Click models evaluations
(IR2 project)

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1 Introduction

Modeling user behavior on a search engine result page is important for understanding the users and supporting simulation experiments. As result pages become more complex, click models have to evolve as well in order to capture additional aspects of user behavior in response to new forms of result presentation. In recent years many models have been proposed that are aimed at predicting the behaviour of web search users.

In this report, we will implement and evaluate different click models using multiple evaluation metrics. These click models include the Click-Through Rate click model (CTR), Position-Based click Model (PBM), Cascade click model (CM) [8], Dependent Click Model (DCM) [5], Dynamic Bayesian Network click model (DBN) [1], User Browsing Model (UBM) [3], Click Chain Model (CCM) [4] and Task-centric Click Model (TCM) [11]. The evaluation metrics include likelihood, perplexity, click through rate prediction, relevance prediction, ranking performance and computation time. By doing this experiment we can know the performance of each different click model and this information can be used as a benchmark of the next click model proposal.

This report is organized as follows. In Section 2 we will describe the different click models and evaluation algorithms used in our experiments. The experiments and data source information will be covered in Section 3, followed by an analysis of these experiments in Section 4.

2 Methodology

The different models we compare UBM, DCM, DBN and TCM on the Yandex dataset [9]. In this section, we briefly describe their main characteristics and differences. We have implemented TCM ourselves, the other algorithms were taken from PyClick [10].

2.1 Click-through rate click model

The most simple click model, abbreviated as CTR, that actually tries to predict relevance for a query-documenent pair. The relevance is the only parameter in this model. The relevance in this model is simple defined as $a_u = \frac{\#\, of\, clicks}{\#\, of\, times\, shown}$. We included this model as a simple baseline that the models should improve upon.

2.2 Position based model

This model builds upon the CTR model. It adds a `position bias` where documents in a higher position are examined more often. A document can only be clicked if it was examined. The values for this position bias are taken from [7] and are $[.68, .61, .48, .34, .28, .2, .11, .1, .08, .06]$.

2.3 Cascade model

Cascade model assumes that users abandon the query session after the first click and hence does not provide a complete picture of how multiple clicks arise in a query session and how to estimate document relevance from such data [8, 2]. This model also assume that the user views search results from top to bottom, deciding whether to click each result before moving to the next. To observe a click, the user must have decided both to click and skip the ranks above.

2.4 Dependent Click Model

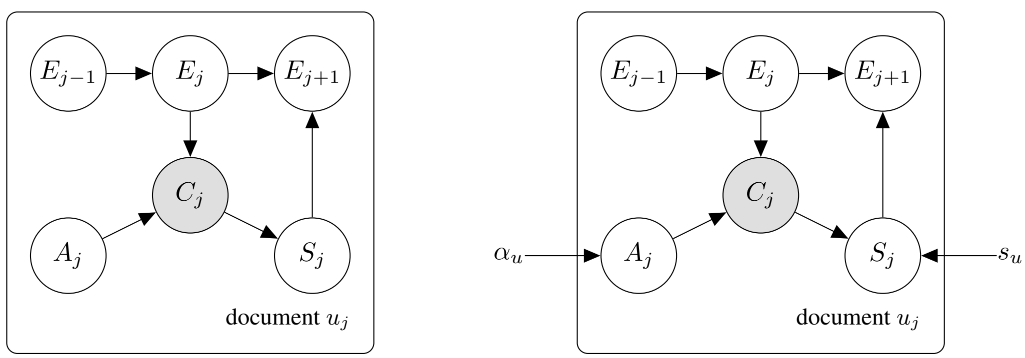
The dependent click model was first proposed by Guo et al. in [**?**]. In the paper they propose a new click model which can handle multiple clicks per query by introducing a position dependent parameter $lambda_j$ to reflect the chance that the user would like to see more results after a click at position j. A graphical representation of the model is presented in Figure1a

2.5 Dynamic Bayesion Network

The dynamic bayesian network click model is an extension to the traditional cascade model proposed by Chapelle and Zhang in [11]. For a given position j, in addition to observed variable C_j indicating whether there was a click or not at this position, the following latent variable are defined to model examination, perceived relevance and actual relevance, respectedly:

- E_j: Did the user $examine$ the document?
- A_j: Was the user $attracted$ by the document?
- S_j: Was the user $satisfied$ by the clicked document?

