Instantiating Bunched Dependent Type Theory for Monoidal Classifying Toposes

finegeometer

June 30, 2025

Abstract

In their PhD thesis, Schöpp introduces a type theory designed to capture the monoidal structure in the Schanuel topos. I provide a strikingly simple characterization of the models of this type theory in arbitrary classifying toposes.

Todo list

1 Intro

Dependent type theory, in its standard form, has a model in every topos. By working within the type theory, you can therefore reason "internally" to the topos.

To capture an additional monoidal structure on the topos, [Sch06] extends the type theory to a system they call $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*)$. While that paper is specific to the Schanuel topos, the type theory described within is applicable to many others.

I characterize the models of $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*)$ in an arbitrary classifying topos. This characterization is extremely easy to use; describing a topos model typically becomes a single-paragraph task.

For simplicity, I describe only the case of single-sorted type theories. While both the theorem and proof extend naturally to multi-sort theories, the notation becomes more awkward in that case.

2 Main Theorem

Recall from [Sch06], section 3.5 and chapter 6, that topos models of the theories $\mathbf{BT}(1, \Sigma, \Pi, \Pi^*)$, $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*)$, or $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*, B^{*(M:A)})$ come from toposes equipped with *strict affine symmetric monoidal closed structures that preserve pullback*. That is, symmetric monoidal structures $(*, I)$ where:

- I is terminal.
- The canonical map $A * B \to A \times B$ is mono.
- The functor $A * -$ preserves pullbacks, and has a right adjoint.

I characterize these structures in classifying toposes.

Theorem 2.1. *Let $\mathbb{T}$ be a single-sorted geometric theory, with sort X. Monoidal structures on topos $\mathbf{Set}[\mathbb{T}]$ of the above form are given by symmetric relations $(\sim) : X \to X \to \mathbf{Prop}$, satisfying the following property.*

> *For any $x : X$, the $\mathbb{T}$-model structure on X restricts to a sub-$\mathbb{T}$-model on $\{a : X \mid a \sim x\}$.*

Under this interpretation, if U is the generic $\mathbb{T}$-model, we will have $U * U = \{(a, b) : U \times U \mid a \sim b\}$.

For example, to interpret the type theory into $\mathbf{Set}[\mathbf{Ring}]$, let $a \sim b$ mean $ab = ba$. We need only check that $\{a : R \mid ax = xa\}$ is a ring for any x, and this holds by standard ring theory. The monoidal product of the generic ring with itself will be the set of commuting pairs of ring elements.

2.1 Details

The above makes more sense intuitively than formally. So to be perfectly precise, when I say "$\{a : X \mid a \sim x\}$ is a sub-$\mathbb{T}$-model of X", I mean the following:

- For every function symbol $f : X^n \to X$, we require:

$$x : X, {}^{a_1 : X}_{a_1 \sim x}, \ldots, {}^{a_n : X}_{a_n \sim x} \vdash f(a_1 \ldots a_n) \sim x$$

- For every axiom of $\mathbb{T}$, run the following steps. We require the resulting sequent to hold.
  - Prepend a variable $x : X$ to the context.
  - For each other variable $a : X$ in the context, add an additional hypothesis $a \sim x$.
  - Replace every instance of $\exists a : X, P$ with $\exists a : X, a \sim x \wedge P$.

As an example of the latter condition, say $\mathbb{T}$ is the theory of inhabited sets, with axiom $\vdash \exists a : X, \top$. Then the required sequent is $x : X \vdash \exists a : X, a \sim x \wedge \top$.

3 Proof of Main Theorem

3.1 Notation

We use the following notation.

