Communication System Laboratory: Basic Quantum Circuits

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Outline

1. History of Quantum Computation
2. Quantum Bits and Gates
3. Multiple Qubits
4. Concluding Remarks

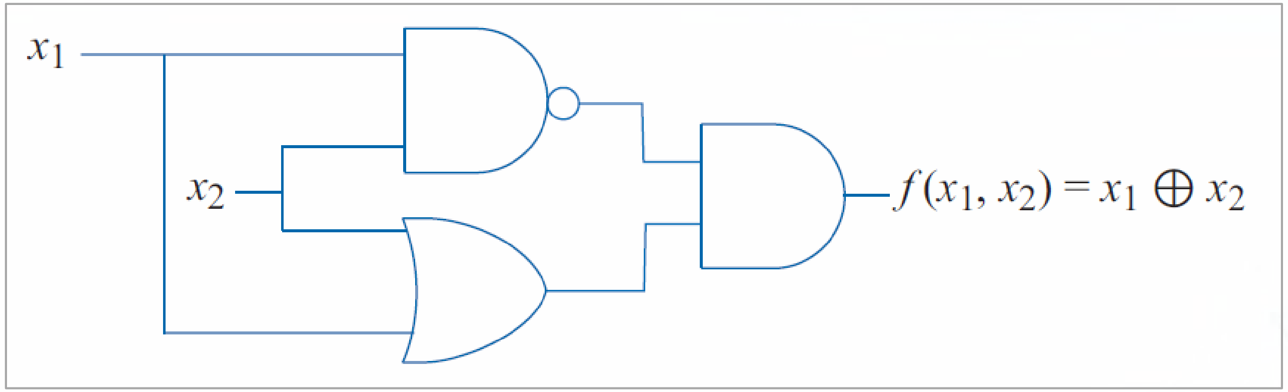
Prologue – History of Quantum Computation

"Any computing machine that is to solve a complex mathematical problem must be 'programmed' for this task. This means that the complex operation of solving that problem must be replaced by a combination of the basic operations of the machine."

John von Neumann (1903-1951)

What's Computation?

- Classical digital computers (under the Boolean circuit model) uses binary digit (*bits*, 0s or 1s) to store, transfer, manipulate data

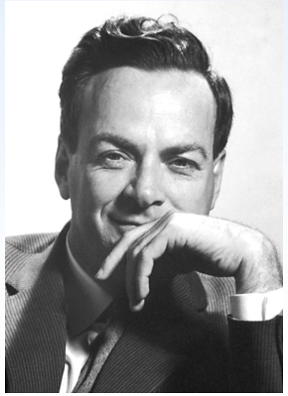


Alan Turing (1912-1954)

- A *bit* can possibly be only one of two states: it is either a one or a zero
- The two states of each bit are represented in the computer by a two-level system
- A quantum computer is a device that leverages specific properties described by quantum mechanics to perform computation

  → Quantum computer uses quantum bits (*qubits*)

Brief History of Quantum Computation (1/2)



Richard Feynman
(1918-1988)

- Paul Benioff (1979):
  "The computer as a physical system: A microscopic quantum mechanical Hamiltonian model of computers as represented by Turing machines.
- Feynman (1981): "Why don't we store information on individual particles that already follow the very rules of quantum mechanics that we try to simulate?

"Nature Isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical."

- David Deutsch (1985) described what a quantum algorithm would look like, and with Richard Jozsa (1992) demonstrated a *deterministic* quantum advantage.
- Umesh Vazirani and Ethan Bernstein (1993) pushed it forward (bounded error).
- Daniel Simon (1994) demonstrated an exponential speedup.

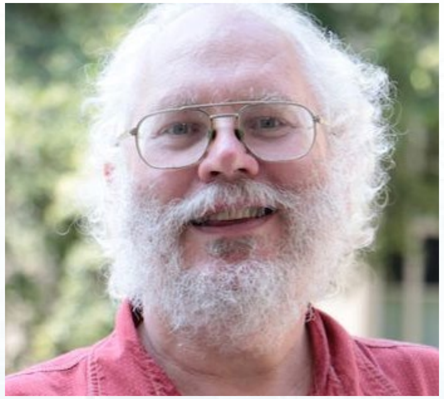
Brief History of Quantum Computation (2/2)

- Seth Lloyd (1993) described a method of building a working quantum computer.
- Peter Shor (1994) invented a polynomial-time quantum algorithm for factoring.
- David DiVincenzo (1996) outlined the key criteria of a quantum computer.
- Isaac Chuang *et al.* (2001) implemented Shor's algorithm on a nuclear magnetic resonance (NMR) system to factor the number 15 as a demonstration.

⋮

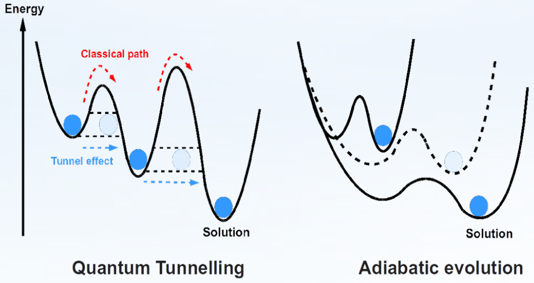
- → A variety of interdisciplinary fields such as Quantum Computation, Quantum Communication, Quantum Simulation, Quantum Sensing, Quantum Chemistry, etc.

Quantum Information Science



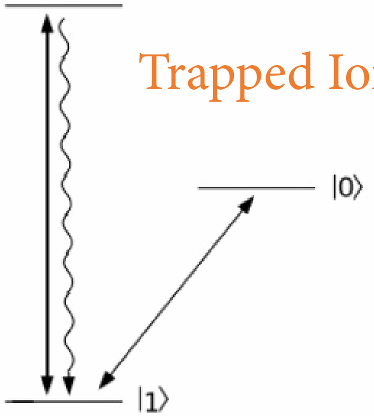
Peter Shor (1959 -)

Realizing Quantum Computers



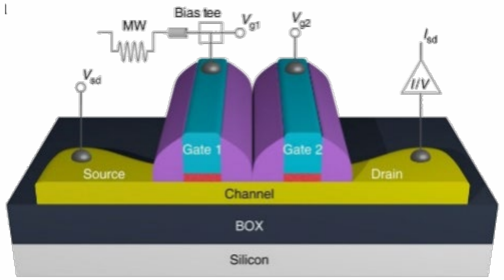
- Models of Quantum Computation
  - Analog: quantum annealing
  - Gate-based: information is encoded onto a discrete set of qubits, and quantum operations are broken down to a sequence of a few basic quantum logic gates.
  - Measurement-based
- Hardware:

Trapped Ion



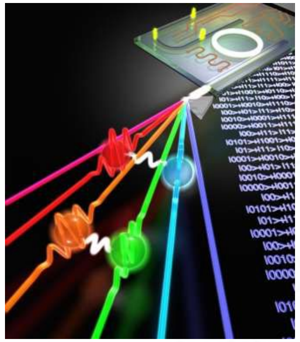
IONQ

Silicon-based Spin



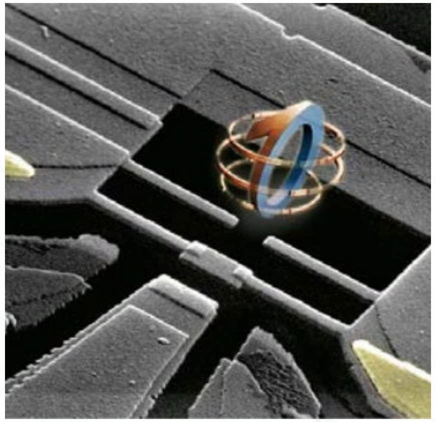
Silicon Quantum Computing

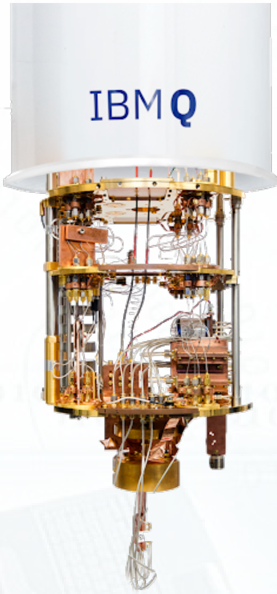
Photonics

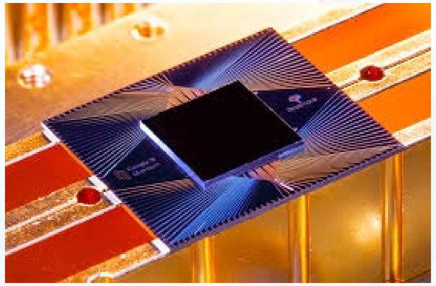


Ψ PsiQuantum

Superconducting

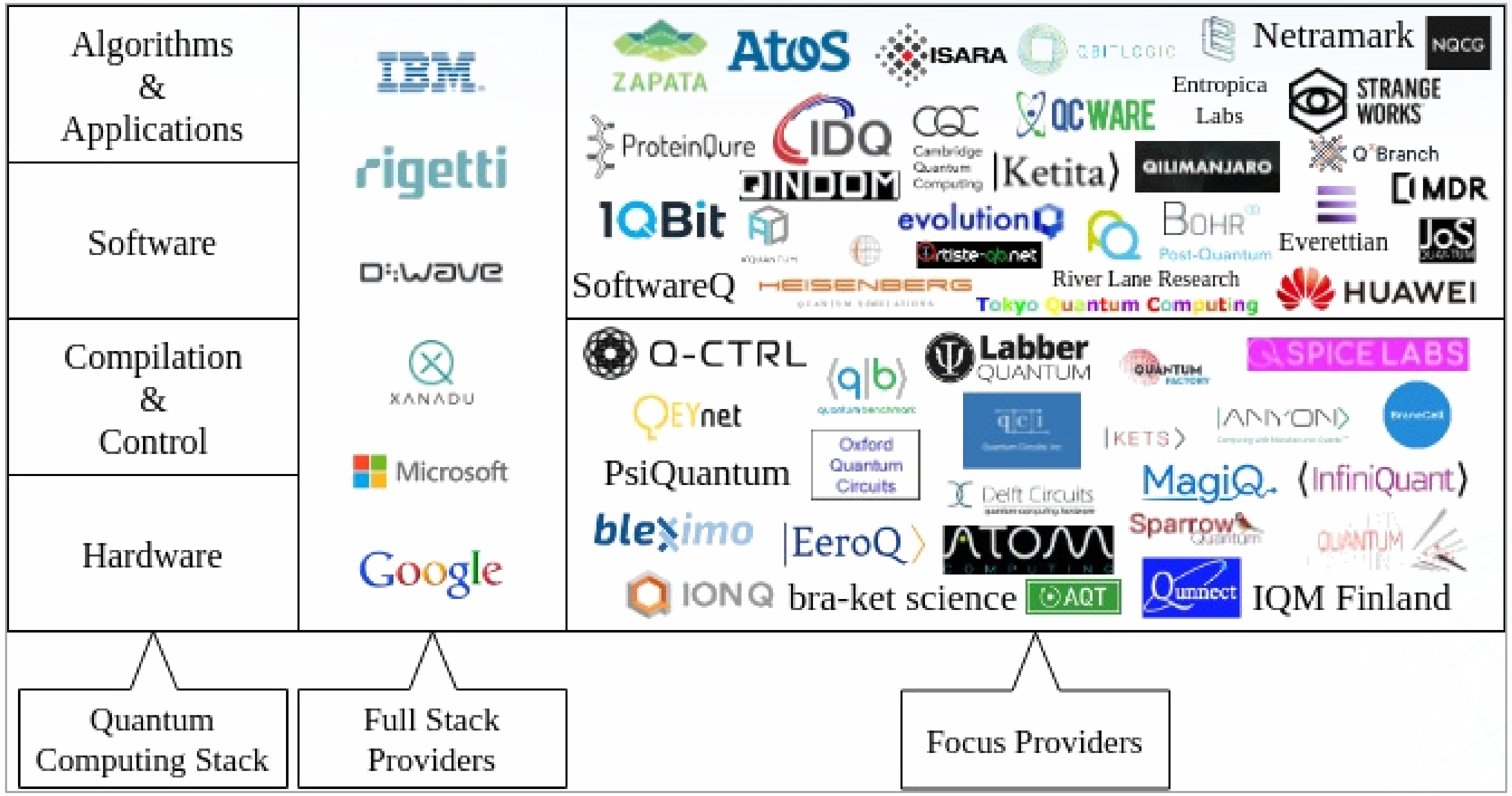


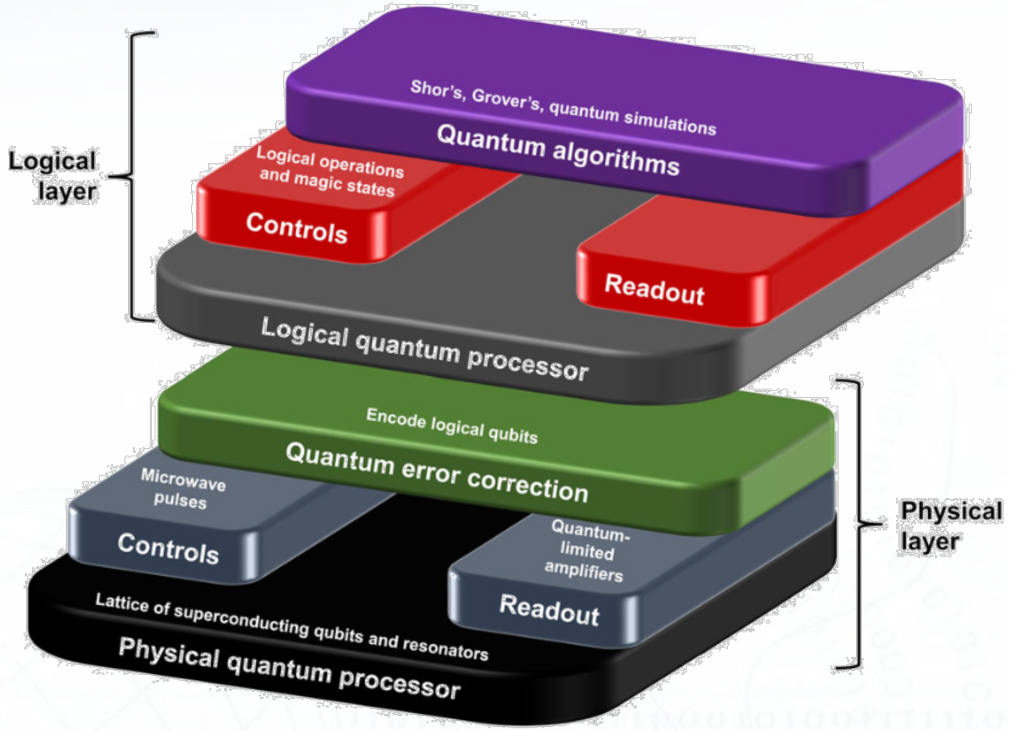




Google AI Quantum

Quantum Computer Stacks

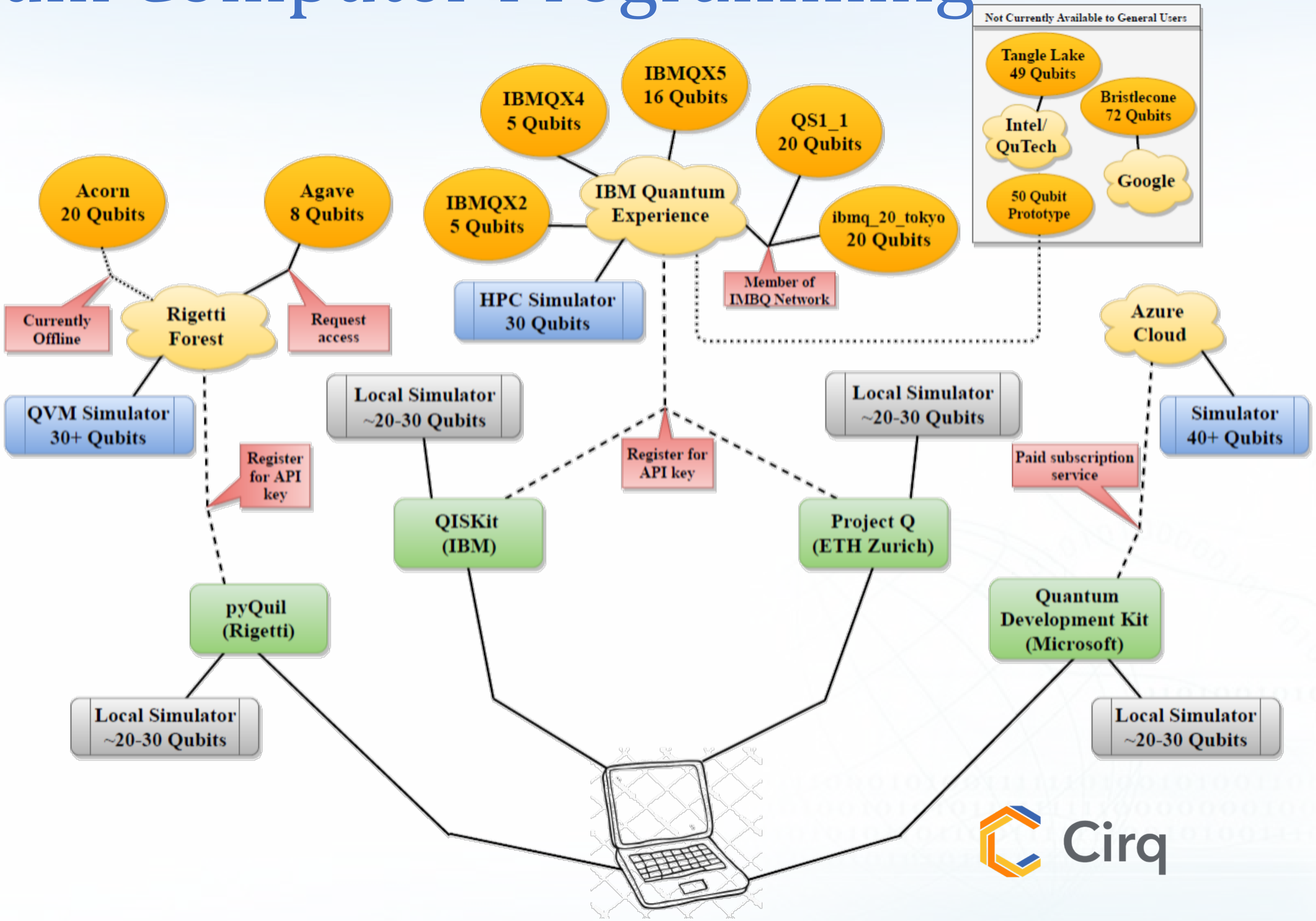




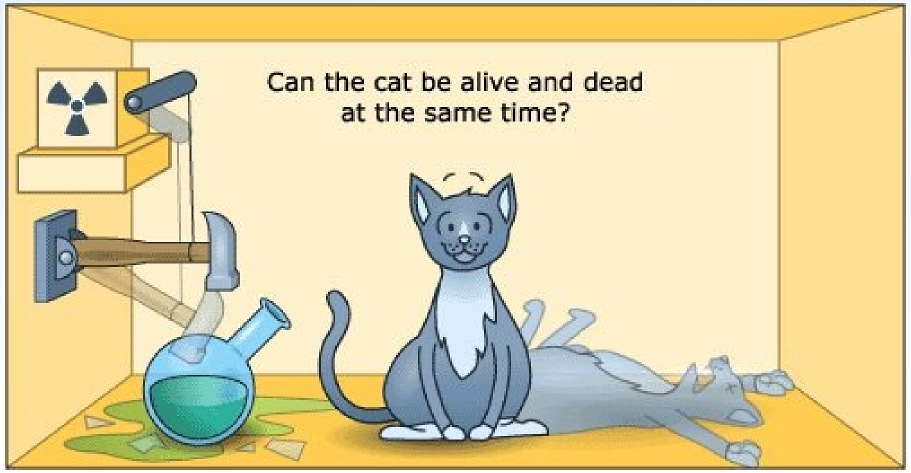
[https://quantumcomputingreport.com/review-of-the-cirq-quantum-software-framework/]

[J. Gambetta, J. Chow, M. Steffen, "Building logical qubits in a superconducting quantum computing system," *npj Quan. Info.*, 2017]

Quantum Computer Programming



[R. LaRose, “Overview and comparison of gate level quantum software platforms," *Quantum*, 3:130, 2019]



Main Course – Quantum Bits and Gates

"The world is given to me only once, not one existing and one perceived. Subject and object are only one."

Erwin Schrödinger (1902-1984)

Recap of Linear Algebras (1/2)

- Let $\mathbb{C}^d$ be a d-dimensional complex Euclidean space.
