Instantiating Bunched Type Theory for Monoidal Classifying Toposes

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1 Setup

Let $\mathbb{T}$ be a geometric theory.

Suppose that for each pair A, B of sorts of $\mathbb{T}$, we have a relation $\sim: A \times B \to \mathbf{Prop}$, geometrically defined in terms of $\mathbb{T}$'s signature. For instance, if $\mathbb{T}$ is the theory of rings, then $\sim: R \times R \to \mathbf{Prop}$ might be defined as $a \sim b \iff ab = ba$.

For each sort A, define the shorthand $A_{\sim x} = \{a : A \mid a \sim x\}$.

We want to require these subsets to themselves form a $\mathbb{T}$-model. If $\mathbb{T}$ is the theory of rings, for instance, we want $R_{\sim x}$ to be a ring, for any $x : R$.

To state this precisely, say that $\sim$ *cuts out a sub-*$\mathbb{T}$*-model* if the following two conditions hold.

- For any function symbol $f : A_1 \times \cdots \times A_n \to B$ and any $x : X$, the restriction $f_{\sim x} : (A_1)_{\sim x} \times \cdots \times (A_n)_{\sim x} \to B_{\sim x}$ is well-defined for all x.
- For any $x : X$, every axiom of $\mathbb{T}$ continues to hold when each sort A is replaced by $A_{\sim x}$, each function symbol f is replaced by $f_{\sim x}$, and each relation symbol is replaced by its restriction.

Let's look at the example where $\mathbb{T}$ is the theory of *local* rings, where we define $a \sim b \iff ab = ba$.

The first condition expands to the following five sequents.

$$\begin{aligned} x : R, a : R, ax = xa, b : R, bx = xb &\vdash (a + b)x = x(a + b) \\ x : R, a : R, ax = xa, b : R, bx = xb &\vdash (ab)x = x(ab) \\ x : R, a : R, ax = xa &\vdash (-a)x = x(-a) \\ x : R &\vdash 0x = x0 \\ x : R &\vdash 1x = x1 \end{aligned}$$

In other words, it says that $R_{\sim x} = \{a : R \mid a \sim x\}$ is a subring of R, for all $x : R$.

The second condition generates one requirement per axiom in the theory, but most of these are trivial. The interesting one comes from the following axiom of local rings.

$$a : R, b : R, (\exists c : R, (a+b)c = 1) \vdash (\exists c : R, ac = 1) \vee (\exists c : R, bc = 1)$$

The requirement generated is as follows.

$$\begin{aligned} &x : R, a : R, ax = xa, b : R, bx = xb, (\exists c : R, cx = xc \wedge (a+b)c = 1) \\ &\vdash (\exists c : R, cx = xc \wedge ac = 1) \vee (\exists c : R, cx = xc \wedge bc = 1) \end{aligned}$$

This is exactly the condition that the subring $R_{\sim x} \subseteq R$ is local.

2 Theorem

Theorem 2.1. *Let $\mathbb{T}$ be a geometric theory. For each pair of sorts A, B of $\mathbb{T}$, let $\sim : A \times B \to \mathbf{Prop}$ be a geometrically-defined relation which cuts out a sub-$\mathbb{T}$-model. Further suppose $\sim$ is symmetric; we have $a : A, b : B, a \sim b \vdash b \sim a$ for any sorts A and B. Then we can interpret the type theory $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*)$, as defined in [Sch06, Section 5.1], in $\mathbf{Set}[\mathbb{T}]$.*

3 Proof

We begin by bringing in some results from [Sch06].

Lemma 3.1. *The type theory $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*)$ can be modeled in any $(*, 1, \Sigma, \Pi, \Pi^*)$-type-category.*

Proof. [Sch06], section 6. □

Lemma 3.2. *Let $\mathbb{B}$ be a topos equipped with a strict affine symmetric monoidal closed structure $(*, -\!*)$, such that $- * A$ preserves pullbacks for all A. Further suppose we have a canonical choice of pullback for every span in $\mathbb{B}$. Then the family fibration over $\mathbb{B}$ is a $(*, 1, \Sigma, \Pi, \Pi^*)$-type-category.*

Proof. [Sch06], section 3.5. Since we are in a topos, the condition that the monomorphism $A * B \hookrightarrow A \times B$ is strong is trivial. □

Next, we specialize to sheaf toposes.

