcal{M}}| = N$

- Efficiency is the extent to which the network is able to diffuse the information within itself.
- It is defined as $E(\mathcal{M}) = \frac{1}{N(N-1)} \sum_{u,v \in X_{\mathcal{M}}, u \neq v} \frac{1}{d_{uv}}$

When interlayers $\neq$ intralayer I

- If we consider interlayer and intralayer edges not to be equivalent, we need to give alternative definitions of the metric quantities.
- Let $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ be a multilayer network, and $\omega = \{x_1^{\alpha_1}, l_1, x_2^{\alpha_2}, l_2, \ldots, l_q, x_{q+1}^{\alpha_{q+1}}\}$ be a path in $\mathcal{M}$.
- The length of ω can be defined as the non-negative value

$$l(\omega) = q + \beta \sum_{j=2}^{q} \Delta(j)$$

, where

$$\Delta(j) = \begin{cases} 1 & \text{if } l_j \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

and β is an arbitrarily chosen non-negative parameter.

When interlayers $\neq$ intralayer II

- In this case, the *distance* in $\mathcal{M}$ between two nodes i and j is the minimal length among all possible paths from i to j.
- When $\beta = 0$, it reduces to the natural metric in $\mathrm{proj}(\mathcal{M})$;
- When $\beta > 0$, it corresponds to metrics that take into account also the interplay between the different layers.

When interlayers $\neq$ intralayer III

- If we want to account for different distances between layers, then we can generalize $l(\omega)$ as

$$\tilde{l}(\omega) = q + \sum_{j=2}^{q} \tilde{\Delta}(j)$$

, where

$$\tilde{\Delta}(j) = \begin{cases} \beta(G_{\sigma(j-1)}, G_{\sigma(j)}) & \text{if } G_{\sigma(j)} \neq G_{\sigma(j-1)} \\ 0 & \text{otherwise} \end{cases}$$

with $\beta(G_{\sigma(j-1)}, G_{\sigma(j)})$ being the element of an $M \times M$ non-negative matrix.

Reachability and Interdependence I

- In multiplex networks, it is often important to quantify the participation of single nodes to the structure of each layer in terms of node reachability.
  - In single-layer networks, reachability depends on the existence and length of shortest paths.
  - In multilayer networks, shortest paths may significantly differ between layers, as well as within each layer.

- The interdependence λ_i of a node i is defined as

$$\lambda_i = \sum_{j \neq i} \frac{\psi_{ij}}{\sigma_{ij}}$$

  where σ_{ij} is the total number of shortest paths between node i and node j on the multiplex network, and ψ_{ij} is the number of shortest paths between node i and node j which make use of links in two or more layers.
  - $\lambda_i = 1$ when all shortest paths make use of edges laying on at least two layers, $\lambda_i = 0$ when all shortest paths use only one layer of the system.
  - Averaging λ_i over all nodes, we obtain the network interdependence $\lambda_{\mathcal{M}}$.

Reachability and Interdependence II

- λ_i provides information in terms of reachability
  - It anti-correlates to degree measures such as the overlapping degree: a node with a high overlapping degree has a higher number of links that can be the first step in a path; as such, it is likely to have a low λ_i.

Clustering I

- The classical graph clustering coefficient [7] can be extended to multilayer networks in many ways.
  - It quantifies the tendency of nodes to form triangles: *the friend of your friend is my friend*.
  - In a single-layer network $\mathcal{G} = (X, E)$, the clustering coefficient of a given node i is given by

$$c_{\mathcal{G}}(i) = \frac{\text{\# of links between the neighbors of } i}{\text{largest possible \# of links between the neighbors of } i}$$

  - The global clustering coefficient of $\mathcal{G}$ is the average of the clustering coefficient of all nodes.
  - The global clustering coefficient is often expressed as a function of network features, albeit not being equivalent to the previous one,

$$T = \frac{\text{\# of triangles in the network}}{\text{\# of triads in the network}}$$

