Logistic Regression

MGTF 495

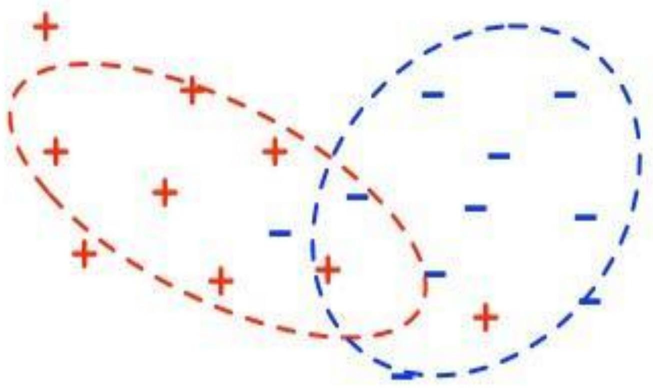
Class Outline

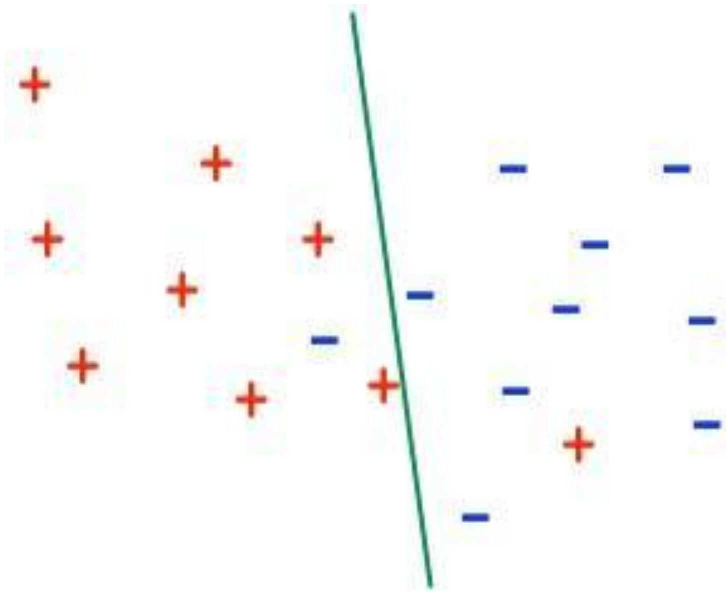
- Generative vs Discriminative Models
- Discriminative Models
    - Logistic Regression
    - SVM
    - Perceptron
- Kernels

Classification with parametrized models

Classifiers with a fixed no. of parameters can represent a limited set of functions. Learning a model is about picking a good approximation.

Typically the x 's are points in p-dimensional Euclidean space, R^p





Two ways to classify:

- **Generative**: model the individual classes.
- **Discriminative**: model the decision boundary between the classes.

Generative models: pros and cons

Advantages:

- Multiclass is a breeze
- For many common models: converges fast
- Returns not just a classification but also a confidence $\Pr(y|x)$

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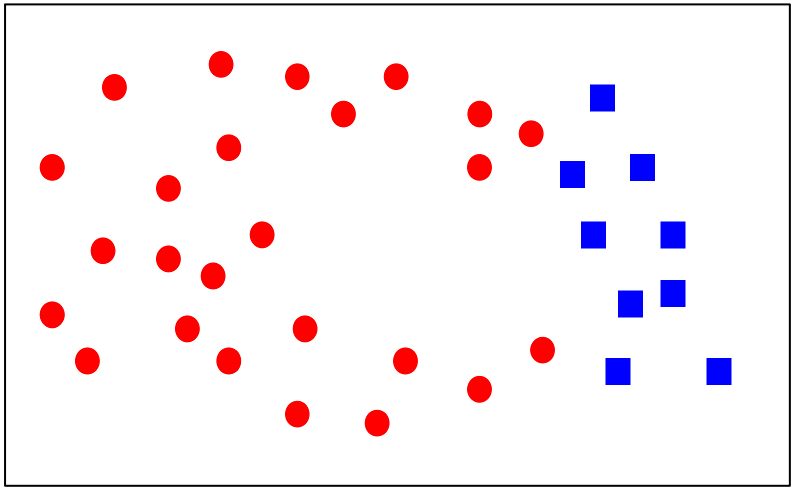
If we only care about classification, shouldn't we focus on the decision boundary rather than trying to model other aspects of the distribution of x ?