They introduce a variable s_u for each document u which describes the relevance of the document for this query. When the user clicks on this document, there is a certain chance that the user will be satisfied. If the user is not satisfied, he continues to examine the next document with a probability γ and stops otherwise. The parameter γ is known as the 'perseverance'. A graphical representation of the model is presented in Figure 1b.



(a) DCM (b) DBN

Figure 1: The graphical model of DCM and DBN.

2.6 User Browsing Model

In [3], Dupret and Piwowarski propose a new click model called the User Browsing Model (UBM). The main difference between UBM and other models is that UBM takes the distance into account from the current document u_j to the last clicked document $u_{j'}$ for determining the probability that the user continues browsing:

$$P(E_j = 1 \mid C_{j'} = 1, C_{j'+1} = 0, \ldots, C_{j-1} = 0) = \gamma_{jj'}$$

The probability that a document at rank j is examined E_j therefore depends on all possible paths the user could have taken to arrive at this document:

$$P(E_j = 1) = \sum_{j'=1}^{j-1} \gamma_{jj'}$$

A graphical representation of the model is presented in Figure 2.

2.7 Click Chain Model

In 2009, Fan Guo et al, proposed a bayesian based click model [4]. CCM shares the following assumptions with the cascade model and DCM: (1) users are homogeneous: their information needs are similar given the same query; (2)

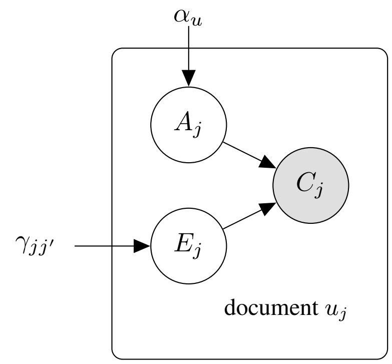


Figure 2: The graphical model of UBM.

decoupled examination and click events: click probability is solely determined by the examination probability and the document relevance at a given position; (3) cascade examination: examination is in strictly sequential order with no breaks.

This model can be formalized with the following conditional probabilities in which R_j is the relevance variable of document u at position j, E_j is the examination variable, and αs form the set of user behavior parameters:

$$P(C_j = 1|E_j = 0) = 0 \tag{1}$$
$$P(C_j = 1|E_j = 1, R_j) = R_j \tag{2}$$
$$P(E_{j+1} = 1|E_j = 0) = 0 \tag{3}$$
$$P(E_{j+1} = 1|E_j = 1, C_j = 0) = \alpha_1 \tag{4}$$
$$P(E_{j+1} = 1|E_j = 1, C_j = 1, R_j) = \alpha_2(1 - R_j) + \alpha_3 R_j \tag{5}$$
$$\tag{6}$$

A graphical representation of the model is presented in Figure 3.

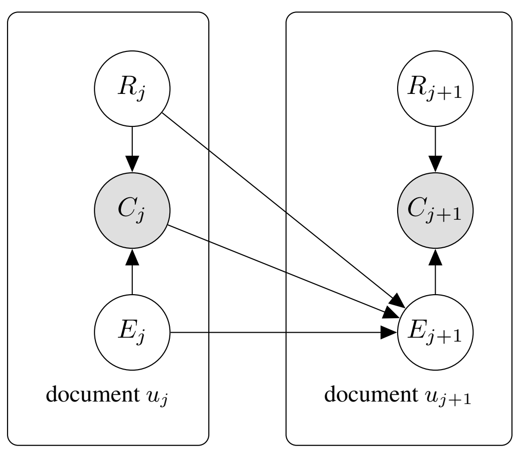


Figure 3: The graphical model of CCM.