Clustering II

- To extend it in multilayer networks, we need to consider not only intralayer links but also the interlayer ones.
- For every node $i \in X$, let $\mathcal{N}(i)$ be the set of all neighbors of i in the projection network $proj(\mathcal{M})$.
- For every $\alpha \in \{1, \ldots, M\}$, let
  - $\mathcal{N}_\alpha(i) = \mathcal{N}(i) \cap X_\alpha$, and
  - $\overline{S_\alpha}(i)$ be the subgraph of the layer G_α induced by $\mathcal{N}_\alpha(i)$, i.e., $\overline{S_\alpha}(i) = (\mathcal{N}_\alpha, \overline{E_\alpha}(i))$, where

$$\overline{E_\alpha}(i) = \{(k,j) \in E_\alpha : k,j \in \mathcal{N}_\alpha(i)\}$$

- Similarly, $\overline{S}(i)$ is the subgraph of $proj(\mathcal{M})$ induced by $\mathcal{N}(i)$.
- Finally, the clustering coefficient of a node i in $\mathcal{M}$ is defined as

$$\mathbb{C}_{\mathcal{M}}(i) = \frac{2\sum_{\alpha=1}^{M} |\overline{E_\alpha}(i)|}{\sum_{\alpha=1}^{M} |\mathcal{N}_\alpha(i)|(|\mathcal{N}_\alpha(i)| - 1)}$$

Clustering III

- The clustering coefficient of $\mathcal{M}$ is the average of all $\mathbb{C}_{\mathcal{M}}(i)$.
- Also in this case, the clustering coefficient may be defined in different ways [12, 2, 3].

Correlations

- Multiplex networks encode significantly more information than their single layers taken in isolation.
    - They include correlations between the role of the nodes in different layers, and between statistical properties of the single layers.
- Interesting types of correlations to study are:
    - **Interlayer degree correlations** – *they indicate if the hubs in one layer are also hubs in another layer.*
    - **Overlap and multidegree** – *are connectivity patterns consistent across layers? For instance, in social networks, we could have friends to whom we talk through multiple means of communication (phone, instant messaging, direct messages).*
    - **Multistrengths of weighted multiplex** [18] – *the weights of the links in the different layers can be correlated with other structural properties.*
    - **Node and layers pairwise multiplexity** [20] – *are nodes active in all layers? Do they exhibit correlated activity patterns? Are layers' activity patterns correlated?*

Examples of Correlations I

- Degree correlation – the degrees of the same node in different layers can be correlated.
    - Each node $i = 1, 2, \ldots, N$ has a degree $k_i^{[\alpha]}$ in each layer $\alpha = 1, 2, \ldots, M$.
    - To evaluate the degree correlation between two layers α and β we can use different methods.
    - Characterization of $P(k^\alpha, k^\beta) = \frac{N(k^\alpha, k^\beta)}{N}$.
        - Here, $N(k^\alpha, k^\beta)$ is the number of nodes that have degree k^α in layer α and degree k^β in layer β.
        - Thus, $P(k^\alpha, k^\beta)$ is a matrix from which the full pattern of correlations can be determined.
    - Average degree in layer α conditioned on the degree of the node in layer β [5].
    - Pearson, Spearman, and Kendall correlation coefficients [5].

Examples of Correlations II

- Overlap degree – studying overlap links can shed light on properties shared by layers.
  - The total overlap $O^{\alpha\beta}$ between two layers α and β is defined as the number of links that are in common between layer α and β, that is

$$O^{\alpha\beta} = \sum_{i<j} a_{ij}^{\alpha} a_{ij}^{\beta}$$

    where $\alpha \neq \beta$.
  - The local overlap $o_i^{\alpha\beta}$ between two layers α and β is the total number of neighbors of node i that are neighbors in both layer α and β, that is

$$o_i^{\alpha\beta} = \sum_{j=1}^{N} a_{ij}^{\alpha} a_{ij}^{\beta}$$

  - These measures can be very significant. For instance, in [24] the authors found that in an online social game studied, the friendship and communication layers have a significant overlap.

