Informative Projections (2)

MGTF 495

Class Outline

- Representation Learning
  - k-means
  - EM
  - Agglomerative hierarchical clustering
  - Hands-On
- Informative Projections
  - PCA
  - SVD
  - Latent semantic indexing (LSI)
  - Hands-On

Singular value decomposition (SVD)

For **symmetric** matrices, such as covariance matrices, we have seen:

- Results about existence of eigenvalues and eigenvectors
- The fact that the eigenvectors form an alternative basis
- The resulting spectral decomposition, which is used in PCA

But what about arbitrary matrices $M \in R^{pxq}$?

Any $p \times q$ matrix (say $p \leq q$) has a **singular value decomposition**:

$$M = \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_p \\ \downarrow & & \downarrow \end{pmatrix}}_{p \times p \text{ matrix } U} \underbrace{\begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{pmatrix}}_{p \times p \text{ matrix } \Lambda} \underbrace{\begin{pmatrix} \longleftarrow & v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & v_p & \longrightarrow \end{pmatrix}}_{p \times q \text{ matrix } V^T}$$

- $u_1, \ldots, u_p$ are orthonormal vectors in $\mathbb{R}^p$
- $v_1, \ldots, v_p$ are orthonormal vectors in $\mathbb{R}^q$
- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ are **singular values**

Matrix approximation

We can **factor** any $p \times q$ matrix as $M = UW^T$:

$$M = \begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_p \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{pmatrix} \begin{pmatrix} \longleftarrow & v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & v_p & \longrightarrow \end{pmatrix}$$

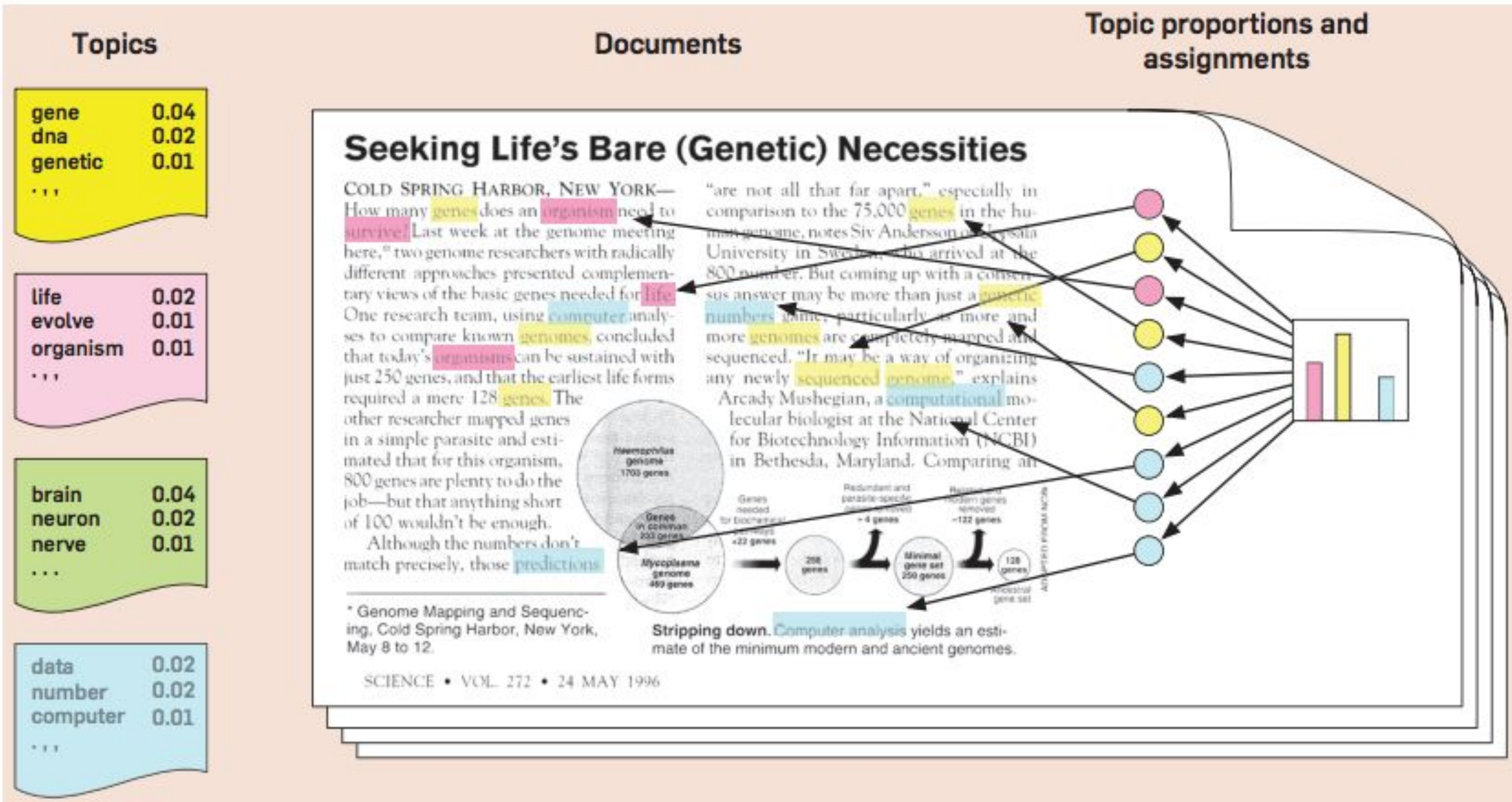
$$= \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_p \\ \downarrow & & \downarrow \end{pmatrix}}_{p \times p \text{ matrix } U} \underbrace{\begin{pmatrix} \longleftarrow & \sigma_1 v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \sigma_p v_p & \longrightarrow \end{pmatrix}}_{p \times q \text{ matrix } W^T}$$