Generative versus discriminative



The generative way:

- Fit: π_0, π_1, P_0, P_1
- This determines a full joint distribution $\Pr(x, y)$
- Use Bayes' rule to obtain $\Pr(y|x)$

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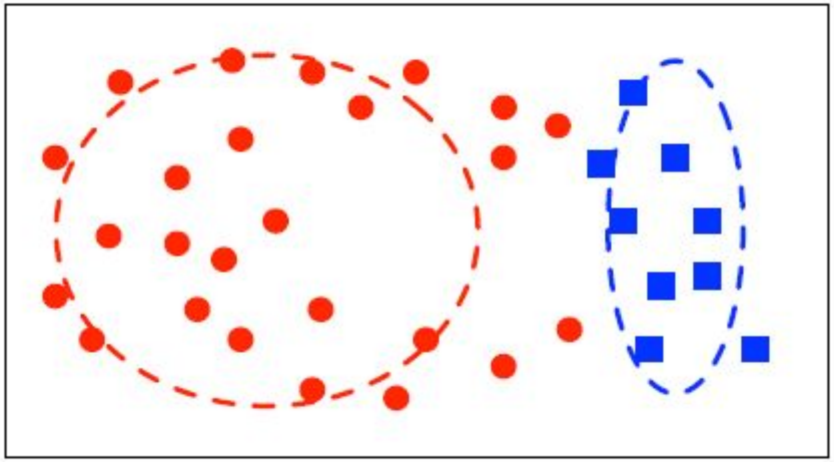

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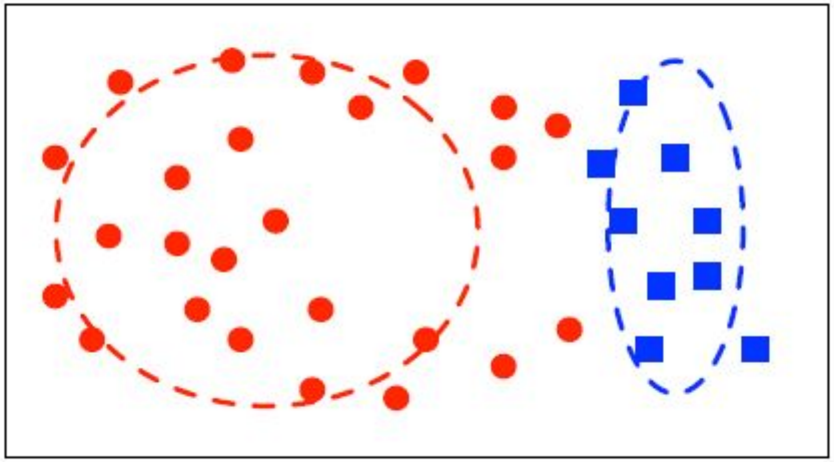

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The generative way: model Pr($y|x$) directly.
In our earlier terminology: forget about the μ (Prob. Distribution), just learn the η (likelihood)

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- Say $\mathcal{Y} = \{-1, 1\}$. Recall: for Gaussians with common covariance,

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- Can drop θ by adding an extra feature to x.

