

Prob/Stats Cheatsheet

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ABSTRACT: Everything I know about prob/stats/maybe information theory too..

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1 Conventions

Math Notation

2 Distributions

2.1 Gaussians

2.1.1 Basics

1. To start with, *memorize* that

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \pi^{1/2} \quad (2.1)$$

2. Next, anything multiplying the x^2 in the integrand is present in inverse under the square root.

$$\int_{-\infty}^{\infty} dx e^{-\text{stuff } x^2} = \left(\frac{\pi}{\text{stuff}} \right)^{1/2} \quad (2.2)$$

so, for example:

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a} \right)^{1/2} \quad (2.3)$$

3. The traditional Gaussian pdf has $a = 1/\sigma^2$, and is easily seen to be

$$\mathcal{N}(x|0, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}x^2} \quad (2.4)$$

2.1.2 Differentiation moment trick

By differentiating Eq. 2.3 wrt a , we obtain an expression for integrals of the form $\int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}ax^2}$, with $n \in \mathbb{Z}^+$.

e.g. for $n = 1$:

$$-2 \frac{d}{da} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \int_{-\infty}^{\infty} dx x^2 e^{-\frac{1}{2}ax^2} = -2 \frac{d}{da} \left(\frac{2\pi}{a} \right)^{1/2} = \left(\frac{2\pi}{a} \right)^{1/2} \frac{1}{a} \quad (2.5)$$

For $n = 2$:

$$\left(-2 \frac{d}{da} \right)^2 \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \int_{-\infty}^{\infty} dx x^4 e^{-\frac{1}{2}ax^2} = \left(-2 \frac{d}{da} \right)^2 \left(\frac{2\pi}{a} \right)^{1/2} = \left(\frac{2\pi}{a} \right)^{1/2} \frac{1}{a} \frac{3}{a} \quad (2.6)$$

Generally:

$$\int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a} \right)^{1/2} \frac{1}{a^n} (2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1 \quad (2.7)$$

We thus obtain an expression for the expectation value of x^{2n} under the Gaussian distribution:

$$\langle x^{2n} \rangle = \frac{\int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}ax^2}}{\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n} (2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1 \quad (2.8)$$

2.1.3 Gaussian with Linear Term

To evaluate integrals of the form

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + Jx}, \quad (2.9)$$

first complete the square in the exponent

$$-\frac{a}{2}x^2 + Jx = -\frac{a}{2} \left(x^2 - \frac{2Jx}{a} \right) = -\frac{a}{2} \left(x - \frac{J}{a} \right)^2 + \frac{J^2}{2a} \quad (2.10)$$

which gives

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+Jx} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}a(x-J/a)} e^{J^2/2a} = \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a} \quad (2.11)$$

where the integral is done by shifting $x \rightarrow x + J/a$ (or noting that the infinite integral of a Gaussian is independent of its mean.)

By differentiating this expression wrt J repeatedly, and finally setting $J = 0$, we obtain another way of deriving the moments of the Gaussian, Eq. (2.8). This motivates the introduction of the **moment generating function**: given a pdf $p(x)$, the moment generating function is

$$\psi_x(J) = \mathbb{E}_x [e^{Jx}] = \int_{-\infty}^{\infty} dx e^{Jx} p(x) \quad (2.12)$$

which satisfies

$$\langle x^n \rangle = \left. \frac{d^n \psi_x(J)}{dJ^n} \right|_{J=0} \quad (2.13)$$

TODO: Finish. Figure out clear way to include normalization factor of pdf in exposition

2.1.4 Multivariate Gaussians

Promoting a to a real $N \times N$ symmetric matrix \mathbf{A} , and x and J to a N -dim vectors \vec{x} and \vec{J} with components x_i and J_i , we have the multivariate Gaussian integral

$$\prod_{i=1}^N \left(\int_{-\infty}^{\infty} dx_i \right) e^{-\frac{1}{2}\vec{x}^T \mathbf{A} \vec{x} + \vec{J}^T \vec{x}} = \left(\frac{(2\pi)^{N/2}}{|\mathbf{A}|^{1/2}} \right) e^{\frac{1}{2}\vec{J}^T \mathbf{A}^{-1} \vec{J}} \quad (2.14)$$

TODO: finish —

A detailed walkthrough of multivariate Gaussian integrals is in [viXra:1404.0026](#).