- $\mathbb{T}$ will be a single-sorted geometric theory, with sort X.
- $\mathcal{C}_\mathbb{T}$ will be the syntactic site of $\mathbb{T}$.
- $\mathrm{Psh}(\mathcal{C}_\mathbb{T})$ and $\mathrm{Sh}(\mathcal{C}_\mathbb{T})$ are the categories of presheaves and sheaves on the syntactic site. The latter is the classifying topos of $\mathbb{T}$, so we also call it $\mathbf{Set}[\mathbb{T}]$.
- We have the following diagram of functors:

$$\begin{tikzcd} \mathcal{C}_\mathbb{T} \arrow[r, "\text{よ}"] & \mathrm{Psh}(\mathcal{C}_\mathbb{T}) \arrow[r, shift left=1.5, "\mathbf{a}"] \arrow[r, phantom, "\perp"] & \mathrm{Sh}(\mathcal{C}_\mathbb{T}) = \mathbf{Set}[\mathbb{T}] \arrow[l, shift left=1.5, "i"] \end{tikzcd}$$

  よ is the Yoneda embedding, $\mathbf{a}$ is sheafification, and i is the inclusion of presheaves into sheaves. The composite functor $\mathbf{a} \circ \text{よ}$ interprets the syntax of the theory into the generic model.
- $U = \mathbf{a}\text{よ}(x : X)$ is the interpretation of the $\mathbb{T}$'s sort in the generic model.

3.2 Forwards Direction

For the duration of this section, let $(\sim) : X \to X \to \mathbf{Prop}$ be a relation satisfying the property described in the main theorem.

Lemma 3.1. *The transformation of sequents described in section 2.1, used there to transform the axioms of $\mathbb{T}$, preserves derivability. Any provable sequent remains provable after the transformation.*

Proof. First, prove that for any term t in any context, $t \sim x$ is provable in the transformed context. This is immediate by induction on the structure of t — for variables, the transformation added $t \sim x$ to the context; for compound terms, $f(t_1 \dots t_n) \sim x$ follows from the statements $t_i \sim x$.

Now the lemma follows from a second induction, this time on the structure of proofs. There are a number of cases; all of them easy, so I'll provide just two examples.

Consider the introduction and elimination rules for $\exists$.

$$\frac{\Gamma \vdash t : X \qquad \Gamma \vdash P[t/a]}{\Gamma \vdash \exists a : X, P} \qquad\qquad \frac{\Gamma \vdash \exists a : X, P \qquad \Gamma, a : X, P \vdash Q}{\Gamma \vdash Q}$$

After transformation, these become:

$$\frac{\Gamma' \vdash t : X \qquad \Gamma' \vdash P'[t/a]}{\Gamma' \vdash \exists a : X, a \sim x \wedge P'}$$

$$\frac{\Gamma' \vdash \exists a : X, a \sim x \wedge P' \qquad \Gamma', a : X, a \sim x, P' \vdash Q'}{\Gamma' \vdash Q'}$$

Both of which are immediate; the first because we've already proven $\Gamma' \vdash t \sim x$. $\square$

Construction 3.2. *We place a monoidal structure on $\mathbb{T}$'s syntactic category.*

Proof. Given two contexts A and B, define their product by extending $A \times B$ with the hypotheses $a \sim b$ for each variable a in the first context and each variable b in the second. For example:

$$(a, b : X, \phi(a, b)) * (c : X, \psi(c)) = (a, b, c : X, \phi(a, b), \psi(c), a \sim c, b \sim c)$$

We'll handle functoriality one argument at a time.

Given a morphism $f : A \to A'$, we must produce a morphism $f * 1_B : A * B \to A' * B$. That is, a provably functional relation on the combined context $(A * B) \times (A' * B)$.

But $f \times 1_B : A \times B \to A' \times B$ is similarly a relation on the context $(A \times B) \times (A' \times B)$. We choose its restriction.