Lemma 3.3. *Let (C, J) be a (small) site, such that C is finitely complete. In fact, suppose we have a canonical choice for pullbacks in C. Let $*$ be an affine symmetric monoidal structure on C that preserves pullbacks and covers. Then $\mathrm{Sh}(C, J)$ satisfies the conditions of the above lemma.*

Proof. $\mathrm{Sh}(C, J)$ is a Grothendieck topos, hence a topos.

Theorem 4.3.2 from [Bie04] says we get a monoidal structure $\otimes^{\mathrm{Sh}}$ on $\mathrm{Sh}(C, J)$ by transporting Day convolution across the adjunction $(\mathbf{a} \dashv i) : \mathrm{Psh}(C, J) \rightleftarrows$

$\mathrm{Sh}(C, J)$. Its monoidal unit is the sheafification of the Yoneda embedding of $*$'s unit, which simplifies to the terminal object. Chasing definitions quickly shows $\otimes^{\mathrm{Sh}}$ is symmetric. And corollary 4.3.10 from [Bie04] says this monoidal structure is closed.

In other words, $\otimes^{\mathrm{Sh}}$ is an affine symmetric monoidal closed structure. We must show this is strict — that the canonical map $A \otimes^{\mathrm{Sh}} B \to A \times B$ is always a monomorphism.

But, letting $\otimes^{\mathrm{Psh}}$ represent Day convolution on $\mathrm{Psh}(C)$, this map is just $iA \otimes^{\mathrm{Psh}} iB \to iA \times iB$, restricted to sheaves. Expanding both $\otimes^{\mathrm{Psh}}$ and $\times$ as Day convolutions, this is a map, natural in $c \in C^{\mathrm{op}}$, of the following type.

$$\left(\int^{c_1, c_2 \in C} A(c_1) \times B(c_2) \times (c \xrightarrow{C} c_1 * c_2)\right) \to \left(\int^{c_1, c_2 \in C} A(c_1) \times B(c_2) \times (c \xrightarrow{C} c_1 \times c_2)\right)$$

This map is exactly what you'd expect it to be; it composes the $c \xrightarrow{C} c_1 * c_2$ component with the canonical map $c_1 * c_2 \xrightarrow{C} c_1 \times c_2$, and leaves everything else alone. So since that canonical map is mono, this is too.

Finally, we have a canonical construction for pullbacks in $\mathrm{Sh}(C, J)$, since we have one for C. □

And finally, we set the site to a syntactic site, to understand classifying toposes.

Lemma 3.4. *Let $\mathbb{T}$ be a geometric theory. For each pair of sorts A, B of $\mathbb{T}$, let $\sim: A \times B \to$ **Prop** be a symmetric, geometrically-defined relation, which cuts out a sub-$\mathbb{T}$-model. Then the syntactic site for $\mathbb{T}$ satisfies the conditions of the above lemma.*

Proof. Let us begin with a few shorthands.

We'll allow ourselves to write sequents involving compound types, such as $p : A \times B \vdash \pi_1 p \sim \pi_2 p$. These can be straightforwardly "compiled out" to ordinary sequents, such as $a : A, b : B \vdash a \sim b$.

Next, if we have a context $\Gamma = (a_1 : A_1, \ldots, a_n : A_n, \phi_1 \ldots \phi_p)$, we'll reuse the name Γ for the type $\{(a_1, \ldots, a_n) : A_1 \times \cdots \times A_n \mid \phi_1 \wedge \cdots \wedge \phi_p\}$. If we have a substitution $f : \Gamma \to \Delta$, there is then a natural way to define the term $\gamma : \Gamma \vdash f\gamma : \Delta$.

Finally, if $\gamma = (a_1, \ldots, a_n) : \Gamma$ and $\delta = (b_1, \ldots, b_m) : \Delta$, we write $\gamma \sim \delta$ as a shorthand for $\bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} a_i \sim b_j$.

With that out of the way, let's begin the proof.