- A *column vector* in it is $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \in \mathbb{C}^d$, and its *conjugate transpose* is $\boldsymbol{v}^\dagger = \begin{pmatrix} v_1^* & \cdots & v_d^* \end{pmatrix}$.
- Inner product: $\boldsymbol{u}^\dagger \boldsymbol{v} \coloneqq \begin{pmatrix} u_1^* & \cdots & u_d^* \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} = \sum_{i=1}^d u_i^* v_i \in \mathbb{C}$.
- Euclidean norm of $\boldsymbol{v}$ is $\|\boldsymbol{v}\|_2 \coloneqq \sqrt{\boldsymbol{v}^\dagger \boldsymbol{v}}$.
- Outer product: $\boldsymbol{u}\boldsymbol{v}^\dagger \coloneqq \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \begin{pmatrix} v_1^* & \cdots & v_d^* \end{pmatrix} = \begin{pmatrix} u_1 v_1^* & \cdots & u_1 v_d^* \\ \vdots & \ddots & \vdots \\ u_d v_1^* & \cdots & u_d v_d^* \end{pmatrix} \in \mathbb{C}^{d \times d}$.

Recap of Linear Algebras (2/2)

- Basic properties
  - An outer product $\boldsymbol{u}\boldsymbol{v}^\dagger$ is a $d \times d$ matrix with rank at most 1.
  - The outer product $\boldsymbol{v}\boldsymbol{v}^\dagger$ is a rank-1 *projection matrix* onto the subspace spanned by $\boldsymbol{v}$.