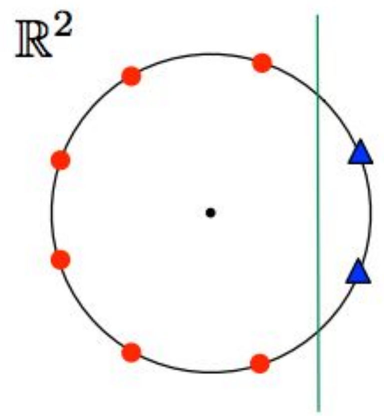
Homogeneous linear separators

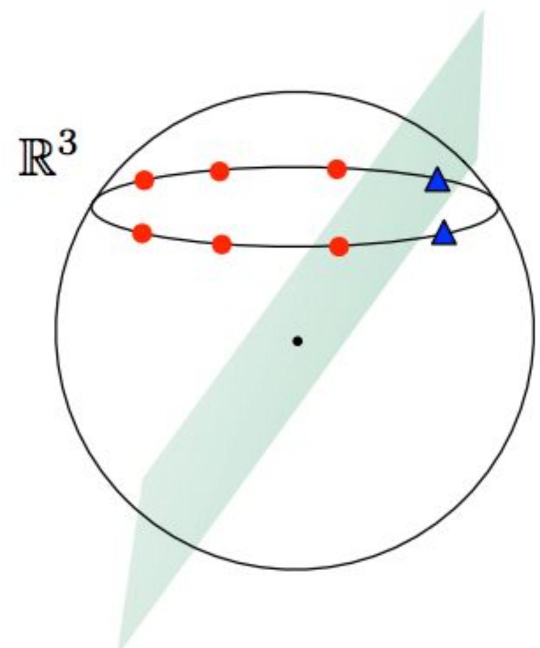
Hyperplanes that pass through the origin have no offset, $b = 0$.

Reduce to this case by adding an extra feature to x :

$$\widetilde{x} = (x, 1) \in \mathbb{R}^{p+1}$$

Then $\{x : w \cdot x = b\} \equiv \{x : \widetilde{w} \cdot \widetilde{x} = 0\}$ where $\widetilde{w} = (w, -b)$.





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$$\Pr(y = -1 \mid x) = \frac{1}{1 + e^{w \cdot x}}$$

$$\Pr(y = 1 \mid x) = 1 - \frac{1}{1 + e^{w \cdot x}} = \frac{e^{w \cdot x}}{1 + e^{w \cdot x}} = \frac{1}{1 + e^{-w \cdot x}}$$

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- More concisely,

$$\Pr(y \mid x) = \frac{1}{1 + e^{-y(w \cdot x)}}$$

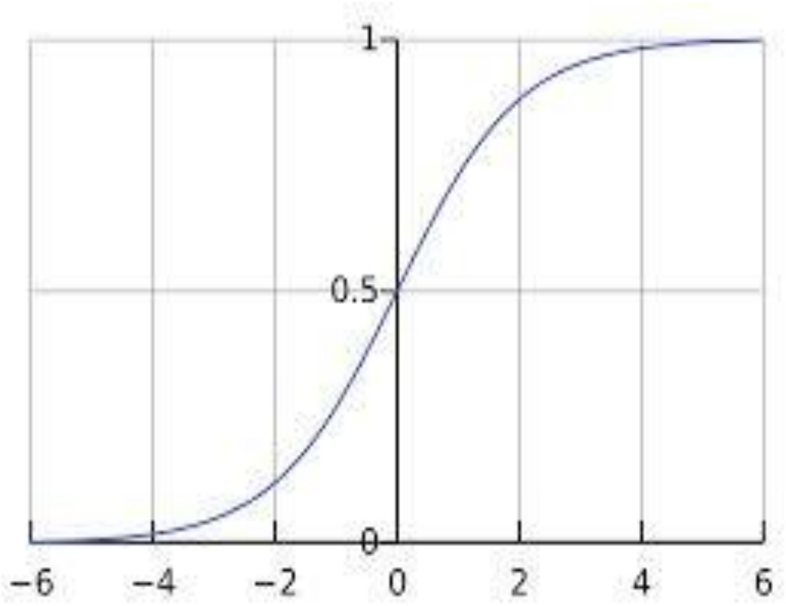
This is the **logistic regression model**, parametrized by w.

The squashing function

Take $X = \mathbf{R}^p$ and $Y = \{-1, 1\}$. The model specified by $w \in \mathbf{R}^p$ is

$$\Pr_w (y \mid x) = \frac{1}{1 + e^{-y(w \cdot x)}} = g(y(w \cdot x)),$$

where $g(z) = 1/(1 + e^{-z})$ is the *squashing function*.



