Prob/Stats Cheatsheet

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ABSTRACT: Everything I know about prob/stats/maybe information theory too..

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Math Notation

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2.1 Gaussians

2.1.1 Basics

1. To start with, *memorize* that

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-x^2} = \pi^{1/2}} \tag{2.1}$$

2. Next, anything multiplying the x^2 in the integrand is present in inverse under the square root.

$$\int_{-\infty}^{\infty} dx\, e^{-\text{stuff}\, x^2} = \left(\frac{\pi}{\text{stuff}}\right)^{1/2} \tag{2.2}$$

so, for example:

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{1/2} \tag{2.3}$$

3. The traditional Gaussian pdf has $a = 1/\sigma^2$, and is easily seen to be

$$\mathcal{N}(x|0,\sigma^2) = \frac{1}{\left(2\pi\sigma^2\right)^{1/2}}\, e^{-\frac{1}{2\sigma^2}x^2} \tag{2.4}$$

2.1.2 Differentiation moment trick

By differentiating Eq. 2.3 wrt a, we obtain an expression for integrals of the form $\int_{-\infty}^{\infty} dx\, x^{2n} e^{-\frac{1}{2}ax^2}$, with $n \in \mathbb{Z}^+$.

e.g. for $n = 1$:

$$-2\frac{d}{da}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{1}{2}ax^2} = -2\frac{d}{da}\left(\frac{2\pi}{a}\right)^{1/2} = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a} \tag{2.5}$$

For $n = 2$:

$$\left(-2\frac{d}{da}\right)^2\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \int_{-\infty}^{\infty} dx\, x^4 e^{-\frac{1}{2}ax^2} = \left(-2\frac{d}{da}\right)^2\left(\frac{2\pi}{a}\right)^{1/2} = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a}\frac{3}{a} \tag{2.6}$$

Generally:

$$\int_{-\infty}^{\infty} dx\, x^{2n} e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a^n}(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1 \tag{2.7}$$

We thus obtain an expression for the expectation value of x^{2n} under the Gaussian distribution:

$$\langle x^{2n}\rangle = \frac{\int_{-\infty}^{\infty} dx\, x^{2n} e^{-\frac{1}{2}ax^2}}{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1 \tag{2.8}$$

2.1.3 Gaussian with Linear Term

To evaluate integrals of the form

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 + Jx}, \tag{2.9}$$

first complete the square in the exponent

$$-\frac{a}{2}x^2 + Jx = -\frac{a}{2}\left(x^2 - \frac{2Jx}{a}\right) = -\frac{a}{2}\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a} \tag{2.10}$$

which gives

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+Jx} = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}a\,(x-J/a)} e^{J^2/2a} = \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a} \tag{2.11}$$

where the integral is done by shifting $x \to x + Ja$ (or noting that the infinite integral of a Gaussian is independent of its mean.)

By differentiating this expression wrt J repeatedly, and finally setting $J = 0$, we obtain another way of deriving the moments of the Gaussian, Eq. (2.8). This motivates the introduction of the ***moment generating function***: given a pdf $p(x)$, the moment generating function is

$$\psi_x(J) = \mathbb{E}_x\left[e^{Jx}\right] = \int_{-\infty}^{\infty} dx\, e^{Jx} p(x) \tag{2.12}$$

which satisfies

$$\langle x^n \rangle = \left.\frac{d^n\, \psi_x(J)}{dJ^n}\right|_{J=0} \tag{2.13}$$

TODO: Finish. Figure out clear way to include normalization factor of pdf in exposition

2.1.4 Multivariate Gaussians

Promoting a to a real $N \times N$ symmetric matrix $\mathbf{A}$, and x and J to a N-dim vectors $\vec{x}$ and $\vec{J}$ with components x_i and J_i, we have the multivariate Gaussian integral

$$\prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} dx_i\right) e^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}+\vec{J}^T\vec{x}} = \left(\frac{(2\pi)^{N/2}}{|\mathbf{A}|^{1/2}}\right) e^{\frac{1}{2}\vec{J}^T\mathbf{A}^{-1}\vec{J}} \tag{2.14}$$

TODO: finish —

A detailed workthrough of multivariate Gaussian integrals is in `viXra:1404.0026`.