A concise approximation to M: just take the first k columns of U and the first k rows of W^T, for $k < p$:

$$\widehat{M} = \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_k \\ \downarrow & & \downarrow \end{pmatrix}}_{p \times k} \underbrace{\begin{pmatrix} \longleftarrow & \sigma_1 v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \sigma_k v_k & \longrightarrow \end{pmatrix}}_{k \times q}$$

Example: topic modeling

Blei (2012):



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Latent semantic indexing (LSI)

Given a large corpus of n documents:

- Fix a vocabulary, say of V words.
- Bag-of-words representation for documents: each document becomes a vector of length V, with one coordinate per word.
- The corpus is an $n \times V$ matrix, one row per document.

	cat	dog	house	boat	garden	…
Doc 1	4	1	1	0	2	
Doc 2	0	0	3	1	0	
Doc 3	0	1	3	0	0	
		⋮				

Let's find a concise approximation to this matrix M.

Latent semantic indexing, cont'd

Use SVD to get an approximation to M: for small k,

$$\underbrace{\begin{pmatrix} \longleftarrow \text{doc } 1 \longrightarrow \\ \longleftarrow \text{doc } 2 \longrightarrow \\ \longleftarrow \text{doc } 3 \longrightarrow \\ \vdots \\ \longleftarrow \text{doc } n \longrightarrow \end{pmatrix}}_{n \times V \text{ matrix } M} \approx \underbrace{\begin{pmatrix} \longleftarrow \theta_1 \longrightarrow \\ \longleftarrow \theta_2 \longrightarrow \\ \longleftarrow \theta_3 \longrightarrow \\ \vdots \\ \longleftarrow \theta_n \longrightarrow \end{pmatrix}}_{n \times k \text{ matrix } \Theta} \underbrace{\begin{pmatrix} \longleftarrow \Psi_1 \longrightarrow \\ \vdots \\ \longleftarrow \Psi_k \longrightarrow \end{pmatrix}}_{k \times V \text{ matrix } \Psi}$$

Think of this as a *topic model* with k topics.

- Ψ_j is a vector of length V describing topic j: coefficient Ψ_{jw} is large if word w appears often in that topic.
- Each document is a combination of topics: θ_{ij} is the weight of topic j in document i.