  - For a matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}$, the matrix multiplication is $A\boldsymbol{v} = \begin{pmatrix} a_{11}v_1 + \cdots + a_{1d}v_d \\ a_{21}v_1 + \cdots + a_{2d}v_d \\ \vdots \\ a_{d1}v_1 + \cdots + a_{dd}v_d \end{pmatrix} \in \mathbb{C}^d$.
  - Multiplication example: $\boldsymbol{u}\boldsymbol{v}^\dagger\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v}^\dagger\boldsymbol{w}) = (\boldsymbol{v}^\dagger\boldsymbol{w})\boldsymbol{u} \in \mathbb{C}^d$.
  - A linear space may have *infinitely many* bases (orthogonal unit vectors spanning the whole space).

- In quantum mechanics, a ***unit column vector*** in $\mathbb{C}^d$ is denoted by $|v\rangle \in \mathbb{C}^d$, and its conjugate transpose is $\langle v| \coloneqq |v\rangle^\dagger$. Other properties hold accordingly.

The complex Euclidean space $\mathbb{C}^d$ here is called a *Hilbert space*.

The Quantum Bit (Qubit)

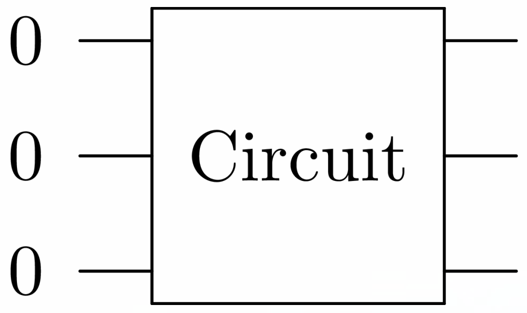
{$|0\rangle, |1\rangle$} is a basis of $\mathbb{C}^2$

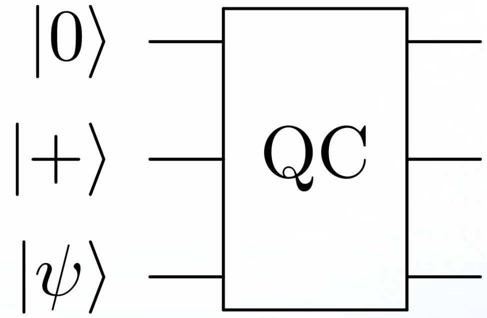
- Definition: A **qubit** is the fundamental unit of quantum information
  It is a *superposition* state represented by a linear combination of $|0\rangle$ and $|1\rangle$ in $\mathbb{C}^2$:

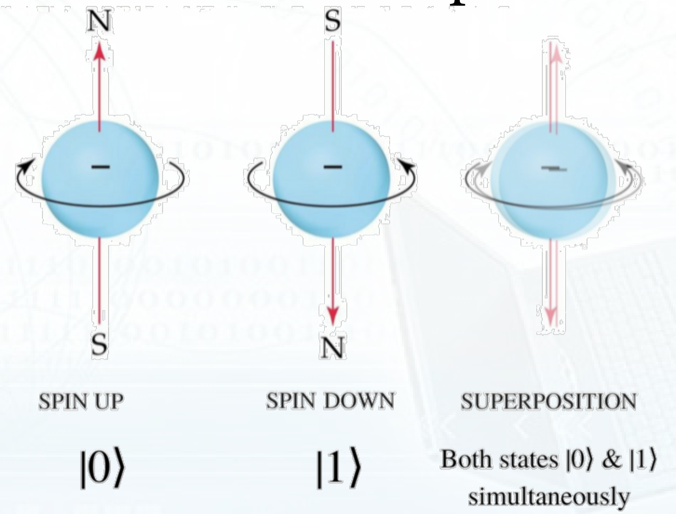
$$|\psi\rangle = a|0\rangle + b|1\rangle, \ a, b \in \mathbb{C}, \ |a|^2 + |b|^2 = 1$$

- Physically, a qubit can be a realized by a two-state (two-level) quantum-mechanical system e.g. a spinning electron or polarized light

- A quantum register (a quantum system) is a collection of qubits we use for computation



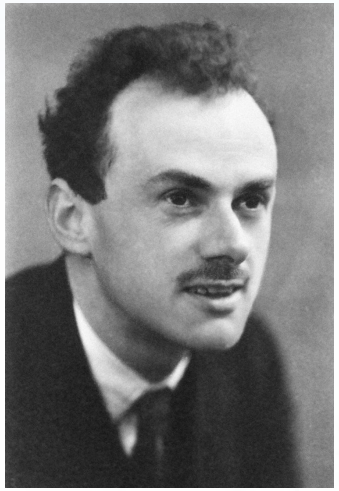




Vector Representation for a Qubit

$\langle\psi|\psi\rangle = 1$

- $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow |\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{C}^2,\ |a|^2 + |b|^2 = 1$

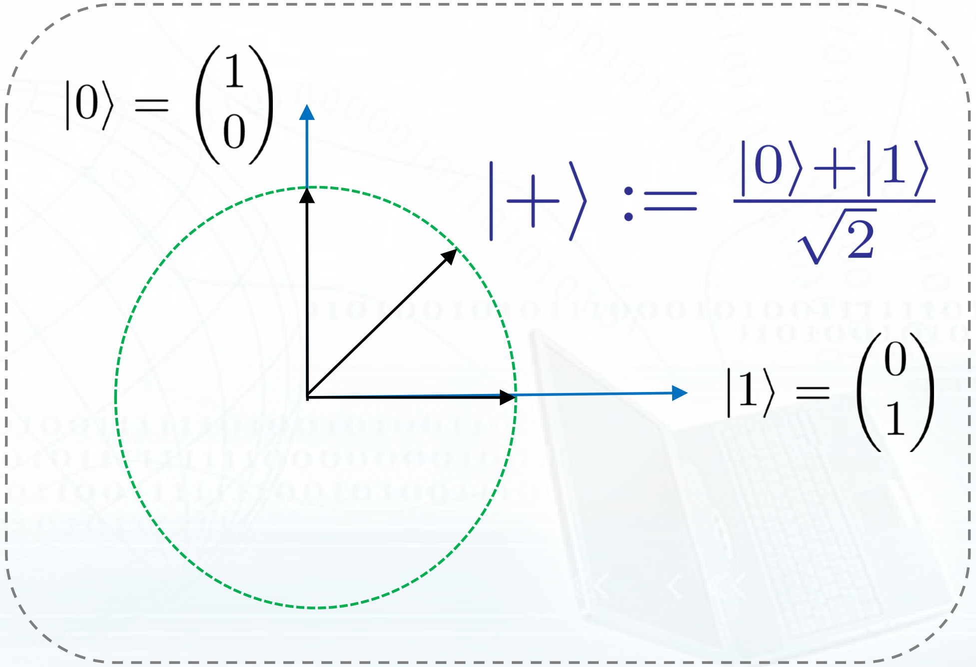


Paul Dirac (1902-1984)

- Bra–Ket notation: the Ket vectors $|\psi\rangle = \begin{pmatrix}a\\b\end{pmatrix}, |\phi\rangle = \begin{pmatrix}c\\d\end{pmatrix}$

- Inner product: $\langle\phi| := |\phi\rangle^\dagger = (c^*,\quad d^*)$

$$\Rightarrow \langle\phi|\psi\rangle = (c^*,\quad d^*)\begin{pmatrix}a\\b\end{pmatrix}$$
$$= c^*a + d^*b \in \mathbb{C}$$



- The global phase does not matter:

$$|q\rangle = a|0\rangle + b|1\rangle \simeq e^{i\phi}(a|0\rangle + b|1\rangle)$$

Measurement

- A **quantum measurement** (with respect to the computational basis $\{|0\rangle, |1\rangle\}$) gives you the readout of '0' or '1' with certain probability

- The Born rule:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Probability amplitude

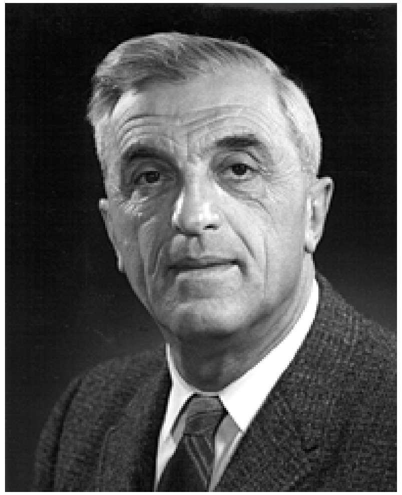
$$\Pr(0) = |\langle 0|\psi\rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = |a|^2$$

$$\Pr(1) = |\langle 1|\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = |b|^2$$

Max Born (1882-1970)

- After measurement, a qubit is forced to *collapse* (irreversibly) through projection to one of the basis

Bloch Representation for a Qubit (1/3)



Felix Bloch (1905-1983)

$$|\psi\rangle = a|0\rangle + b|1\rangle = r_1 e^{\mathrm{i}\phi_1}|0\rangle + r_2 e^{\mathrm{i}\phi_2}|1\rangle$$

$$= e^{\mathrm{i}\phi_1}\left(r_1|0\rangle + r_2 e^{\mathrm{i}(\phi_2-\phi_1)}|1\rangle\right)$$