Generative Models

- **Generative models refer to algorithms that produce instances of networks with desired properties.**
- For single-layer networks, models can be generally divided into two major categories:
  - *Growing networks models* – models that exhibit the emergence of mechanisms such as preferential attachment (Barabási-Albert model).
  - *Static models* – models that describe network ensembles with given structural properties, e.g., the degree sequence, and the expected degree sequence (aka exponential random graph).
- Similarly, for multilayer networks, two classes of models can be observed in the literature:
  - *Growing multilayer networks models* – the number of nodes grows, and there is a generalized preferential attachment rule [15, 19].
  - *Multilayer network ensembles* – ensembles of networks with N nodes in each layer satisfying a certain set of structural constraints [4, 21].
- For multiplex networks, different models have already been presented and discussed in detail [15, 19, 16, 12, 20].

Resilience and Percolation

- Resilience is the ability of a network to maintain its main topological structure under the action of some kind of damage.
    - While some single-layer networks seem to be particularly resilient to random damage because *they must be robust*, the same conclusion does not hold for multilayer networks.
        - Technological and biological networks display a scale-free nature that makes them robust to damage [9, 13].
    - Percolation refers to the exhibited ability of a network to allow information to pass through it.
        - Generally, it is described by studying the emergence of a giant connected component in the network having a size of the order of the size of the entire network [8, 23]

Applications

- Multilayer networks find applications in a plethora of different contexts.
- The reason is that traditional network representation is an oversimplification of a corresponding system.
- The research in several topics has greatly advanced in the last years thanks to the *multilayer network* modeling tool.

Field	Topic	References
Social	Online communities	Pardus: [63,419–422] Netflix: [423,424] Flickr: [66,88,425] Facebook: [68,426–428] Youtube: [429] Other online communities: [54,89,430] Merging multiple communities: [122,123, 431,432]
	Internet	[109,110,433]
	Citation networks	DBLP: [31,33,434–439] Scottish Community Alliance: [440]
	Other social networks	Politics: [68,441] Terrorism: [23] Bible: [442] Mobile communication: [443]
Technical	Interdependent systems	Power grids: [25,81,444] Space networks: [445]
	Transportation systems	Multimodal: [149,184] Cargo ships: [446] Air transport: [16,78]
	Other technical networks	Warfare: [447]
Economy	Trade networks	International Trade Network: [70,71,448] Maritime flows: [449]
	Interbank market	[450]
	Organizational networks	[451–453]
Other applications	Biomedicine	[454–459]
	Climate	[24,460]
	Ecology	[64,461]
	Psychology	[462]

Figure: Main applications topics. Source and references: [5]

Exercise

- Let's try playing with multilayer networks
- The exercises should be done in Python using iPython Notebook.
- Five steps:
  1. Define a `MultilayerNetwork` class
     - It could be based on one of the simple network classes given by a Python library of your choice.
     - The layers of the network must be discerned by an attribute of the node. Also, indicate how to specify intra-layer and inter-layer connections.
  2. Make an instance of such class.
  3. Create a function that returns `true` if the represented network is a multiplex one.
  4. Create a function that, given a node, returns its vector-type node degree.
  5. Create a function that plots the overlapping degree of all nodes. The overlapping degree of a node is an aggregation (e.g., average) of its vector-type node degree.
- A notebook to start with is available at `exercises/multilayer-network-1.ipynb`

Thanks for your attention!

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References I

Alberto Aleta and Yamir Moreno.
Multilayer networks in a nutshell.
Annual Review of Condensed Matter Physics, 10:45–62, 2019.

Federico Battiston, Vincenzo Nicosia, and Vito Latora.
Structural measures for multiplex networks.
Physical Review E, 89(3):032804, 2014.

Federico Battiston, Vincenzo Nicosia, and Vito Latora.
Structural measures for multiplex networks.
Physical Review E, 89(3):032804, 2014.

Ginestra Bianconi.
Statistical mechanics of multiplex networks: Entropy and overlap.
Physical Review E, 87(6):062806, 2013.

References II

Stefano Boccaletti, Ginestra Bianconi, Regino Criado, Charo I Del Genio, Jesús Gómez-Gardenes, Miguel Romance, Irene Sendina-Nadal, Zhen Wang, and Massimiliano Zanin.
The structure and dynamics of multilayer networks.
Physics reports, 544(1):1–122, 2014.