Document i originally represented by ith row of M, a vector in $\mathbb{R}^V$.
Can instead use $\theta_i \in \mathbb{R}^k$, a more concise "semantic" representation.

The rank of a matrix

Suppose we want to approximate a matrix M by a simpler matrix $\widehat{M}$. What is a suitable notion of "simple"?

- Let's say M and $\widehat{M}$ are $p \times q$, where $p \leq q$.
- Treat each row of $\widehat{M}$ as a data point in $\mathbb{R}^q$.
- We can think of the data as "simple" if it actually lies in a low-dimensional subspace.
- If the rows lie in k-dimensional subspace, we say that $\widehat{M}$ has **rank** k.

The **rank** of a matrix is the number of linearly independent rows.

Low-rank approximation: given $M \in \mathbb{R}^{p \times q}$ and an integer k, find the matrix $\widehat{M} \in \mathbb{R}^{p \times q}$ that is the best rank-k approximation to M.

That is, find $\widehat{M}$ so that

- $\widehat{M}$ has rank $\leq k$
- The approximation error $\sum_{i,j}(M_{ij} - \widehat{M}_{ij})^2$ is minimized.

We can get $\widehat{M}$ directly from the singular value decomposition of M.

Low-rank approximation

Recall: Singular value decomposition of $p \times q$ matrix M (with $p \leq q$):

$$M = \begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_p \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{pmatrix} \begin{pmatrix} \longleftarrow & v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & v_p & \longrightarrow \end{pmatrix}$$

- $u_1, \ldots, u_p$ is an orthonormal basis of $\mathbb{R}^p$
- $v_1, \ldots, v_q$ is an orthonormal basis of $\mathbb{R}^q$
- $\sigma_1 \geq \cdots \geq \sigma_p$ are **singular values**

The **best rank-k approximation** to M, for any $k \leq p$, is then

$$\widehat{M} = \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_k \\ \downarrow & & \downarrow \end{pmatrix}}_{p \times k} \underbrace{\begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_k \end{pmatrix}}_{k \times k} \underbrace{\begin{pmatrix} \longleftarrow & v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & v_k & \longrightarrow \end{pmatrix}}_{k \times q}$$

Example: Collaborative filtering

Details and images from Koren, Bell, Volinksy (2009).

Recommender systems: matching customers with products.

- Given: data on prior purchases/interests of users
- Recommend: further products of interest

Prototypical example: Netflix.