Relative phase

$$\simeq r_1|0\rangle + r_2 e^{\mathrm{i}(\phi_2-\phi_1)}|1\rangle$$

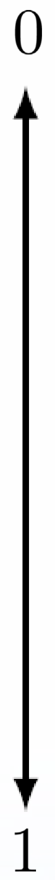
$$r_1^2 + r_2^2 = 1$$

- Polar form: $\Rightarrow |\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})e^{\mathrm{i}\varphi}|1\rangle,\ \theta \in [0,\pi],\ \varphi \in [0, 2\pi]$

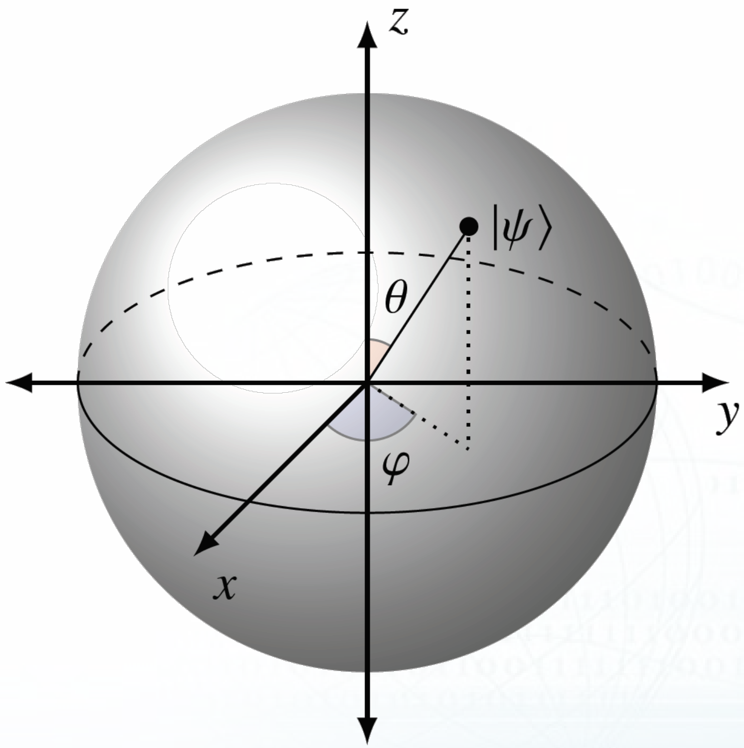
Bloch Representation for a Qubit (2/3)

$$\Rightarrow |\psi\rangle = \cos(\tfrac{\theta}{2})|0\rangle + \sin(\tfrac{\theta}{2})e^{i\varphi}|1\rangle,\ \theta \in [0, \pi],\ \varphi \in [0, 2\pi]$$

Classical Bit



$|0\rangle \leftarrow (\theta, \varphi) = (0, 0)$



$|1\rangle \leftarrow (\theta, \varphi) = (\pi, 0)$

Quantum Bit

Bloch Representation for a Qubit (3/3)

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$(\theta, \varphi) = \left(\tfrac{\pi}{2}, \pi\right)$$

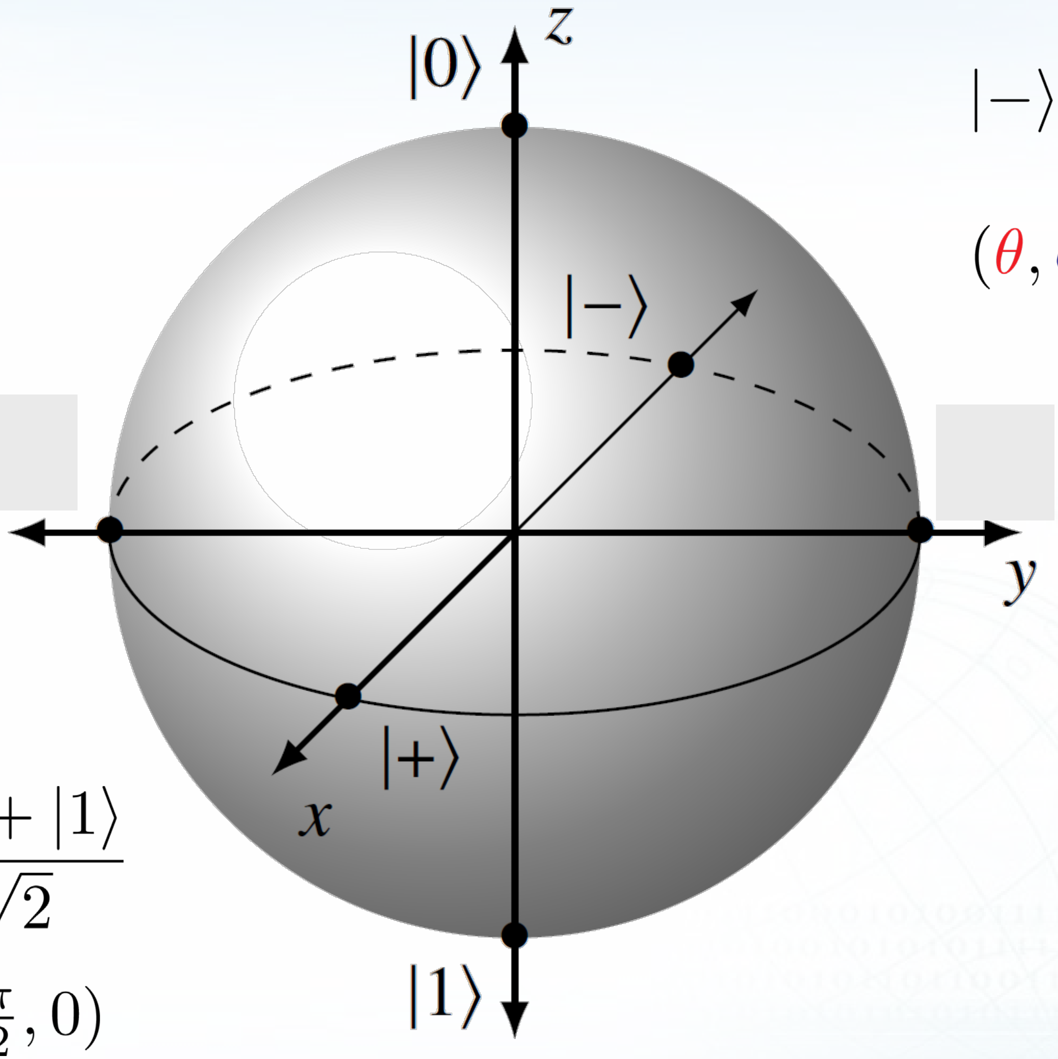
Y-Basis

$$|{-i}\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

$$(\theta, \varphi) = \left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)$$

