# Intermediate Axis Theorem

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Throughout this article, I describe linear-algebraic concepts in graphical notation. To avoid confusion, I use blue wires to represent vectors in the body frame, and red to represent vectors in the fixed frame.

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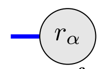
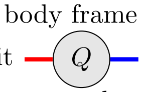
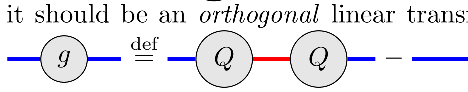
Consider a rigid body, made of masses $m_\alpha$ at locations  relative to the center of mass. We allow it to rotate freely, but fix its center of mass to the origin.

How do we represent the state of the system? The thing that's changing over time is the object's orientation. This is equivalently *the relationship between the body frame and the fixed frame.*

This relationship is an isometry, hence an affine transformation. Since it maps the origin of the body frame to the origin of the fixed frame, it's a linear transformation. I call it , since it's our "position" coordinate.

In fact, it should be an *orthogonal* linear transformation. So we impose a constraint:  $= 0$,

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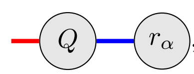
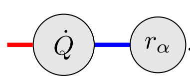
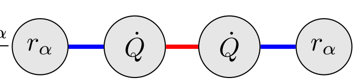
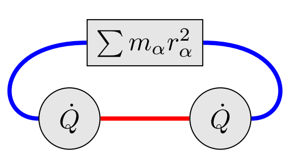
We apply Lagrangian mechanics.

In the fixed frame, the position of the mass $m_\alpha$ is , so the velocity is . Therefore, the kinetic energy of the rigid body is:

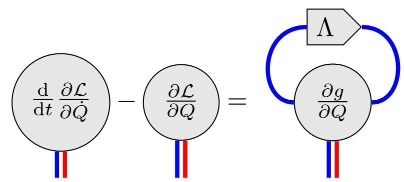
$$\sum_\alpha \frac{m_\alpha}{2}$$ 

Since there is no potential, the Lagrangian is the same.

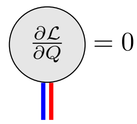
It will be useful to write this in a slightly different form:

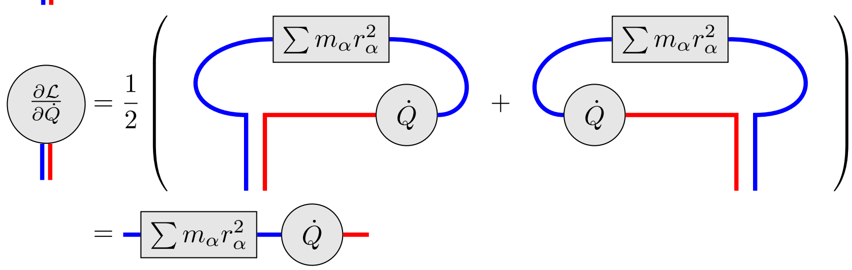
$$\mathcal{L} = \frac{1}{2}$$ 

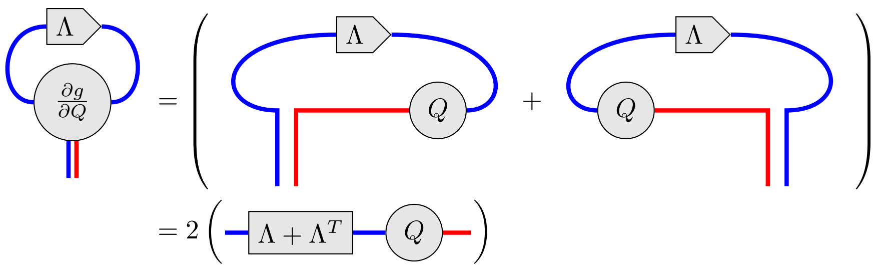
The constrained Euler-Lagrange equations says the following holds for some unknown time-dependent matrix $\Lambda$.



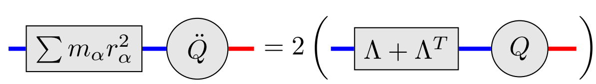

Evaluating the derivatives:



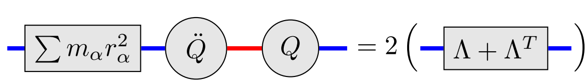



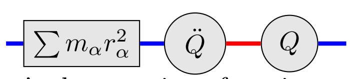




So:

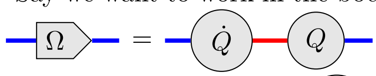
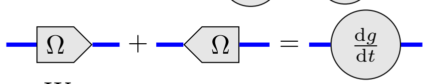



In other words:

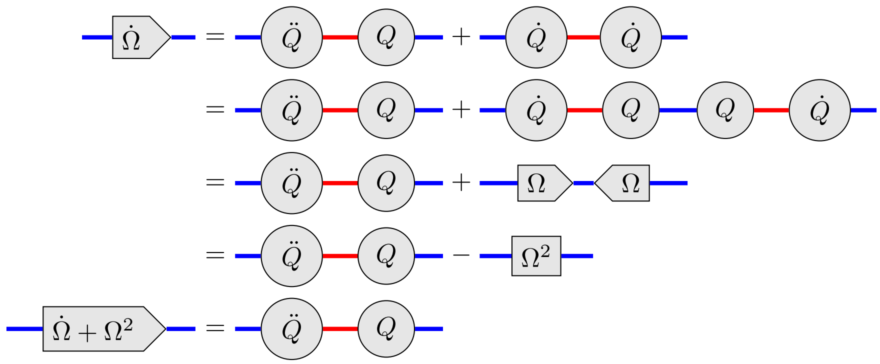


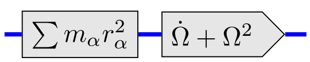

So  is symmetric.


That's the equation of motion.

Say we want to work in the body frame. Then we'll say the angular velocity is . This is an antisymmetric matrix, because  $= 0$.

We compute:



So  is symmetric.

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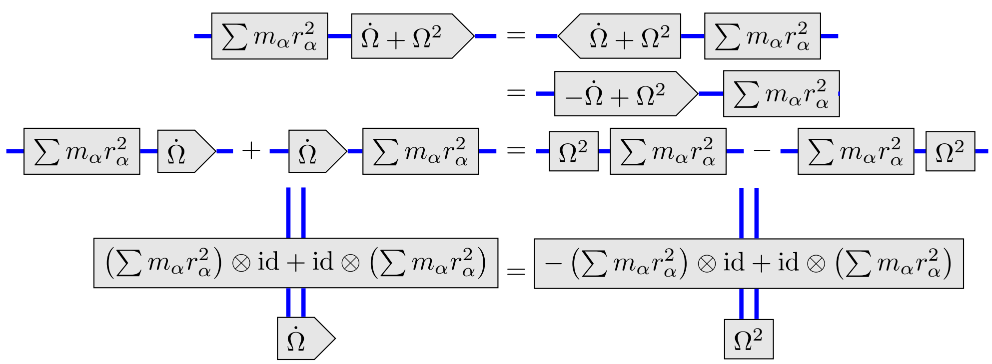
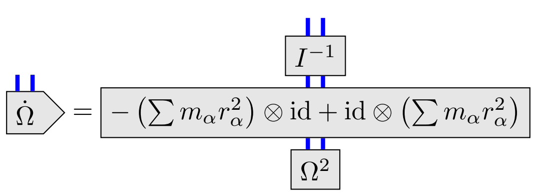
If we want, we can solve for $\dot{\Omega}$.



The big box on the left, treated as a map from antisymmetric matrices to antisymmetric matrices, is the inertia tensor $I$. It is invertible in a suitable sense, so we get an equation for $\dot{\Omega}$.



Let's work in a convenient basis.

$$\sum m_\alpha r_\alpha^2 = \begin{bmatrix} a_0 & & & & \\ & a_1 & & & \\ & & a_2 & & \\ & & & a_3 & \\ & & & & \ddots \end{bmatrix}$$

Then this formula simplifies. We compute:

$$\left(\left(\sum m_\alpha r_\alpha^2\right) \otimes \mathrm{id} + \mathrm{id} \otimes \left(\sum m_\alpha r_\alpha^2\right)\right)(e_i \otimes e_j - e_j \otimes e_i) = (a_i + a_j)(e_i \otimes e_j - e_j \otimes e_i)$$

$$\left(-\left(\sum m_\alpha r_\alpha^2\right) \otimes \mathrm{id} + \mathrm{id} \otimes \left(\sum m_\alpha r_\alpha^2\right)\right)(e_i \otimes e_j + e_j \otimes e_i) = (a_j - a_i)(e_i \otimes e_j - e_j \otimes e_i)$$

$$\dot{\Omega}_{ij} = \frac{a_j - a_i}{a_i + a_j}(\Omega^2)_{ij}$$

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It is clear, then, that if the body is fully asymmetric, then equilibrium occurs when $\Omega^2$ is diagonal. If $\Omega$'s eigenvalues are likewise maximally distinct, then we have, up to permuting rows and columns:

$$\Omega = \begin{bmatrix} & & & & \omega_0 \\ & & & \omega_1 & \\ & & \iddots & & \\ & \omega_{n-1} & & & \\ \omega_n & & & & \end{bmatrix}$$

(where $\omega_k = -\omega_{n-k}$).

That is, the planes of rotation are spanned by pairs of principal axes.