2.8 Task-centric Click Model

The Task-centric Click Model (TCM) was first proposed by Zhang et al. in [11]. In the paper they propose a new click model which can handle multiple clicks of multiple queries in a task by introducing two new biases. The first bias indicates that users tend to express their information needs incrementally in a task, thus perform more clicks as their needs become clearer. The other bias indicates that users tend to click fresh documents that are not included in the results of previous queries. In their paper, they named the first assumption as `query bias`, and the second assumption as `duplicate bias`. A graphical representation of the state-of-the-art of the model is presented in Figure1a and the notations used in TCM are described in Table1.

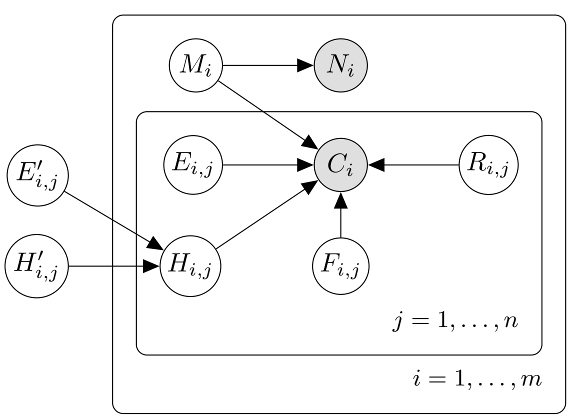
Symbol	Description
(i,j)	j-th ranking position in i-th query session.
M_i	Whether the i-th query matches the user's intent.
N_i	Whether the the user submits another query after i-th query session.
$E_{i,j}$	Examination of the document at (i,j).
$H_{i,j}$	Previous Examination of the document at (i,j).
$F_{i,j}$	Freshness of the document at (i,j).
$R_{i,j}$	Relevance of the document at (i,j).
$C_{i,j}$	Whether the the document at (i,j) is clicked.
(i',j')	Assume that d is the document at (i,j). i' is the latest query session where d has appeared in previous query sessions, and j' is the ranking position of this appearance.

Table 1: Notations used in TCM

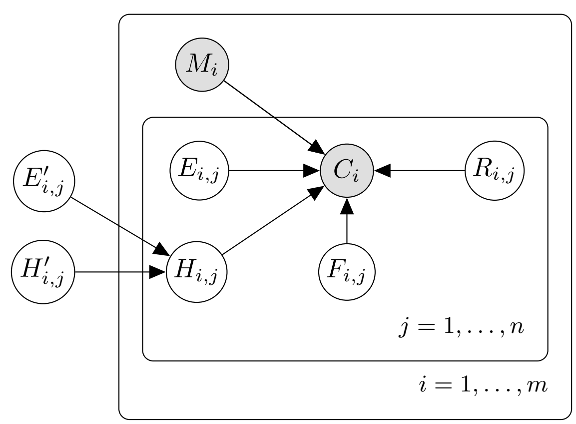
This model can be formalized with the following conditional probabilities:

$$P(M_i = 1) = \alpha_1 \tag{7}$$
$$P(N_i | M_i = 1) = \alpha_2 \tag{8}$$
$$P(F_{i,j} = 1 | M_{i,j} = 1) = \alpha_3 \tag{9}$$
$$P(E_{i,j} = 1) = \beta_j \tag{10}$$
$$P(R_{i,j} = 1) = r_d \tag{11}$$
$$M_i = 0 \Rightarrow N_i = 1 \tag{12}$$
$$H_{i,j} = 0 \Rightarrow F_{i,j} = 1 \tag{13}$$
$$H_{i,j} = 0 \Leftrightarrow H_{i',j'} = 0, E_{i',j'} = 0 \tag{14}$$
$$C_{i,j} = 1 \Leftrightarrow M_i = 1, E_{i,j} = 1, R_{i,j} = 1, F_{i,j} = 1 \tag{15}$$

In our implementation, we simplified TCM model by assuming that M_i is observed from the click log data, thus eq.8 can be removed. Our second assumptions is that $M_i, E_{i,j}, R_{i,j}$ and $F_{i,j}$ are independent. The detail calculation for updating EM parameters of the simplified TCM can be found in the appendices section. The graphical model of our TCM implementation is presented in Fig 4b.



(a) state-of-the-art TCM.



(b) simplified TCM

Figure 4: The graphical model of state-of-the-art TCM and simplified TCM.