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$(\theta, \varphi) = \left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$



$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

X-Basis

$$(\theta, \varphi) = \left(\tfrac{\pi}{2}, 0\right)$$

Z-Basis or the Computation Basis

Single-Qubit Gates (1/2)

A kind of rotation

- In gate-based quantum computer, a quantum operation is a unitary operation

- The quantum X gate is given by the Pauli X matrix $\quad |0\rangle - \boxed{X} - \quad |1\rangle$
  → It is called the *NOT* gate or the "*bit flip*" gate since it rotates π around the x-axis

Matrix representation

$$X := |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Rightarrow X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

- The quantum Z gate rotates π around the z-axis $\quad Z := |0\rangle\langle 0| - |1\rangle\langle 1|$
  → It is called the "*phase flip*" gate $\quad \Rightarrow Z|\psi\rangle = r_1|0\rangle + r_2 e^{i(\pi+\psi)}|1\rangle$

- The quantum Y gate rotates π around the y-axis $\quad \boxed{Y = iXZ}$
  → It does the bit flip and phase flip at the same time

Wolfgang Pauli (1900-1958)

Single-Qubit Gates (2/2)

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- The Hadamard gate changes the basis from $\{|0\rangle, |1\rangle\}$ to $\{|+\rangle, |-\rangle\}$
  → It creates *superposition*; and it is self-inverse: $HH = I$

Jacques Hadamard (1865-1963)

$$|b\rangle \; -\boxed{H}- \; \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^b|1\rangle\right), \; b \in \{0, 1\}$$

- The quantum R_φ^Z gate rotates φ around the z-axis $\quad R_\varphi^Z := \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix}$
- The quantum R_φ^X gate rotates φ around the x-axis $\quad R_\varphi^X := \begin{pmatrix} \cos(\frac{\varphi}{2}) & -\mathrm{i}\sin(\frac{\varphi}{2}) \\ -\mathrm{i}\sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{pmatrix}$
- The quantum R_φ^Y gate rotates φ around the y-axis $\quad R_\varphi^Y := \begin{pmatrix} \cos(\frac{\varphi}{2}) & -\sin(\frac{\varphi}{2}) \\ \sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{pmatrix}$

What’s more – Multiple Qubits and Gates

“Anyone who is not shocked by quantum theory has not understood it.”

Niels Bohr (1865-1962)

Just a bit Math...

- Definition. Tensor product of vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}^2$

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} := \begin{pmatrix} a_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 \times b_1 \\ a_1 \times b_2 \\ a_2 \times b_1 \\ a_2 \times b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} \in \mathbb{C}^{2 \times 2}$$

- Properties (linearity).

$$(\mathbf{a}_1 + \mathbf{a}_2) \otimes (\mathbf{b}_1 + \mathbf{b}_2) = \mathbf{a}_1 \otimes (\mathbf{b}_1 + \mathbf{b}_2) + \mathbf{a}_2 \otimes (\mathbf{b}_1 + \mathbf{b}_2)$$

$$= \mathbf{a}_1 \otimes \mathbf{b}_1 + \mathbf{a}_1 \otimes \mathbf{b}_2 + \mathbf{a}_2 \otimes \mathbf{b}_1 + \mathbf{a}_2 \otimes \mathbf{b}_2$$

Just a bit Math...

- **Definition.** Tensor product of matrices $A := \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$, $B := \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$

$$A \otimes B = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \otimes B$$

$$:= \begin{pmatrix} a_{1,1}B & a_{1,2}B \\ a_{2,1}B & a_{2,2}B \end{pmatrix} = \begin{pmatrix} a_{1,1}\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} & a_{1,2}\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \\ a_{2,1}\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} & a_{2,2}\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}$$

Multiple Qubits (1/3)

- Recall: a qubit is represented by a unit vector in two-dimensional linear space $\mathbb{C}^2$
- A two-qubit is represented by a unit vector in four-dimensional linear space $\mathbb{C}^{2\times 2}$
  → The computational basis:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

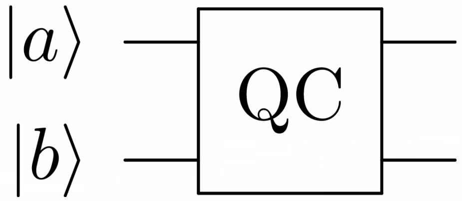
- Tensor product: $|01\rangle \equiv |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} := \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \times 0 \\ 1 \times 1 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Multiple Qubits (2/3)

- Product state: Given two qubits states $|a\rangle = a_1|0\rangle + a_2|1\rangle,\ |b\rangle = b_1|0\rangle + b_2|1\rangle \in \mathbb{C}^2$

$$\Rightarrow |a\rangle \otimes |b\rangle = a_1 b_1 |0\rangle \otimes |0\rangle + a_1 b_2 |0\rangle \otimes |1\rangle + a_2 b_1 |1\rangle \otimes |0\rangle + a_2 b_2 |1\rangle \otimes |1\rangle$$

$$= a_1 b_1 |00\rangle + a_1 b_2 |01\rangle + a_2 b_1 |10\rangle + a_2 b_2 |11\rangle = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

- Remark:
  - Can check $|a_1 b_1|^2 + |a_1 b_2|^2 + |a_2 b_1|^2 + |a_2 b_2|^2 = 1$ using $|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2 = 1$
  - 

Multiple Qubits (3/3)

- General 2-qubit state:

Probability amplitude

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle \in \mathbb{C}^4 \qquad \sum_{i,j\in\{0,1\}} |a_{ij}|^2 = 1$$

- Entangled state: there are states that cannot be expressed as the product form, i.e.

$$\nexists |q_1\rangle, |q_2\rangle \in \mathbb{C}^2, \text{ s.t. } |\psi\rangle = |q_1\rangle \otimes |q_2\rangle$$

- The *Bell states*:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

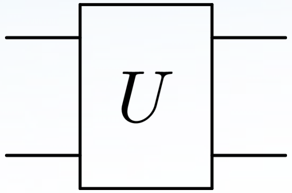
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \qquad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

John S. Bell (1928-1990)