Is this restriction provably functional? The only potential point of failure is the existence of an output value. So we must show:

$$(\vec{a}, \vec{b}) : (A * B) \vdash \exists (\vec{a'}, \vec{b'}) : (A' * B), (f * 1_B)((\vec{a}, \vec{b}), (\vec{a'}, \vec{b'}))$$

Or equivalently:

$$\vec{a} : A, \vec{b} : B, \vec{a} \sim \vec{b} \vdash \exists \vec{a'} : A', \exists \vec{b'} : B, \vec{a'} \sim \vec{b'} \wedge f(\vec{a}, \vec{a'}) \wedge \vec{b} = \vec{b'}$$

$$\vec{a} : A, \vec{b} : B, \vec{a} \sim \vec{b} \vdash \exists \vec{a'} : A', \vec{a'} \sim \vec{b} \wedge f(\vec{a}, \vec{a'})$$

But this is obtained from $\vec{a} : A \vdash \exists \vec{a'} : A', f(\vec{a}, \vec{a'})$ by applying Lemma 3.1 once for each variable in B. And that holds because f is provably functional.

We've defined $f * 1_B$. The definition of $1_A * g$ is analogous. All of the functoriality laws, including the interchange condition $(f * 1) \circ (1 * g) = (1 * g) \circ (f * 1)$, follow from relating $f * 1_B$ back to $f \times 1_B$, and using the functoriality of the cartesian product. $\square$

Lemma 3.3.

1. *The monoidal structure is symmetric, with the terminal object as the monoidal unit.*
2. *The canonical map $A * B \to A \times B$ is a monomorphism.*
3. *For any X, the endofunctor $- * X$ preserves pullbacks and covers.*

Proof.

1. Immediate by chasing definitions.
2. $A * B$ is simply $A \times B$ with extra hypotheses.

3. **Pullbacks:** Consider any cospan $A \xrightarrow{f} C \xleftarrow{g} B$. We have:

$$\begin{aligned}
A \times_C B &= \vec{a} : A, \vec{b} : B, \vec{c} : C, f(a,c), g(b,c) \\
(A \times_C B) * X &= \vec{a} : A, \vec{b} : B, \vec{c} : C, f(a,c), g(b,c), \\
&\quad \vec{x} : X, \vec{a} \sim \vec{x}, \vec{b} \sim \vec{x}, \vec{c} \sim \vec{x} \\
(A * X) \times_{C*X} (B * X) &= \vec{a} : (A * X), \vec{b} : (B * X), \vec{c} : (C * X), \\
&\quad (f * 1_X)(a,c), (g * 1_X)(b,c) \\
&= \vec{a} : A, \vec{x}_1 : X, \vec{a} \sim \vec{x}_1, \\
&\quad \vec{b} : A, \vec{x}_2 : X, \vec{b} \sim \vec{x}_2, \\
&\quad \vec{c} : A, \vec{x}_3 : X, \vec{c} \sim \vec{x}_3, \\
&\quad f(a,c), \vec{x}_1 = \vec{x}_3, \\
&\quad g(b,c), \vec{x}_2 = \vec{x}_3
\end{aligned}$$

Covers: Consider any cover $\{f_i : A_i \to B | i \in I\}$. Since this is a cover:

$$\vec{b} : B \vdash \bigvee_{i \in I} \exists \vec{a} : A_i, f_i(\vec{a}, \vec{b})$$

We are to show that $\{f_i * 1_X : A_i * X \to B * X | i \in I\}$ is a cover. That is:

$$\vec{b} : B * X \vdash \bigvee_{i \in I} \exists \vec{a} : A_i * X, (f_i * 1_X)(\vec{a}, \vec{b})$$

Equivalently:

$$\vec{b} : B, \vec{x} : X, \vec{b} \sim \vec{x} \vdash \bigvee_{i \in I} \exists \vec{a} : A_i, \exists \vec{y} : X, \vec{a} \sim \vec{y} \wedge f_i(\vec{a}, \vec{b}) \wedge \vec{y} = \vec{x}$$

$$\vec{b} : B, \vec{x} : X, \vec{b} \sim \vec{x} \vdash \bigvee_{i \in I} \exists \vec{a} : A_i, \vec{a} \sim \vec{x} \wedge f_i(\vec{a}, \vec{b})$$

But this follows from applying the transformation in Lemma 3.1 to the hypothesis, once for each variable in X.