2.2 Bernoulli

For $x \in \{0, 1\}$, Bernoulli dist parameterized by μ , with

$$p(x; \mu) = \mu^x (1 - \mu)^{1-x} \quad (2.15)$$

2.3 Exponential Family

These are pdfs of the form

$$p(x; \theta) = h(x) \exp [\theta^T T(x) - A(\theta)] \quad (2.16)$$

where

- θ is the *natural* or *canonical parameter*
- $T(x)$ is the *sufficient statistic*
- $A(\theta)$ is the *log partition function*
- $h(x)$ determines the distribution at $\theta = 0$

TODO: more detail about the above terms. e.g. the θ are the sources or external fields.

3 Prob and stats

3.1 The Rules of Probability

- **Product Rule:** $p(x, y) = p(x|y)p(y) = p(y|x)p(x)$
- **Sum Rule:** $p(x) = \sum_y p(x, y) = \sum_y p(x|y)p(y)$

3.2 Bayes' Rule

Using $p(y|x)p(x) = p(x, y) = p(x|y)p(y)$, we have

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} = \frac{p(x|y)p(y)}{\sum_y p(x|y)p(y)} \quad (3.1)$$

3.3 Expectation and Variance

- **Expectations of sum of variables add:**

If X_1, \dots, X_n are random variables, and a_1, \dots, a_n are constants, then

$$\mathbb{E} \left(\sum_{i=1}^n a_i X_i \right) = \sum_{i=1}^n a_i \mathbb{E}(X_i) \quad (3.2)$$

- **Variances of sum of independent variables add:**

If X_1, \dots, X_n are *independent* random variables, and a_1, \dots, a_n are constants, then

$$\text{Var} \left(\sum_{i=1}^n a_i X_i \right) = \sum_{i=1}^n a_i^2 \text{Var}(X_i) \quad (3.3)$$

- **Variances of sum of (dependent) variables:**

If X and Y are random variables, then

$$\begin{aligned} \text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ \text{Var}(X - Y) &= \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y) \end{aligned} \quad (3.4)$$

3.4 Central Limit Theorem

Let $S_N = \sum_{i=1}^N X_i$, where each X_i is **iid** with mean μ and variance σ^2 . Then, as $N \rightarrow \infty$, the pdf of S_N approaches a normal distribution:

$$p(S_N = s) = \frac{1}{(2\pi N\sigma^2)^{1/2}} \exp \left[-\frac{(s - N\mu)^2}{2N\sigma^2} \right] \quad (3.5)$$

NB the factors of N in the pdf, which make the pdf mean/variance equal to N times the original mean/variance (*i.e.* means and variances of independent variables add; see section 3.3.)

3.5 (Weak) Law of Large Numbers

Let X_1, \dots, X_n be iid, and $\mu = \mathbb{E}(X_1)$ ¹. Defining the *sample mean* as $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, the WLLN states that \bar{X}_n converges in probability to μ .

4 Information Theory

KL divergence:

$$\begin{aligned} KL[p(x)||q(x)] &= \sum_{x_i} p(x_i) \log \left(\frac{p(x_i)}{q(x_i)} \right) = - \sum_{x_i} p(x_i) \log \left(\frac{q(x_i)}{p(x_i)} \right) \\ &= - \sum_{x_i} p(x_i) \log q(x_i) + \sum_{x_i} p(x_i) \log p(x_i) \\ &= H(p, q) - H(p) \end{aligned} \tag{4.1}$$

where $H(p, q)$ is the cross entropy, and $H(p)$ is the entropy.

5 Bayesian

6 Optimal Stopping Theory

¹ $\mu = \mathbb{E}(X_1) = \mathbb{E}(X_i)$ for any $1 \leq i \leq n$