3 Evaluation

3.1 Evaluation Criteria

To equally evaluate each click model's performance, we use evaluation metrics that are proposed along with the click models. The evaluation metrics used in this experiment are listed as follow:

3.1.1 Likelihood and Perplexity

Likelihood is already implied in each click model during E-step, where it should get better for every EM iteration. However, this evaluation can become interesting when we compare one click model to the others. By comparing the likelihood evaluation per iteration, we can see the performance of each click-model based on its convergent speed. We are not doing this? We are just calculating the likelihood at the end.

Click perplexity is a widely used metric for evaluation click model accuracy. Perplexity can be seen as the log-likelihood powers which are computed independently at each position as also mentioned at [11] and [3]. For example, we assume that q_j^s is the probability of some click calculated from a click model, i.e. $P(C_j^s = 1)$ where C_j^s is a binary value indicating the click event at position j in query session s. Then the click perplexity at position j is computed as follows:

$$p_j = 2^{-\frac{1}{|S|}\sum_{s\in S}(C_j \log_2 q_j + (1-C_j)\log_2(1-q_j))}$$

The perplexity of a data set is defined as the average of perplexities in all positions. Thus, a smaller perplexity value indicates a better consistency between the click model and the actual click data.

3.1.2 Click-Trough-Rate prediction (CTR)

The purpose of click-through rates is to measure the ratio of clicks to impressions of an document. Generally the higher the CTR the higher change of that document being clicked. The click-through rate is defined as:

$$CTR = \frac{\#\, of\; clicks}{\#\, of\; impression}$$

A way to use this as an evaluation measure is proposed in [1, p. 4]. In the same way we calculate the CTR prediction using the following protocol:

1. Retrieve all the sessions related to a given query
2. Consider an url that appeared both in position 1 and some other positions
3. Hold out as test sessions all the sessions in which that url appeared in position 1
4. Train the model on the remaining sessions and predict a_u
5. Compute the test CTR in position 1 on the held-out sessions
6. Compute an error between these two quantities
7. Average the error on all such urls and queries, weighted by the number of test sessions

The error measure we use is the Mean-Square-Error (MSE).

3.1.3 Relevance prediction

Relevance prediction was used to evaluate performance of DBN model [1, p. 6]. The accuracy of CTR prediction may not directly translate to relevance, especially when we were to evaluate TCM. On TCM, the CTR of a particular document can be very low even if the document is relevant. This can happen because on TCM, we look over the whole task instead of single query, and the second time this document reappear, the user tends to ignore it by the freshness assumption. I don't think we want to mention the performance of the TCM here already.? Therefore, relevance prediction can be used as additional inference of relevancy.

TODO:: add more info about general relevance algorithm

3.1.4 Predicted relevance as a ranking feature

The accuracy of CTR prediction may not directly translate to relevance. However we can use these relevances by focusing on the relative order of the relevances. For every query for which we have editorial judgements we use the predicted relevances to rank the documents. To evaluate the performance of a ranker we can use the Normalized Discounted Cumulative Gain (NDCG) [6], for which we use a cutoff at five (NDCG$_5$). All these queries are then averaged to calculate the ranking performance of the click model.

3.1.5 Computation time

Historically in Machine Learning a big problem in creating accurate models was the amount of data that was available. However this is no longer the case, we are mostly restricted by the time that it takes to learn a model from the large amount of data that we currently have. So an important feature of a succesful click model is that it should be able to efficiently compute its parameters. Therefor we also decided to look at the computation time it takes to train the click models.

3.2 Evaluation setup

The experiment was run ...

3.3 Results

In Table 2 one can see the results of the experiments. It can be seen that ...

	Log-likelihood	Perplexity	Computation Time
Click Model A	0	0	
Click Model B	0	0	

Table 2: Results

4 Analysis

After running the experiments we were able to evaluate the different algorithms based on the ...

5 Conclusions

In this paper we showed that ...

In our implementation, we did not ...

References

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A Appendix

We give here some details about the inference in our TCM implementation outlined in section 2.8.