2.2 Bernoulli

For $x \in \{0, 1\}$, Bernoulli dist parameterized by μ, with

$$p(x; \mu) = \mu^x (1-\mu)^{1-x} \tag{2.15}$$

2.3 Exponential Family

These are pdfs of the form

$$p(x; \eta) = b(x) \exp\left[\eta^T T(x) - a(\eta)\right] \tag{2.16}$$

where

- η is the ***natural*** or ***canonical parameter***
- $T(x)$ is the ***sufficient statistic***
- $a(\eta)$ is the ***log partition function***
- $b(x)$ determines the distribution at $\eta = 0$

TODO: more detail about the above terms. *e.g.* the η are the sources or external fields.

3 Prob and stats

3.1 The Rules of Probability

- **Product Rule**: $p(x, y) = p(x|y)p(y) = p(y|x)p(x)$
- **Sum Rule**: $p(x) = \sum_y p(x, y) = \sum_y p(x|y)p(y)$

3.2 Bayes' Rule

Using $p(y|x)p(x) = p(x, y) = p(x|y)p(y)$, we have

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} = \frac{p(x|y)p(y)}{\sum_y p(x|y)p(y)} \tag{3.1}$$

3.3 Expectation and Variance

- **Expectations of sum of variables add**:
  If $X_1, \dots, X_n$ are random variables, and $a_1, \dots, a_n$ are constants, then

$$\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \, \mathbb{E}(X_i) \tag{3.2}$$

- **Variances of sum of independent variables add**:
  If $X_1, \dots, X_n$ are *independent* random variables, and $a_1, \dots, a_n$ are constants, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \, \mathrm{Var}(X_i) \tag{3.3}$$

- **Variances of sum of (dependent) variables**:
  If X and Y are random variables, then

$$\begin{aligned} \mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y) \\ \mathrm{Var}(X-Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X, Y) \end{aligned} \tag{3.4}$$

3.4 Central Limit Theorem

Let $S_N = \sum_{i=1}^{N} X_i$, where each X_i is **iid** with mean μ and variance σ^2. Then, as $N \to \infty$, the pdf of S_N approaches a normal distribution:

$$p(S_N = s) = \frac{1}{\left(2\pi N \sigma^2\right)^{1/2}} \exp\left[-\frac{(s - N\mu)^2}{2N\sigma^2}\right] \tag{3.5}$$

NB the factors of N in the pdf, which make the pdf mean/variance equal to N times the original mean/variance (*i.e.* means and variances of independent variables add; see section 3.3.)

3.5 (Weak) Law of Large Numbers

Let $X_1, \dots, X_n$ be **iid**, and $\mu = \mathbb{E}(X_1)$[1]. Defining the ***sample mean*** as $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$, the WLLN states that $\overline{X}_n$ converges in probability to μ.

4 Information Theory

KL divergence:

$$\begin{aligned} KL\big[p(x)||q(x)\big] &= \sum_{x_i} p(x_i) \log \left(\frac{p(x_i)}{q(x_i)} \right) = - \sum_{x_i} p(x_i) \log \left(\frac{q(x_i)}{p(x_i)} \right) \\ &= - \sum_{x_i} p(x_i) \log q(x_i) + \sum_{x_i} p(x_i) \log p(x_i) \\ &= H(p, q) - H(p) \end{aligned} \tag{4.1}$$

where $H(p, q)$ is the cross entropy, and $H(p)$ is the entropy.

5 Bayesian

6 Optimal Stopping Theory

[1] $\mu = \mathbb{E}(X_1) = \mathbb{E}(X_i)$ for any $1 \le i \le n$