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When is the equilibrium stable? Linearize by setting $\Omega = \Omega_0 + \epsilon\Omega_1$, where $\Omega_0$ is an equilibrium, and take the order $\epsilon$ part.

$$\begin{aligned}(\dot{\Omega}_1)_{ij} &= \frac{a_j - a_i}{a_i + a_j}(\Omega_0\Omega_1 + \Omega_1\Omega_0)_{ij} \\ &= \frac{a_j - a_i}{a_i + a_j}(\omega_i(\Omega_1)_{(n-i)j} + \omega_{n-j}(\Omega_1)_{i(n-j)})\end{aligned}$$

For any $i$ and $j$, this linear transformation $\Omega_1 \mapsto \dot{\Omega}_1$ preserves the subspace of matrices which are nonzero only when both indices are either $i$, $j$, $n-i$, or $n-j$. We'll find all our eigenvectors in those subspaces. So, in essence, *we can reduce the problem to the case where there are only four dimensions.*

Let's work in four dimensions. Then $\Omega_1$ is a six-dimensional vector:

$$\begin{bmatrix} (\Omega_1)_{01} \\ (\Omega_1)_{02} \\ (\Omega_1)_{03} \\ (\Omega_1)_{12} \\ (\Omega_1)_{13} \\ (\Omega_1)_{23} \end{bmatrix}$$

The linearized equations of motion are thus given by a $6 \times 6$ matrix.

$$\dot{\Omega}_1 = \begin{bmatrix} 0 & \frac{a_1-a_0}{a_0+a_1}\omega_2 & 0 & 0 & -\frac{a_1-a_0}{a_0+a_1}\omega_0 & 0 \\ \frac{a_2-a_0}{a_0+a_2}\omega_1 & 0 & 0 & 0 & 0 & -\frac{a_2-a_0}{a_0+a_2}\omega_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{a_3-a_1}{a_1+a_3}\omega_0 & 0 & 0 & 0 & 0 & \frac{a_3-a_1}{a_1+a_3}\omega_1 \\ 0 & -\frac{a_3-a_2}{a_2+a_3}\omega_0 & 0 & 0 & \frac{a_3-a_2}{a_2+a_3}\omega_2 & 0 \end{bmatrix} \Omega_1$$

Simplifying using $\omega_2 = -\omega_1$:

$$\dot{\Omega}_1 = \begin{bmatrix} 0 & -\frac{a_1-a_0}{a_0+a_1}\omega_1 & 0 & 0 & -\frac{a_1-a_0}{a_0+a_1}\omega_0 & 0 \\ \frac{a_2-a_0}{a_0+a_2}\omega_1 & 0 & 0 & 0 & 0 & -\frac{a_2-a_0}{a_0+a_2}\omega_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{a_3-a_1}{a_1+a_3}\omega_0 & 0 & 0 & 0 & 0 & \frac{a_3-a_1}{a_1+a_3}\omega_1 \\ 0 & -\frac{a_3-a_2}{a_2+a_3}\omega_0 & 0 & 0 & -\frac{a_3-a_2}{a_2+a_3}\omega_1 & 0 \end{bmatrix} \Omega_1$$

To simplify, define $\alpha = \frac{a_1-a_0}{a_0+a_1}$, $\beta = \frac{a_2-a_0}{a_0+a_2}$, $\gamma = \frac{a_3-a_1}{a_1+a_3}$, and $\delta = \frac{a_3-a_2}{a_2+a_3}$.

$$\dot{\Omega}_1 = \begin{bmatrix} 0 & -\alpha\omega_1 & 0 & 0 & -\alpha\omega_0 & 0 \\ \beta\omega_1 & 0 & 0 & 0 & 0 & -\beta\omega_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma\omega_0 & 0 & 0 & 0 & 0 & \gamma\omega_1 \\ 0 & -\delta\omega_0 & 0 & 0 & -\delta\omega_1 & 0 \end{bmatrix} \Omega_1$$

We drop the trivial rows and columns, then compute the characteristic polynomial.

$$\begin{aligned} &\det \begin{bmatrix} -\lambda & (-\alpha)\omega_1 & (-\alpha)\omega_0 & \\ (\beta)\omega_1 & -\lambda & & (-\beta)\omega_0 \\ (-\gamma)\omega_0 & & -\lambda & (\gamma)\omega_1 \\ & (-\delta)\omega_0 & (-\delta)\omega_1 & -\lambda \end{bmatrix} \\ &= \lambda^4 - (-\delta)(\gamma)\lambda^2\omega_1^2 - (-\delta)(-\beta)\lambda^2\omega_0^2 \\ &- (\beta)(-\alpha)\lambda^2\omega_1^2 + (\beta)(-\alpha)(-\delta)(\gamma)\omega_1^4 - (\beta)(-\delta)(-\alpha)(\gamma)\omega_0^2\omega_1^2 \\ &- (-\gamma)(-\alpha)(-\delta)(-\beta)\omega_0^2\omega_1^2 - (-\gamma)(-\alpha)\omega_0^2\lambda^2 + (-\gamma)(-\delta)(-\alpha)(-\beta)\omega_0^4 \\ &= \lambda^4 + ((\alpha\beta + \gamma\delta)\omega_1^2 - (\alpha\gamma + \beta\delta)\omega_0^2)\lambda^2 + \alpha\beta\gamma\delta(\omega_1^2 - \omega_0^2)^2 \end{aligned}$$

I have not yet found a clean-enough way to determine when the roots are all imaginary.