A.1 Click probability

For $P(F_{i,j} = 1)$ we introduce a variable $f_{i,j}$, which will be derived later.
By assumption that $M_i, E_{i,j}, R_{i,j}$and$F_{i,j}$ are independent, click probability can be formularize as:

$$P(C_{i,j} = 1) = P(M_i = 1) * P(E_{i,j} = 1) * P(R_{i,j} = 1) * P(F_{i,j} = 1) \tag{16}$$

$$= \alpha_1 * \beta_j * r_{i,j} * f_{i,j} \tag{17}$$

A.2 Probability of the query match user intention

Because we remove equation that depends on α_2, we can now set α_1 as MLE.

$$P(M_i = 1) = \alpha_1$$

A.3 Probability of user submit next query

User submit next query if the query does not match user intention (α_1) or user want to search more.

$$\begin{aligned} P(N_i = 1) &= \frac{1}{|S|} \sum_{i \in S} \mathcal{I}(N_i = 1) \\ &= \frac{q_i}{|S|} \\ &= n_i \end{aligned}$$

q_i is the number of submitted-queries where user submit another query after i-th query session.

$$\begin{aligned} P(N_i = 1|M_i = 1) = \alpha_2 & \\ &= \frac{P(N_i = 1) - P(N_i = 1|M_i = 0)P(M_i = 0)}{P(M_i = 1)} \\ &= \frac{n_i + \alpha_1 - 1}{\alpha_1} \end{aligned}$$

A.4 Relevance probability

$$\begin{aligned} P(R_{i,j} = 1) = r_{i,j} & \\ &= \frac{\sum_{q_{i,j} \in S_{i,j}} P(R_{i,j} = 1|C)}{|S_{i,j}|} \end{aligned}$$

Where $S_{i,j}$ are all sessions (queries) containing the document corresponding with the query i at rank j - document $P(R_{i,j} = 1|C)$ will be derive on eq.20

$$P(R_{i,j} = 1|C) = \mathcal{I}(C_{i,j} = 1)P(R_{i,j}|C_{i,j} = 1) + \mathcal{I}(C_{i,j} = 0)P(R_{i,j}|C_{i,j} = 0) \tag{18}$$

$$= c_{i,j} + (1 - c_{i,j})\frac{P(C_{i,j} = 0|R_{i,j} = 1)P(R_{i,j} = 1)}{P(C_{i,j} = 0)} \tag{19}$$

$$= c_{i,j} + (1 - c_{i,j})\frac{P(C_{i,j} = 0|R_{i,j} = 1)r_{i,j}}{1 - P(C_{i,j} = 1)} \tag{20}$$

Where $c_{i,j} = 1$ if (i,j) was clicked in the current session. $P(C_{i,j} = 0|R_{i,j} = 1)$ is the chance of no click given that it is relevant.

$$\begin{aligned}
P(C_{i,j} = 0|R_{i,j} = 1) &= P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1)P(M_i = 1) + P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0)P(M_i = 0) \\
&= \alpha_1 P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 1)P(E_{i,j} = 1) \\
&+ \alpha_1 P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 0)P(E_{i,j} = 0) \\
&+ (1 - \alpha_1)P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0, E_{i,j} = 1)P(E_{i,j} = 1) \\
&+ (1 - \alpha_1)P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0, E_{i,j} = 0)P(E_{i,j} = 0) \\
\\
&= \alpha_1\beta_j P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 1, F_{i,j} = 1)P(F_{i,j} = 1) \\
&+ \alpha_1\beta_j P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 1, F_{i,j} = 0)P(F_{i,j} = 0) \\
&+ \alpha_1(1 - \beta_j)P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 0, F_{i,j} = 1)P(F_{i,j} = 1) \\
&+ \alpha_1(1 - \beta_j)P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 0, F_{i,j} = 0)P(F_{i,j} = 0) \\
&+ (1 - \alpha_1)\beta_j P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0, E_{i,j} = 1, F_{i,j} = 1)P(F_{i,j} = 1) \\
&+ (1 - \alpha_1)\beta_j P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0, E_{i,j} = 1, F_{i,j} = 0)P(F_{i,j} = 0) \\
&+ (1 - \alpha_1)(1 - \beta_j)P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0, E_{i,j} = 0, F_{i,j} = 1)P(F_{i,j} = 1) \\
&+ (1 - \alpha_1)(1 - \beta_j)P(C_{i,j} = 0|R_{i,j} = 1, M_i = 0, E_{i,j} = 0, F_{i,j} = 0)P(F_{i,j} = 0)
\end{aligned}$$