More on entangled states

- Must use a 2-qubit register to represent an entangled state

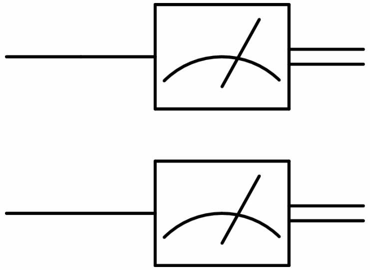
$|\Phi^+\rangle$



- A product state is not entangled, i.e. the first qubit is *independent* of the second one

- Each Bell state is also called an *EPR pair* or the *maximally entangled state*

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



Alice

Bob

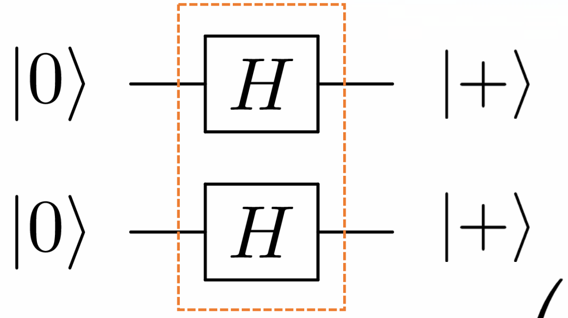
If Alice obtains '0' or '1', Bob gets so, and vice versa

- They could be physically far apart → The Spooky Effect!

Einstein, Podolsky, and Rosen

Multi-Qubit Gates (1/3)

- A qubit gate has a 2 by 2 unitary matrix in a given basis.
  $\rightarrow$ For an n-qubit gate, the matrix is 2^n by 2^n (tensor product of matrices)



A two-qubit gate $H \otimes H$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

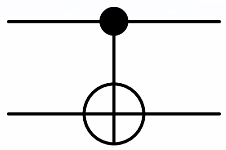
$$\Rightarrow H^{\otimes 2}|00\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = |+\rangle \otimes |+\rangle$$

$$\boxed{(A \otimes B)\mathbf{a} \otimes \mathbf{b} = (A\mathbf{a}) \otimes (B\mathbf{b})}$$

Multi-Qubit Gates (2/3)

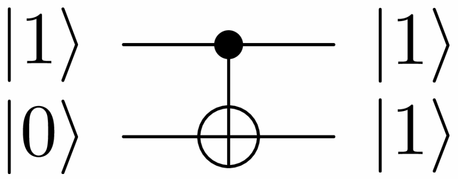
- The previous example is a 2-qubit gate of the *product form*

- Controlled-NOT gate:

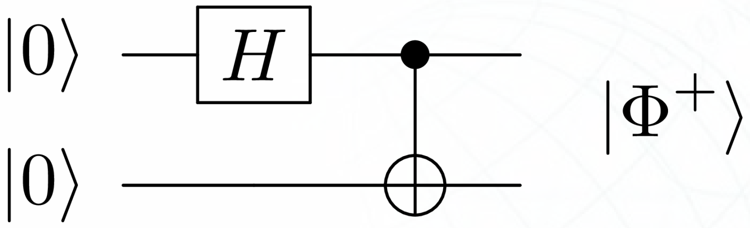


$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{c:c} I_{2,2} & 0_{2,2} \\ \hdashline 0_{2,2} & X \end{array}\right)$$

Ex.



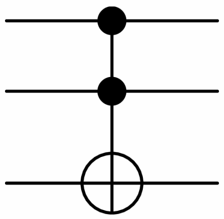
- Used to prepare the Bell state:



$$|00\rangle \overset{H\otimes I}{\mapsto} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \overset{CNOT}{\mapsto} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Multi-Qubit Gates (3/3)

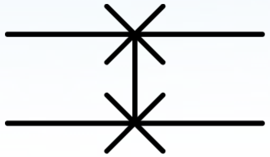
- Toffoli gate (CCNOT gate)
  → *universal* classically



Truth Table

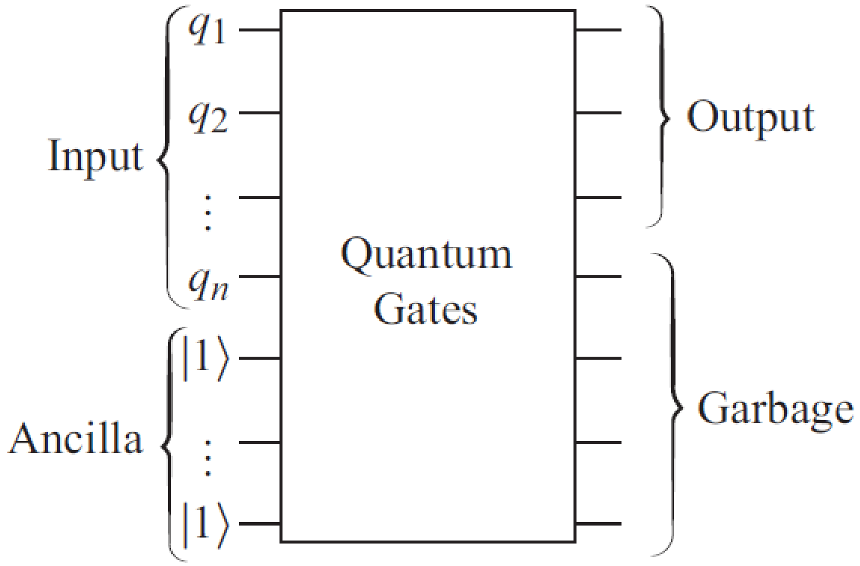
$\lvert 000\rangle \mapsto \lvert 000\rangle$
$\lvert 001\rangle \mapsto \lvert 001\rangle$
$\lvert 010\rangle \mapsto \lvert 010\rangle$
$\lvert 011\rangle \mapsto \lvert 011\rangle$
$\lvert 100\rangle \mapsto \lvert 100\rangle$
$\lvert 101\rangle \mapsto \lvert 101\rangle$
$\lvert 110\rangle \mapsto \lvert 111\rangle$
$\lvert 111\rangle \mapsto \lvert 110\rangle$

- Swap gate



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- A generic quantum circuit



- *Universal* quantum gates sets: $\{CNOT, T, H\}$ or $\{CCNOT, H\}$

Epilogue – Concluding Remarks

"Not only is the Universe stranger than we think,
it is stranger than we can think."

Werner Heisenberg (1865-1962)

Concluding Remarks

- The *state* of a (pure) qubit can be represented by a (unit) vector in $\mathbb{C}^2$
- We didn't consider the case of mixed states → density operators.
- In a nutshell: linear & unitary operation on unit vectors; or just by Truth Table

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Overview and Comparison of Gate Level Quantum Software Platforms

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