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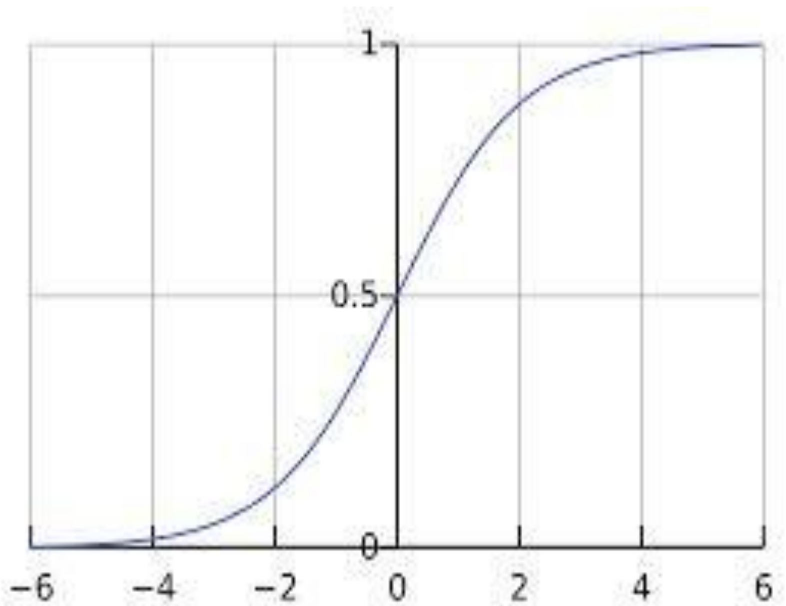

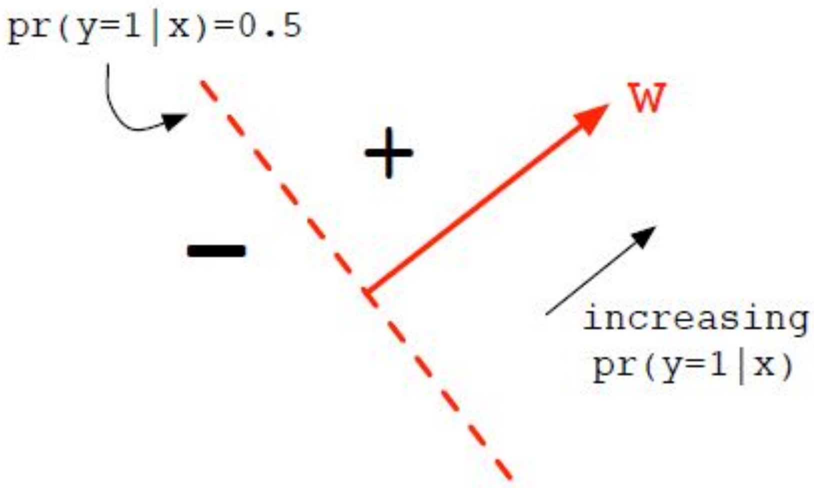

Fitting *w*

The maximum-likelihood principle: given a data set

$$(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^p \times \{-1, 1\},$$

pick the $w \in \mathbb{R}^p$ that maximizes

$$\prod_{i=1}^{n} \Pr_w(y^{(i)} \mid x^{(i)}).$$

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Easier to work with sums, so take negative log to get **loss function**

$$L(w) = -\sum_{i=1}^{n} \ln \Pr_w(y^{(i)} \mid x^{(i)})$$

$$= -\sum_{i=1}^{n} \ln\left(\frac{1}{1 + e^{-y^{(i)}(w \cdot x^{(i)})}}\right) = \sum_{i=1}^{n} \ln(1 + e^{-y^{(i)}(w \cdot x^{(i)})})$$

Our goal is to minimize $L(w)$.