First of all, the syntactic category is guaranteed to be finitely complete, with a canonical construction for pullbacks. Specifically, the empty context is terminal, and the pullback of the span $\Gamma \xrightarrow{f} \Xi \xleftarrow{g} \Delta$ is the context $(\gamma : \Gamma, \delta : \Delta, f\gamma = g\delta)$.

Second, we describe the monoidal structure. Given contexts Γ and Δ, define $\Gamma * \Delta = (\gamma : \Gamma, \delta : \Delta, \gamma \sim \delta)$.

Functoriality of $*$ reduces to the fact that $\sim$ respects substitutions, which further reduces to the fact that $\sim$ respects function symbols. Symmetry reduces to the symmetry of $\sim$. Choosing the empty context as monoidal unit, the unit laws hold on the nose, and associativity up to a rearrangement of contexts. All the coherences hold. And since $\Gamma * \Delta$ is just $\Gamma \times \Delta$ with extra propositional hypotheses, the map $\Gamma * \Delta \hookrightarrow \Gamma \times \Delta$ is a monomorphism. Putting that all together, $*$ is a strict affine symmetric monoidal product on the syntactic category.

Third, we show that starring preserves pullbacks.

Say we have a span $\Gamma \xrightarrow{f} \Xi \xleftarrow{g} \Delta$, and another context Θ. The pullback of the span is the context $(\gamma : \Gamma, \delta : \Delta, f\gamma = g\delta)$. Starring with Θ yields $(\gamma : \Gamma, \delta : \Delta, f\gamma = g\delta, \theta : \Theta, \gamma \sim \theta, \delta \sim \theta)$.

On the other hand, if we star with Θ first, we get the span $(\gamma : \Gamma, \theta : \Theta, \gamma \sim \theta) \xrightarrow{f * \mathrm{id}_\Theta} (\xi : \Xi, \theta : \Theta, \xi \sim \theta) \xleftarrow{g * \mathrm{id}_\Theta} (\delta : \Delta, \theta : \Theta, \delta \sim \theta)$. Taking the pullback yields $(\gamma : \Gamma, \theta_1 : \Theta, \gamma \sim \theta_1, \delta : \Delta, \theta_2 : \Theta, \delta \sim \theta_2, f\gamma = g\delta, \theta_1 = \theta_2)$, which is equivalent to what we got before.

And fourth, we show that starring preserves covers.

Suppose $\{f_i : \Gamma_i \to \Delta \mid i \in I\} \in J$. This means $\delta : \Delta \vdash \bigvee_{i \in I} \exists \gamma : \Gamma_i, f_i\gamma = \delta$ is provable.

By the final hypothesis, the proof is still valid if we, throughout the proof, replace each sort X with $X_{\sim\theta} = \{x : X \mid x \sim \theta\}$, and each function f with its restriction $f_{\sim\theta}$. This is only stated when $\theta : \Theta$ for some *sort* Θ, but the same is true for any *context* Θ, simply by iterating once for each variable in Θ.

So $\theta : \Theta, \delta : \Delta_{\sim\theta} \vdash \bigvee_{i \in I} \exists \gamma : (\Gamma_i)_{\sim\theta}, f_i\gamma = \delta$. But this is equivalent to the statement $\delta : (\Delta * \Theta) \vdash \bigvee_{i \in I} \exists \gamma : (\Gamma_i * \Theta), (f_i * \mathrm{id}_\Theta)\gamma = \delta$, so $\{f_i * \mathrm{id}_\Theta : \Gamma_i * \Theta \to \Delta * \Theta \mid i \in I\}$ is a cover, as required. $\square$

Putting this all together:

Proof of Theorem 2.1. Combine the above four lemmas. $\square$

References

[Bie04] Bodil Biering. "On the Logic of Bunched Implications - and its relation to separation logic". MA thesis. University of Copenhagen, June 2004. Chap. 4. URL: `https://ncatlab.org/nlab/files/Biering-BunchedLogic.pdf`.

[Sch06] Ulrich Schöpp. "Names and binding in type theory". PhD thesis. University of Edinburgh, 2006. URL: `https://ulrichschoepp.de/Docs/th.pdf`.