We note that $P(C_{i,j} = 0|R_{i,j} = 1, M_i = 1, E_{i,j} = 1, F_{i,j} = 1) = 0$. Otherwise it is 1. From eq. 24 from TCM paper. Together with inserting our parameters this gives us the following:

$$P(C_{i,j} = 0|R_{i,j} = 1) = \alpha_1\beta_j f_{i,j} + \alpha_1\beta_j(1 - f_{i,j}) + \alpha_1(1 - \beta_j)f_{i,j} + \alpha_1(1 - \beta_j)(1 - f_{i,j}) \tag{21}$$
$$+ (1 - \alpha_1)\beta_j f_{i,j} + (1 - \alpha_1)\beta_j(1 - f_{i,j}) + (1 - \alpha_1)(1 - \beta_j)(f_{i,j} \tag{22}$$
$$+ (1 - \alpha_1)(1 - \beta_j)(1 - f_{i,j}) \tag{23}$$

expanding this we are only left with

$$P(C_{i,j} = 0|R_{i,j} = 1) = 1 - (\alpha_1\beta_j f_{i,j}) \tag{24}$$

Which seems intuitive as we assumed that all $M_i, R_{i,j}, E_{i,j}$ and $F_{i,j}$ are independent to get $P(C_{i,j} = 1)$. With this information we can calculate

$$P(R_{i,j} = 1|C) = c_{i,j} + (1 - c_{i,j})\frac{(1 - (\alpha_1\beta_j f_{i,j}))r_{i,j}}{1 - \alpha_1\beta_j f_{i,j}r_{i,j}} \tag{25}$$
$$= c_{i,j} + (1 - c_{i,j})\frac{r_{i,j} - \alpha_1\beta_j f_{i,j}r_{i,j}}{1 - \alpha_1\beta_j f_{i,j}r_{i,j}} \tag{26}$$

A.5 Examination probability

$$P(E_{i,j} = 1) = \beta_j \tag{27}$$
$$= \frac{1}{|S|}\sum_{i\in S} P(E_{i,j} = 1|C) \tag{28}$$

Where S is all sessions and i is a query within that session. $P(E_{i,j} = 1|C)$ will be derive on eq.29

$$P(E_{i,j} = 1|C) = \mathcal{I}(C_{i,j} = 1)P(E_{i,j}|C_{i,j} = 1) + \mathcal{I}(C_{i,j} = 0)P(E_{i,j}|C_{i,j} = 0) \tag{29}$$
$$= c_{i,j} + (1 - c_{i,j})\frac{P(C_{i,j} = 0|E_{i,j} = 1)P(E_{i,j} = 1)}{P(C_{i,j} = 0)} \tag{30}$$
$$= c_{i,j} + (1 - c_{i,j})\frac{P(C_{i,j} = 0|E_{i,j} = 1)\beta_j}{1 - P(C_{i,j} = 1)} \tag{31}$$

Where $c_{i,j}$ indicates whether document i,j was clicked. Analog to eq 24 we can show that

$$P(C_{i,j}=0|E_{i,j}=1)=1-(\alpha_1 f_{i,j}r_{i,j}) \tag{32}$$

This gives us

$$P(E_{i,j}=1|C)=c_{i,j}+(1-c_{i,j})\frac{(1-(\alpha_1 f_{i,j}r_{i,j}))\beta_j}{1-\alpha_1\beta_j f_{i,j}r_{i,j}} \tag{33}$$

$$=c_{i,j}+(1-c_{i,j})\frac{\beta_j-\alpha_1\beta_j f_{i,j}r_{i,j}}{1-\alpha_1\beta_j f_{i,j}r_{i,j}} \tag{34}$$

