Quattro pallette e cinque freccette

. . . son solo pallette e freccette.

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Index

Basic definitions

Formally, categories talk about *objects* and *functions* (or *arrows*) between those objects (more or less).

Identities

For each object A there exists a function, that we denote id_A, which is both the left and the right identity for A. In formulas, $\forall A, \exists \mathrm{id}_A : A \to A, \forall f : B \to A, \forall g : A \to B$

$$f \circ \mathrm{id}_A = f \tag{1}$$

$$\mathrm{id}_A(f(b)) = f(b), \forall b \in B$$

$$\mathrm{id}_A \circ g = g \tag{2}$$

$$g(\mathrm{id}_A(a)) = g(a), \forall a \in A$$

Associative

If it is possible to compose functions

$$\forall f, g, h : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

then composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f \tag{3}$$

Trouble every day

- Trouble is, just with objects and arrows things are not interesting.
- Trouble is, to find interesting things we have to move **two levels up**.

Up and up again !

First move up If we have many categories we may be interested in **classifying them**, so we introduce the idea of **functions among categories** (*i.e.* the concept of the **functor**).

Second move up The main reason of giving birth to category theory was the notion of **natural transformation** (*i.e.* how we go from a classification to another, or understanding when two distinct classifications are indeed the same).

Example

The diagram assumes that a, b, c, d are the *literal characters* ‘a’, ‘b’, ‘c’ and ‘d’; the functions α to γ are the functions that change that character in the other (the destination) character.

$$
\begin{tikzcd}
a \arrow[r, "\beta"] \arrow[d, "\gamma"] & b \arrow[d, "\delta_1"] \\
c \arrow[r, "\delta_2"] & d \arrow[ul, "\alpha"]
\end{tikzcd}
\tag{4}
$$

1. Note that we have to discriminate between δ_1 and δ_2.
2. Define the identities.

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Functors

- A functor is a function between two categories (if the category is the same, we call the functor an *endofunctor*).
- The *trick* to define a functor, respect to a general trasformation among categories, is that it must preserve the **structure** of the category.
- This means preserving the **arrow composition** and the **existence of identities**.

Functors

F is a functor if the following properties hold:

Commutativity The diagram 5 commutes

$$\forall A, B, f : A \to B, \exists C, D, F(f) : C \to D$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow F & & \downarrow F \\ C & \xrightarrow{F(f)} & D \end{array} \tag{5}$$

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Functors

Identities F preserves identity morphisms

$$\forall A, F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \tag{6}$$

Associative F preserves composition

$$F(f \circ g) = F(f) \circ F(g) \tag{7}$$

whenever $f \circ g$ is defined.

Example

How many endofunctors are there in the category described by the graph of equation 4 ?

Natural transformations

Once we have functors, we may ask when two given functors are indeed the same. *Sameness* is – again – the preservation of a common structure. Since our structure-preserving machinery are the functions and their properties, we build a new function between functors that preserves mappings.

Definition of natural transformation

Let $F, G : A \to B$ be functors. A natural transformation η from F to G (denoted by $\eta : F \to G$) is a function that assigns to each object A a morphism $\eta_A : F(A) \to G(A)$ in such a way that the square of equation 8 commutes:

$$\forall f : A \to A' \quad \begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow{\eta_{A'}} & G(A') \end{array} \tag{8}$$

Note that η is indeed a *family* of functions, one for each A, so we may also write that $\eta = \cup_A \eta_A$.

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Last question

Given the endofunctors of the previous example (equation 4), are there natural transformations among them ?
Which ones ? How many are they ? What do we *learn* from the natural transformations of those functors, that gives insight on the original category ?