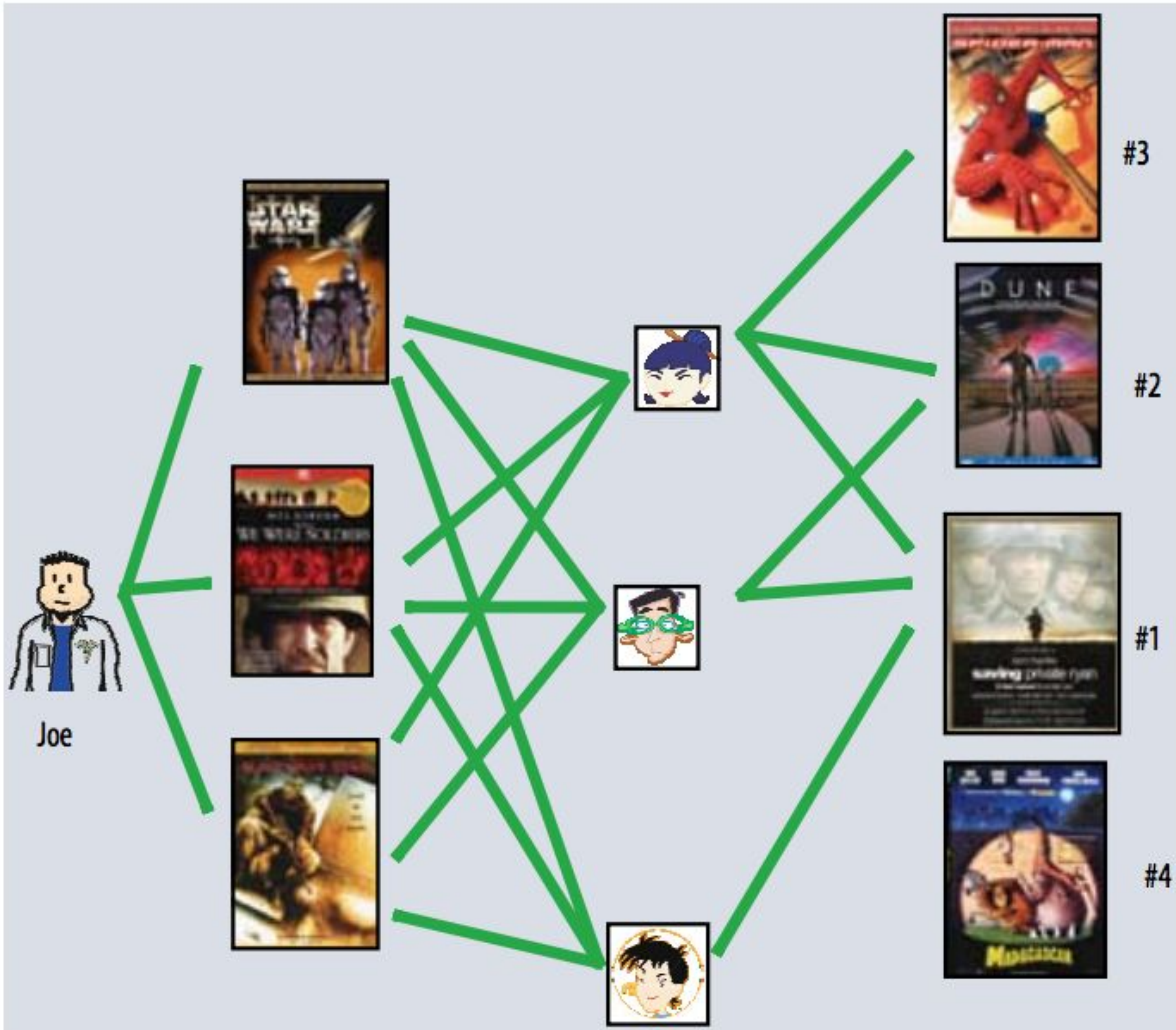
A successful approach: **collaborative filtering**.

- Model dependencies between different products, and between different users.
- Can give reasonable recommendations to a relatively new user.