A.6 Freshness probability

$$P(F_{i,j}=1|H_{i,j}=1)=\alpha_3 \tag{35}$$

$$\alpha_3=\frac{1}{|S_{i,j}|}\sum_{q\in S}\sum_{(i,j)\in q}P(Fi,j=1|H_{i,j}=1,C) \tag{36}$$

Where (i,j) is a query, rank pair identifying a certain document. $P(F_{i,j}=1|C)$ will be derive on eq.37 $P(F_{i,j}=1)$ will be derive on eq.43

$$P(F_{i,j}=1|H_{i,j}=1,C)=\mathcal{I}(C_{i,j}=1)P(F_{i,j}=1|H_{i,j}=1,C_{i,j}=1) \tag{37}$$

$$+\mathcal{I}(C_{i,j}=0)P(F_{i,j}=1|H_{i,j}=1,C_{i,j}=0) \tag{38}$$

$$=c_{i,j}+(1-c_{i,j})\frac{P(C_{i,j}=0|F_{i,j}=1,H_{i,j}=1)P(F_{i,j}=1|H_{i,j}=1)}{P(C_{i,j}=0|H_{i,j}=1)} \tag{39}$$

Analog to eq 24 we can show that

$$P(C_{i,j}=0|F_{i,j}=1,H_{i,j}=1)=1-(\alpha_1\beta_j r_{i,j}) \tag{40}$$

We can also show

$$P(C_{i,j}=0|H_{i,j}=1)=1-P(C_{i,j}=1|H_{i,j}=1) \tag{41}$$

$$=1-(\alpha_1\alpha_3\beta_j r_{i,j}) \tag{42}$$

The only difference between this and eq. 17 is that it is given that $H_{i,j}=1$ and because $H_{i,j}=1$ only has an influence on $P(F_{i,j}=1)$, namely that $P(F_{i,j}=1|H_{i,j}=1)=1$, we can substitute $f_{i,j}$ with $\alpha 3$ in eq. 17

Now we only need to calculate $f_{i,j}=P(F_{i,j})=1$

$$P(F_{i,j}=1)=\mathcal{I}(H_{i,j}=1)P(F_{i,j}=1|H_{i,j}=1)+\mathcal{I}(H_{i,j}=0)P(F_{i,j}=1|H_{i,j}=0) \tag{43}$$

$$=\mathcal{I}(H_{i,j}=1)\alpha_3+\mathcal{I}(H_{i,j}=0) \tag{44}$$

Where $\mathcal{I}(H_{i,j}=1)$ is a binary indicator function from the data specifying whether document (i,j) was shown before in the current (q from eq. 36) session.

We could replace this indicator function with the probability that the document was examined the last time it was shown. This probability, called $H_{i,j}$ would depend on the probability that it was examined and $H_{i',j'}$ where i',j' is the last time this document was shown in the current session. It would look like this

$$P(H_{i,j}=1)=P(E_{i',j'}=1)P(H_{i',j'}=1) \tag{45}$$

then eq. 43 becomes:

$$P(F_{i,j}=1)=P(H_{i,j}=1)\alpha_3+P(H_{i,j}=0) \tag{46}$$

$$=P(H_{i,j}=1)\alpha_3+(1-P(H_{i,j}=1)) \tag{47}$$

$$=\alpha_3 P(E_{i',j'}=1)P(H_{i',j'}=1)+(1-P(E_{i',j'}=1)P(H_{i',j'}=1)) \tag{48}$$

Note that this discards the information that if (i', j') was clicked it surely was examined.

With eq 40 we can calculate $P(F_{i,j} = 1|C)$

$$P(F_{i,j} = 1|H = 1, C) = c_{i,j} + (1 - c_{i,j}) \frac{(1 - (\alpha_1 \beta_j r_{i,j}))\alpha_3}{1 - \alpha_1 \alpha_3 \beta_j r_{i,j}} \tag{49}$$

$$= c_{i,j} + (1 - c_{i,j}) \frac{\alpha_3 - \alpha_1 \alpha_3 \beta_j r_{i,j}}{1 - \alpha_1 \alpha_3 \beta_j r_{i,j}} \tag{50}$$