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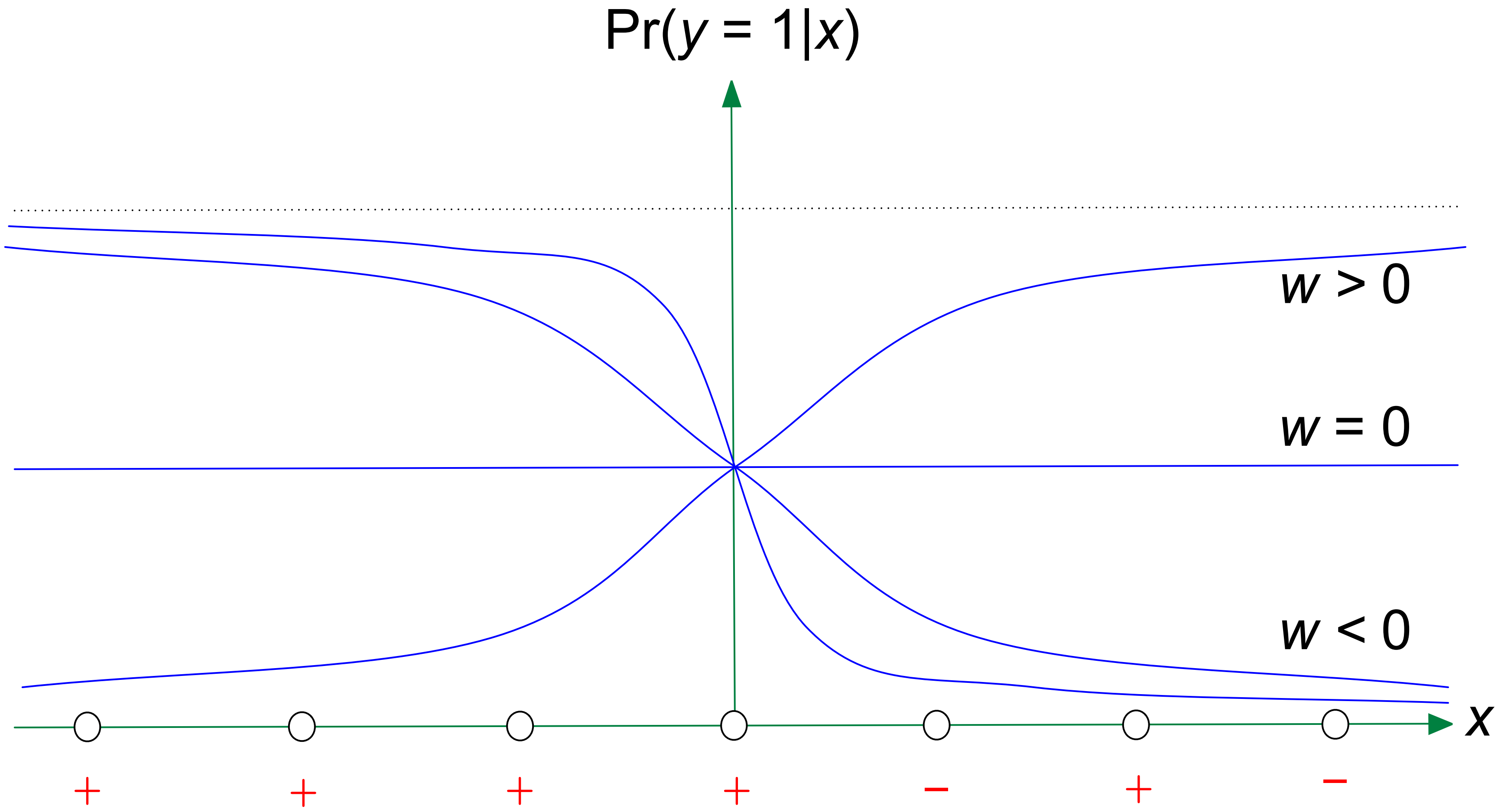