□

Construction 3.4. *We equip* **Set**$[\mathbb{T}]$ *with a pullback-preserving strict affine symmetric monoidal closed structure.*

Proof. Let $\mathcal{C}_\mathbb{T}$ be $\mathbb{T}'s$ syntactic category. Recall that $\mathbf{Set}[\mathbb{T}] = \mathrm{Sh}(\mathcal{C}_\mathbb{T})$.

Theorem 4.3.2 from [Bie04] defines a monidal structure $\otimes^{\mathrm{Sh}}$ on $\mathrm{Sh}(\mathcal{C}_\mathbb{T})$, given by transporting Day convolution across the adjunction $(\mathbf{a} \dashv i) : \mathrm{Psh}(\mathcal{C}_\mathbb{T}) \rightleftarrows \mathrm{Sh}(\mathcal{C}_\mathbb{T})$. Corollary 4.3.10 from the same shows that it is closed.

The monoidal unit is the sheafification of the Yoneda embedding of $*$'s unit. Since $*$ is affine, this simplifies to the terminal object.

Symmetry is a simple definition chase. So it remains only to show that that the canonical map $A \otimes^{\mathrm{Sh}} B \to A \times B$ in $\mathrm{Sh}(C, J)$ is always a monomorphism.

Letting $\otimes^{\mathrm{Psh}}$ represent Day convolution on $\mathrm{Psh}(C)$, the map of interest is just the result of applying the sheafification functor, which preserves monomorphisms, to the canonical map $iA \otimes^{\mathrm{Psh}} iB \to iA \times iB$.

Expanding both $\otimes^{\mathrm{Psh}}$ and $\times$ as Day convolutions, this is a **Set**-map, natural in $c \in C^{\mathrm{op}}$, of the following type.

$$\left(\int^{c_1, c_2 \in C} A(c_1) \times B(c_2) \times (c \xrightarrow{C} c_1 * c_2)\right) \to \left(\int^{c_1, c_2 \in C} A(c_1) \times B(c_2) \times (c \xrightarrow{C} c_1 \times c_2)\right)$$

This map is exactly what you'd expect it to be; it composes the $c \xrightarrow{C} c_1 * c_2$ component with the canonical map $c_1 * c_2 \xrightarrow{C} c_1 \times c_2$, and leaves everything else alone. Since that canonical map is mono, this is too.

Missing condition: Pullback preservation. I'm not sure how to prove this, or if I need to change the theorem statement to make it work.

$\square$

Theorem 3.5. *Let U be the generic $\mathbb{T}$-model in* $\mathbf{Set}[\mathbb{T}]$*. Then $U \otimes^{\mathrm{Sh}} U = \{(a, b) : U \times U \mid a \sim b\}$, as subsets of $U \times U$.*

Proof.

$$\begin{aligned}
U \otimes^{\mathrm{Sh}} U &\simeq \mathbf{a}\text{よ}(x : X) \otimes^{\mathrm{Sh}} \mathbf{a}\text{よ}(x : X) \\
&\simeq \mathbf{a}(i\mathbf{a}\text{よ}(x : X) \otimes^{\mathrm{Psh}} i\mathbf{a}\text{よ}(x : X)) \\
&\simeq \mathbf{a}(\text{よ}(x : X) \otimes^{\mathrm{Psh}} \text{よ}(x : X)) \\
&\simeq \mathbf{a}\text{よ}((x : X) * (x : X)) \\
&\simeq \mathbf{a}\text{よ}(a : X, b : X, a \sim b) \\
&\simeq \{(a, b) : U \times U \mid a \sim b\}
\end{aligned}$$

Tracing the first projection $1_U \otimes^{\mathrm{Sh}} ! : U \otimes^{\mathrm{Sh}} U \to U \otimes^{\mathrm{Sh}} \mathbf{1}$ through the chain of isomorphisms, we see that it becomes the map $(a, b) \mapsto a$ from $\{(a, b) : U \times U \mid a \sim b\}$ to U. Similarly for the second projection. The canonical map $U \otimes^{\mathrm{Sh}} U \to U \times U$ is the product of these, so it becomes the inclusion $\{(a, b) : U \times U \mid a \sim b\} \hookrightarrow U \times U$, as required. $\square$

3.3 Backwards Direction

Consider any pullback-preserving strict affine symmetric monoidal structure on $\mathbf{Set}[\mathbb{T}]$.

