Duration of a Perpetuity

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Let C and K be the perpetuity's cash flow and discount rate per period respectively. Let P denote the present value of the perpetuity, $P = \frac{C}{K}$. Then the duration D of the perpetuity is:

$$D = 1\left(\frac{\frac{C}{1+K}}{\frac{C}{K}}\right)+2\left(\frac{\frac{C}{(1+K)^2}}{\frac{C}{K}}\right)+3\left(\frac{\frac{C}{(1+K)^3}}{\frac{C}{K}}\right)... = \sum_{n=1}^{\infty} n\left(\frac{\frac{C}{(1+K)^n}}{\frac{C}{K}}\right) = K\sum_{n=1}^{\infty} n\left(\frac{1}{1+K}\right)^n$$

So we need to know the value of the above infinite sum for $0 < \frac{1}{1+K} < 1$. I don't know the answer off the top of my head so let's try and figure it out. For simplicity let $x = \frac{1}{1+K}$. We have:

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + ...$$

And

$$x\sum_{n=1}^{\infty} nx^n = x^2 + 2x^3 + 3x^4 + ...$$

Which may not seem useful until you notice the difference (set equal to y):

$$\sum_{n=1}^{\infty} nx^n - x\sum_{n=1}^{\infty} nx^n = x + x^2 + x^3 + ... = y$$

Now notice:

$$xy = x^2 + x^3 + x^4 + ... \Rightarrow y - xy = x \Rightarrow y = \frac{x}{1-x}$$

So we have:

$$\sum_{n=1}^{\infty} nx^n - x\sum_{n=1}^{\infty} nx^n = \frac{x}{1-x} \Rightarrow \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

We can now use this fact in our duration formula (remembering $x = \frac{1}{1+K}$).

$$D = K\sum_{n=1}^{\infty} n\left(\frac{1}{1+K}\right)^n = K\frac{\frac{1}{1+K}}{\left(1-\frac{1}{1+K}\right)^2} = \frac{\frac{K}{1+K}}{\left(\frac{K}{1+K}\right)^2} = \frac{1+K}{K}$$

And there we have it; the duration of a perpetuity is $D = \frac{1+K}{K}$