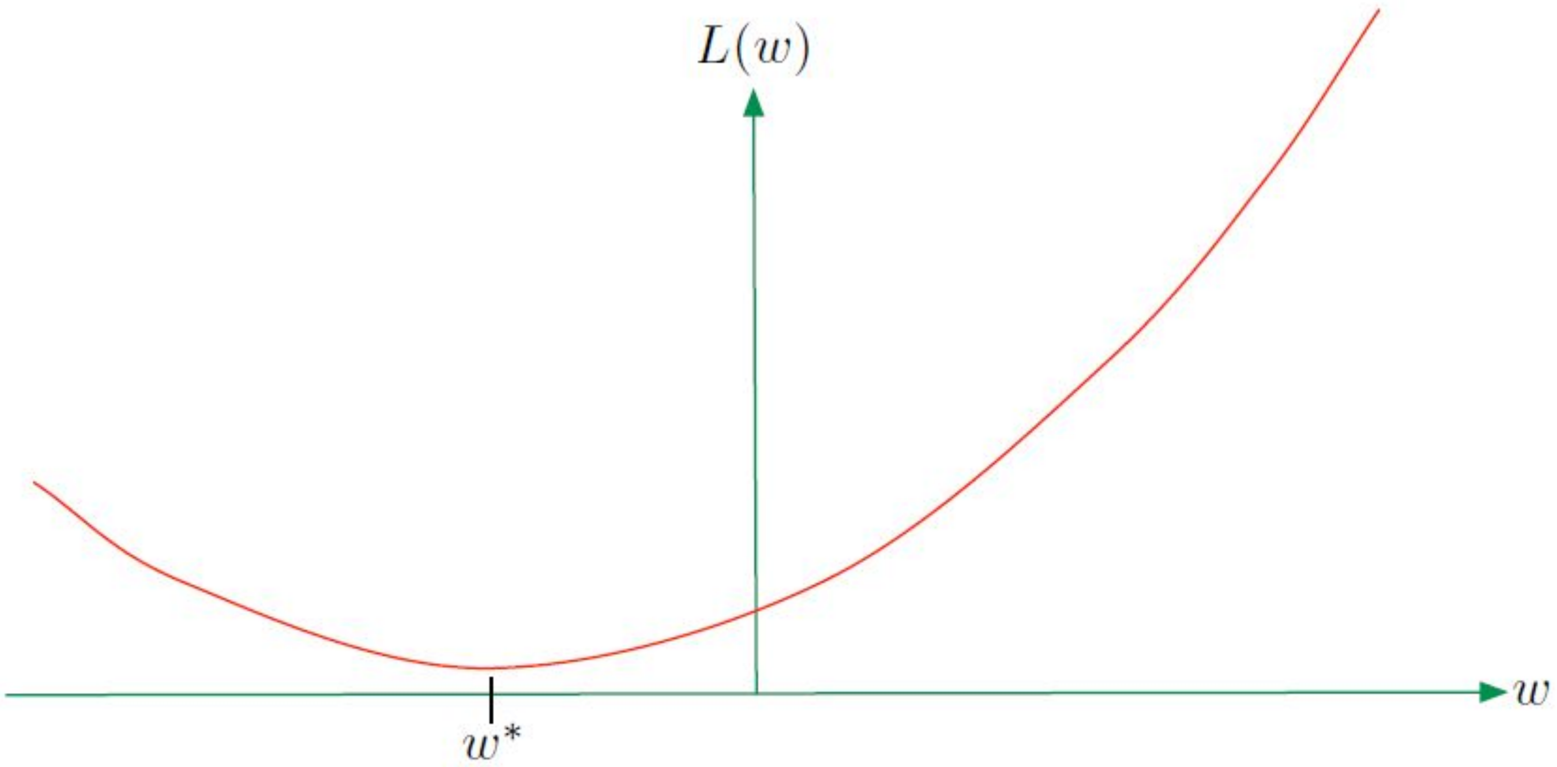
The good news: $L(w)$ is **convex** in w.