Lemma 3.6. *For any $A \in \mathbf{Set}[\mathbb{T}]$, there is a geometric morphism $W_A \dashv \Pi_A^* : \mathbf{Set}[\mathbb{T}]_{/A} \leftrightarrows \mathbf{Set}[\mathbb{T}]$, whose composition with $\Sigma_A \dashv A^* : \mathbf{Set}[\mathbb{T}] \leftrightarrows \mathbf{Set}[\mathbb{T}]_{/A}$ yields the adjunction $(A * -) \dashv (A \mathrel{-\!\!*} -)$.*

Proof. Following the construction in section 2.5.1 of [Sch06], we define:

$$\begin{aligned}
W_A(B) &= \begin{smallmatrix} A*B \\ \downarrow \\ A \end{smallmatrix} \\
\Pi_A^* \left(\begin{smallmatrix} B \\ \downarrow f \\ A \end{smallmatrix} \right) &= \text{Pullback of } (A \mathrel{-\!\!*} B \xrightarrow{A \mathrel{-\!\!*} f} A \mathrel{-\!\!*} A \xleftarrow{id} 1)
\end{aligned}$$

$\square$

Corollary 3.7. *For any $A \in \mathbf{Set}[\mathbb{T}]$, the functor $A * -$ is determined by $A * U$.*

Proof. The above geometric morphism corresponds to a $\mathbb{T}$-model in $\mathbf{Set}[\mathbb{T}]_{/A}$, given by the restriction of the obvious model $\begin{smallmatrix} A\times U \\ \downarrow \\ A \end{smallmatrix}$ to $\begin{smallmatrix} A * U \\ \downarrow \\ A \end{smallmatrix}$. $\square$

Corollary 3.8. *The monoidal structure is determined by $U * U$.*

Proof. Reduce from $A * B$ to $A * U$, then to $U * U$. $\square$

Next, we relate the classifying topos back to the syntactic category.

Lemma 3.9. *If an object of $\mathbf{Set}[\mathbb{T}]$ is the interpretation of some context $A \in \mathcal{C}_{\mathbb{T}}$ in the generic model, then every subobject is the interpretation of some restriction $(\vec{a} : A, \phi(\vec{a}))$. The choice of ϕ is unique up to logical equivalence.*

Proof. Writing Ω for the subobject classifier of $\mathbf{Set}[\mathbb{T}]$, we have:

$$\begin{aligned}
&\mathrm{Sub}(\mathbf{a}\text{よ}A) \\
&= \mathbf{a}\text{よ}A \to \Omega \\
&= \hom_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}(\mathbf{a}\text{よ}A, \Omega) \\
&= \hom_{\mathrm{Psh}(\mathcal{C}_{\mathbb{T}})}(\text{よ}A, i\Omega) \\
&= \Omega A
\end{aligned}$$

This is the collection of covering-closed sieves on A in the syntactic category.

A sieve has the required form when it contains exactly those morphisms that factor through the inclusion of $(\vec{a} : A, \phi(a))$ into A. Uniqueness is immediate; if ϕ_1 and ϕ_2 are both solutions, then the inclusions of $(\vec{a} : A, \phi_1(a, b))$ and $(\vec{a} : A, \phi_2(a, b))$ factor through each other, producing a bi-implication.