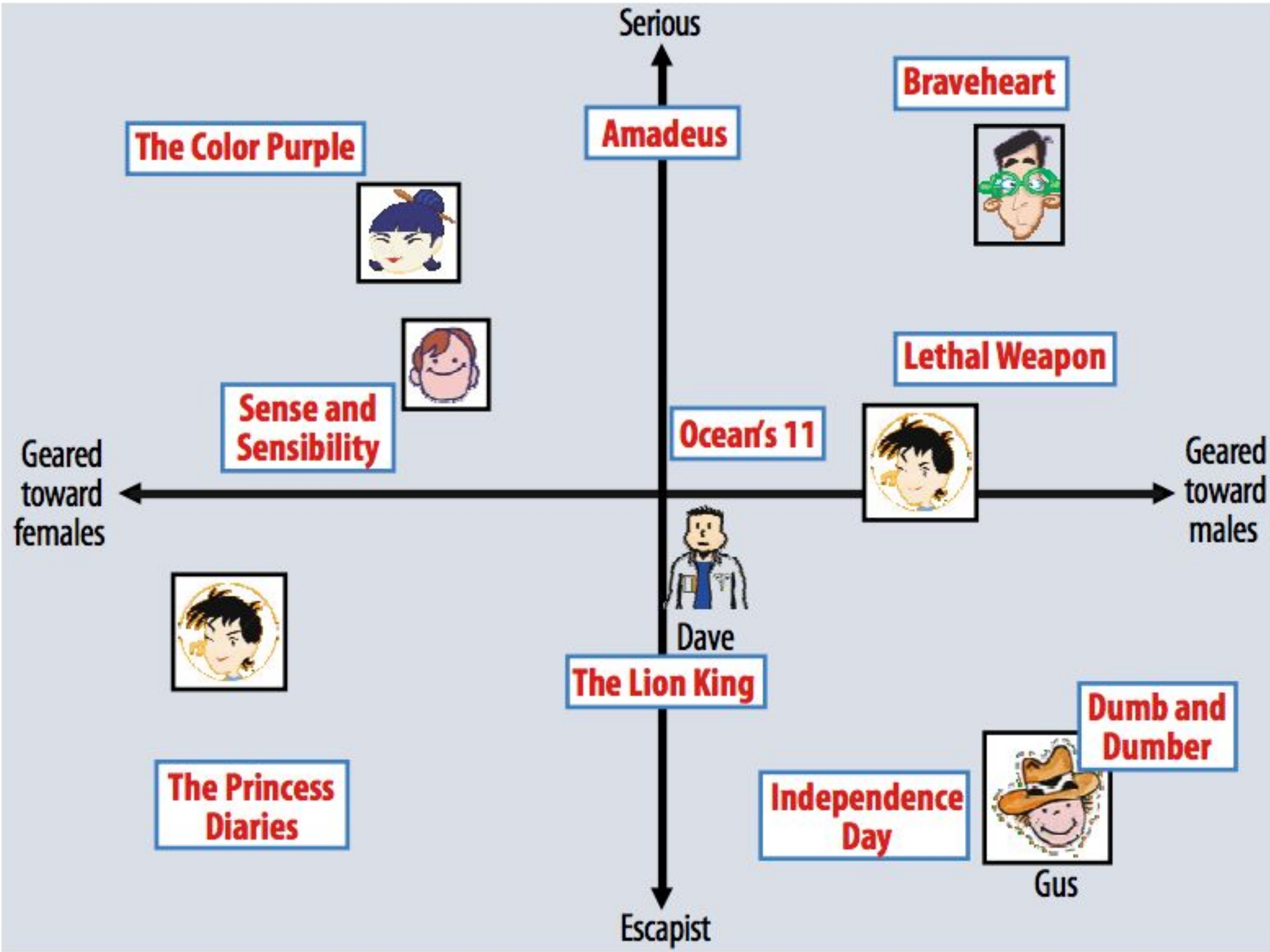
Two strategies for collaborative filtering:

- Neighborhood methods
- Latent factor methods

Neighborhood methods



Latent factor methods



The matrix factorization approach

User ratings are assembled in a large matrix *M*:

	Star Wars	Matrix	Casablanca	Camelot	Godfather	...
User 1	5	5	2	0	0	
User 2	0	0	3	4	5	
User 3	0	0	5	0	0	
		⋮				

- Not rated = 0, otherwise scores 1-5.
- For n users and p movies, this has size $n \times p$.
- Most of the entries are unavailable, and we'd like to predict these

Idea: Find the best low-rank approximation of M, and use it to fill in the missing entries.

User and movie factors

Best rank-k approximation is of the form $M \approx UW^T$:

$$\underbrace{\begin{pmatrix} \longleftarrow & \text{user } 1 & \longrightarrow \\ \longleftarrow & \text{user } 2 & \longrightarrow \\ \longleftarrow & \text{user } 3 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \text{user } n & \longrightarrow \end{pmatrix}}_{n \times p \text{ matrix } M} \approx \underbrace{\begin{pmatrix} \longleftarrow & u_1 & \longrightarrow \\ \longleftarrow & u_2 & \longrightarrow \\ \longleftarrow & u_3 & \longrightarrow \\ & \vdots & \\ \longleftarrow & u_n & \longrightarrow \end{pmatrix}}_{n \times k \text{ matrix } U} \underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ w_1 & w_2 & \cdots & w_p \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{k \times p \text{ matrix } W^T}$$

Thus user i's rating of movie j is approximated as

$$M_{ij} \approx u_i \cdot w_j$$

This "latent" representation embeds users and movies within the same k-dimensional space:

- Represent ith user by $u_i \in \mathbb{R}^k$
- Represent jth movie by $w_j \in \mathbb{R}^k$

Top two Netflix factors