One dimensional example

$$\Pr_w(y \mid x) = \frac{1}{1 + e^{-ywx}}, \quad w \in \mathbf{R}$$



Example, cont'd



How to find the minimum of this convex function? A variety of options:

- Gradient descent
- Newton-Raphson

and many others.

Gradient descent procedure for LR

Given $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^p \times \{-1, 1\}$, find

$$\arg\min_{w \in \mathbb{R}^p} L(w) \;=\; \sum_{i=1}^{n} \ln(1 + e^{-y^{(i)}(w \cdot x^{(i)})})$$

- Set $w_0 = 0$
- For $t = 0, 1, 2, \ldots,$ until convergence:

$$w_{t+1} \;=\; w_t \;+\; \eta_t \sum_{i=1}^{n} y^{(i)} x^{(i)} \underbrace{\mathrm{Pr}_{w_t}(-y^{(i)} | x^{(i)})}_{\text{doubt}_t(x^{(i)}, y^{(i)})},$$

where η_t is a step size chosen by line search to minimize $L(w_{t+1})$.

Newton-Raphson procedure for LR

- Set $w_0 = 0$
- For $t = 0, 1, 2, \ldots,$ until convergence:

$$w_{t+1} \;=\; w_t \;+\; \eta_t \,(X^T D_t X)^{-1} \sum_{i=1}^{n} y^{(i)} x^{(i)} \mathrm{Pr}_{w_t}(-y^{(i)}|x^{(i)}),$$

  where

  - X is the $n \times p$ data matrix with one point per row
  - D_t is an $n \times n$ diagonal matrix with (i, i) entry

$$D_{t,ii} \;=\; \mathrm{Pr}_{w_t}(1|x^{(i)})\,\mathrm{Pr}_{w_t}(-1|x^{(i)})$$

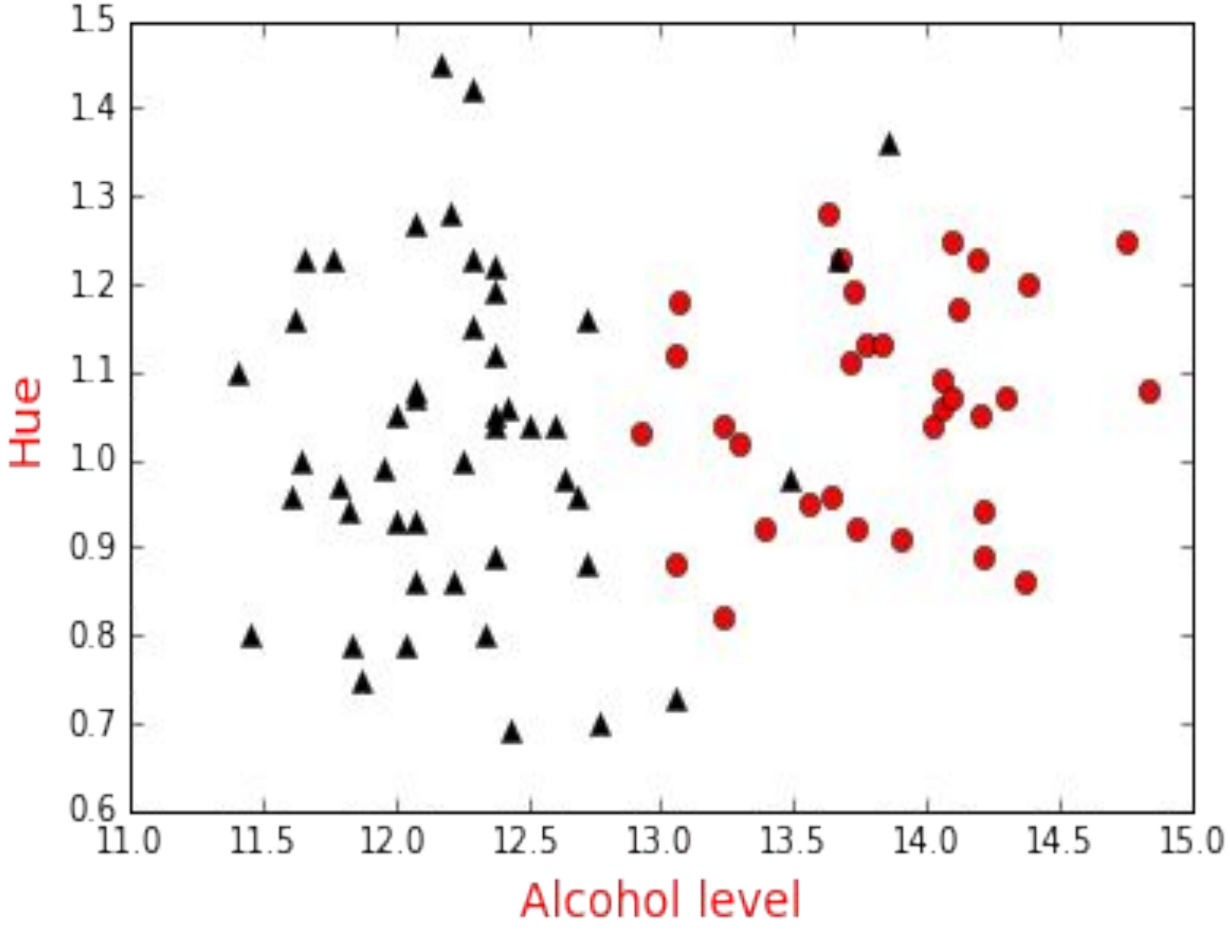
  - η_t is a step size that is either fixed to 1 ("iterative reweighted least squares") or chosen by line search to minimize $L(w_{t+1})$.