Gianluca Bonifazi, Francesco Cauteruccio, Enrico Corradini, Edoardo Giannelli, Michele Marchetti, Domenico Ursino, and Luca Virgili.
A multilayer network-based framework for investigating the evolution and resilience of multimodal social networks.
Social Network Analysis and Mining, 14(1):5, 2023.

Sergey Brin and Lawrence Page.
The anatomy of a large-scale hypertextual web search engine.
Computer networks and ISDN systems, 30(1-7):107–117, 1998.

Sergey V Buldyrev, Roni Parshani, Gerald Paul, H Eugene Stanley, and Shlomo Havlin.
Catastrophic cascade of failures in interdependent networks.
Nature, 464(7291):1025–1028, 2010.

References III

Duncan S Callaway, Mark EJ Newman, Steven H Strogatz, and Duncan J Watts.
Network robustness and fragility: Percolation on random graphs.
Physical review letters, 85(25):5468, 2000.

Michele Coscia.
Multidimensional Network analysis.
PhD thesis, University of Pisa, Italy, 2012.

Michele Coscia, Giulio Rossetti, Diego Pennacchioli, Damiano Ceccarelli, and Fosca Giannotti.
" you know because i know" a multidimensional network approach to human resources problem.
In *Proceedings of the 2013 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining*, pages 434–441, 2013.

Regino Criado, Julio Flores, Alejandro García del Amo, Jesús Gómez-Gardenes, and Miguel Romance.
A mathematical model for networks with structures in the mesoscale.
International Journal of Computer Mathematics, 89(3):291–309, 2012.

References IV

Paolo Crucitti, Vito Latora, Massimo Marchiori, and Andrea Rapisarda.
Error and attack tolerance of complex networks.
Physica A: Statistical mechanics and its applications, 340(1-3):388–394, 2004.

Zaynab Hammoud and Frank Kramer.
Multilayer networks: aspects, implementations, and application in biomedicine.
Big Data Analytics, 5(1):2, 2020.

Jung Yeol Kim and K-I Goh.
Coevolution and correlated multiplexity in multiplex networks.
Physical review letters, 111(5):058702, 2013.

Matteo Magnani and Luca Rossi.
The ml-model for multi-layer social networks.
In *2011 International conference on advances in social networks analysis and mining*, pages 5–12. IEEE, 2011.

References V

Fintan McGee, Mohammad Ghoniem, Guy Melançon, Benoit Otjacques, and Bruno Pinaud.
The state of the art in multilayer network visualization.
In *Computer Graphics Forum*, volume 38, pages 125–149. Wiley Online Library, 2019.

Giulia Menichetti, Daniel Remondini, Pietro Panzarasa, Raúl J Mondragón, and Ginestra Bianconi.
Weighted multiplex networks.
PloS one, 9(6):e97857, 2014.

Vincenzo Nicosia, Ginestra Bianconi, Vito Latora, and Marc Barthelemy.
Growing multiplex networks.
Physical review letters, 111(5):058701, 2013.

Vincenzo Nicosia and Vito Latora.
Measuring and modeling correlations in multiplex networks.
Physical Review E, 92(3):032805, 2015.

References VI

Philippa Pattison and Stanley Wasserman.
Logit models and logistic regressions for social networks: Ii. multivariate relations.
British journal of mathematical and statistical psychology, 52(2):169–193, 1999.

Luis Solá, Miguel Romance, Regino Criado, Julio Flores, Alejandro García del Amo, and Stefano Boccaletti.
Eigenvector centrality of nodes in multiplex networks.
Chaos: An Interdisciplinary Journal of Nonlinear Science, 23(3), 2013.

Seung-Woo Son, Peter Grassberger, and Maya Paczuski.
Percolation transitions are not always sharpened by making networks interdependent.
Physical review letters, 107(19):195702, 2011.

Michael Szell, Renaud Lambiotte, and Stefan Thurner.
Multirelational organization of large-scale social networks in an online world.
Proceedings of the National Academy of Sciences, 107(31):13636–13641, 2010.