As for existince, fix any sieve S. Every morphism it contains factors through the inclusion for the following choice of ϕ.

$$\phi(\vec{a}) = \bigvee_{(B \in \mathcal{C}_{\mathbb{T}}, f : \hom_{\mathcal{C}_{\mathbb{T}}}(B, A)) \in S} \exists \vec{b} : B, f(\vec{b}, \vec{a})$$

And conversely, this inclusion is covered by the sieve, so covering-closedness ensures it it in the sieve. $\square$

Corollary 3.10. *A geometric sequent $\vec{a} : A \vdash \phi(\vec{a})$ is provable if it holds in the generic model.*

Proof. Uniqueness, from the above lemma, shows that ϕ and $\top$ are logically equivalent. $\square$

Corollary 3.11. *A morphism in $\mathbf{Set}[\mathbb{T}]$ is the interpretation of one in the generic model if its domain and codomain are.*

Proof. A morphism in $\mathbf{Set}[\mathbb{T}]$ induces a provably functional relation. Both the relation and the proof of functionality are interpretations of the same in the syntactic category. This yields the data required to produce a morphism in $\mathcal{C}_{\mathbb{T}}$. $\square$

Theorem 3.12. *$U * U$ is the interpretation of a unique geometrically-definable relation $(\sim) : X \times X \to \mathbf{Prop}$. This relation satisfies the assumptions of the main theorem.*

Proof. $U * U$ is a subobject of $U \times U$, which interprets $X \times X$. So $U \times U$ is the interpretation of some context $(a : X, b : X, a \sim b)$.

It remains to show that $(\sim)$ is symmetric, and that $\{a : X \mid a \sim x\}$ is a sub-$\mathbb{T}$-model of X for any $x : X$.

By the above lemma and its corollaries, it suffices to show that $(a, b) \in U * U \Leftrightarrow (b, a) \in (U * U)$, and that $\{a : U \mid a \sim x\}$ is a sub-$\mathbb{T}$-model of U for any $x : U$.

The former is immediate. The latter holds because the geometric morphism $W_U \dashv \Pi_U^*$ induces a $\mathbb{T}$-model in $\mathbf{Set}[\mathbb{T}]_{/U}$. $\square$

4 Future Work

The purpose of this paper is to *enable* future work. Namely, the exploration of monoidal toposes via their internal languages.

But beyond that, there are a few other directions to pursue.

First, [Ble] provides a powerful reasoning principle for the generic model of an arbitrary theory. It would be useful to either generalize or leverage this to prove facts involving the monoidal structure. For instance, calculations in the interpretation sugggest that the generic local ring should support the isomorphism $(\{x : R | x^2 = 0\} \twoheadrightarrow R) \simeq R^2$, analogous to the Kock-Lawvere axiom $(\{x : R | x^2 = 0\} \to R) \simeq R^2$ supported by the generic *commutative* local ring. The commutative case follows immediately from [Ble, Theorem 4.10]; is there a similarly slick way to derive the noncommutative version?

Second, it would be worthwhile to extend this result to other type theories. For instance, [Sch06] defines the type theory $\mathbf{BT}(*, 1, \Sigma, \Pi, \Pi^*, B^{*(M:A)}, \Sigma^*)$, which provides monoidal sigma types as well as pi types. Furthermore, it appears that [Sch06]'s type theories are not the end of the story, based both on their comments in section 13 and the existence of many competing approaches in the literature. As new and better type theories are designed, it will be valuable to have a quick way to know when they apply.

References

[Bie04] Bodil Biering. "On the Logic of Bunched Implications - and its relation to separation logic". MA thesis. University of Copenhagen, June 2004. Chap. 4. URL: `https://ncatlab.org/nlab/files/Biering-BunchedLogic.pdf`.

[Ble] Ingo Blechschmidt. "A General Nullstellensatz for Generalized Spaces". URL: https://rawgit.com/iblech/internal-methods/master/paper-qcoh.pdf.

[Sch06] Ulrich Schöpp. "Names and binding in type theory". PhD thesis. University of Edinburgh, 2006. URL: https://ulrichschoepp.de/Docs/th.pdf.