Example: "wine" data set

Recall: data from three wineries from the same region of Italy.

- 13 attributes: hue, color intensity, flavanoids, ash content, ...
- 178 instances in all: split into 118 train, 60 test

Pick two classes and just two attributes (hue, alcohol content).



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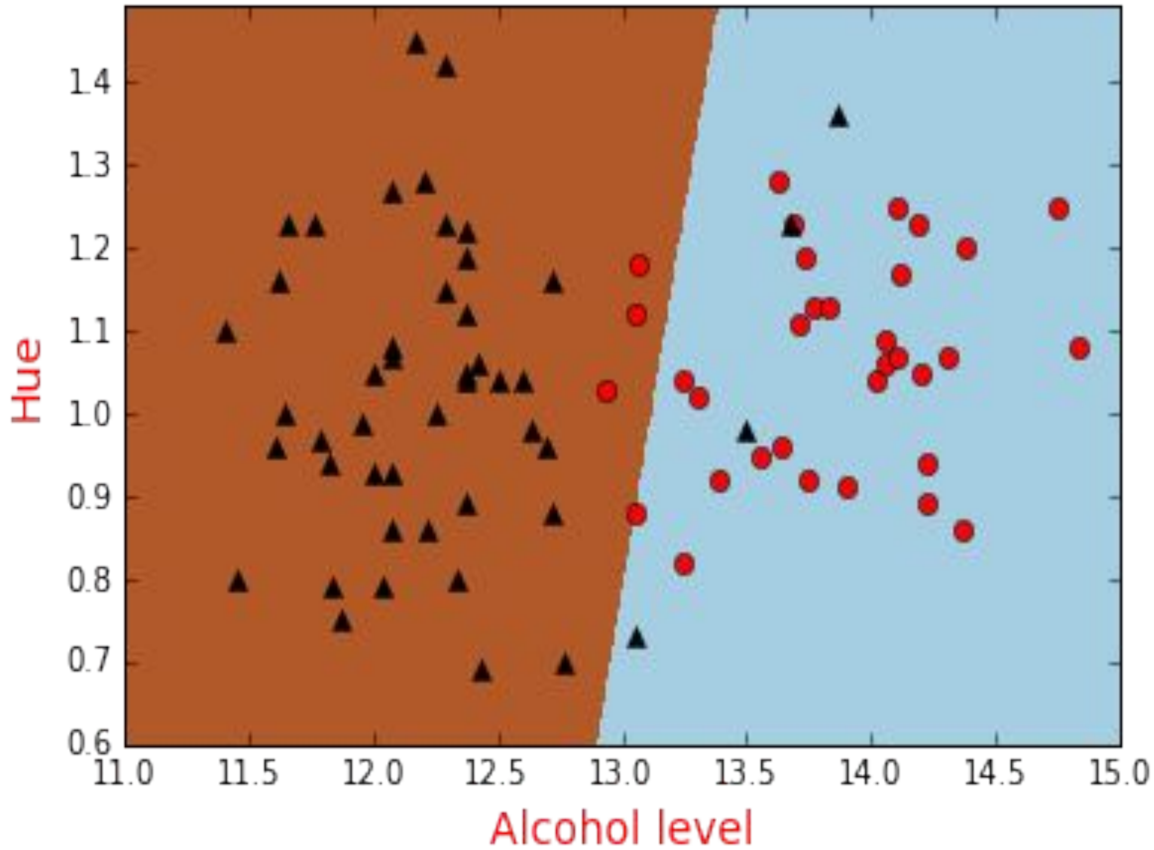

Test error using logistic regression: 10%.