# Quattro pallette e cinque freccette ...son solo pallette e freccette.

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## **Basic definitions**

Formally, categories talk about *objects* and *functions* (or *arrows*) between those objects (more or less).

## Identities

For each object A there exists a function, that we denote  $\mathrm{id}_A$ , which is both the left and the right identity for A. In formulas,  $\forall A, \exists \mathrm{id}_A : A \to A, \forall f : B \to A, \forall g : A \to B$ 

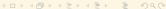
$$f \circ \mathsf{id}_{\mathcal{A}} = f \tag{1}$$

$$id_A(f(b)) = f(b), \forall b \in B$$

$$id_A \circ g = g$$
 (2)

$$g(id_A(a)) = g(a), \forall a \in A$$





## **Associative**

If it is possible to compose functions

$$\forall f, g, h: A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

then composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f \tag{3}$$

# Trouble every day

- Trouble is, just with objects and arrows things are not interesting.
- ► Trouble is, to find interesting things we have to move two levels up.

# Up and up again!

First move up If we have many categories we may be interested in **classifying them**, so we introduce the idea of **functions among categories** (*i.e.* the concept of the **functor**).

Second move up The main reason of giving birth to category theory was the notion of **natural transformation** (*i.e.* how we go from a classification to another, or understanding when two distinct classifications are indeed the same).

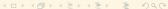
## Example

The diagram assumes that a, b, c, d are the *literal characters* 'a', 'b', 'c' and 'd'; the functions  $\alpha$  to  $\gamma$  are the functions that change that character in the other (the destination) character.

$$\begin{array}{ccc}
a & \xrightarrow{\beta} & b \\
\downarrow^{\gamma} & \downarrow^{\delta_1} & \downarrow^{\delta_1} \\
c & \xrightarrow{\delta_2} & d
\end{array} \tag{4}$$

- 1. Note that we have to discriminate between  $\delta_1$  and  $\delta_2$ .
- 2. Define the identities.





#### **Functors**

- ► A functor is a function between two categories (if the category is the same, we call the functor an *endofunctor*).
- The trick to define a functor, respect to a general trasformation among categories, is that it must preserve the structure of the category.
- ► This means preserving the **arrow composition** and the **existence of identities**.

#### **Functors**

*F* is a functor if the following properties hold:

Commutativity The diagram 5 commutes

$$\forall A, B, f : A \to B, \exists C, D, F(f) : C \to D$$

$$A \xrightarrow{f} B$$

$$\downarrow_{F} \qquad \downarrow_{F}$$

$$C \xrightarrow{F(f)} D$$
(5)

#### **Functors**

Identities *F* preserves identity morphisms

$$\forall A, F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \tag{6}$$

Associative *F* preserves composition

$$F(f \circ g) = F(f) \circ F(g) \tag{7}$$

whenever  $f \circ g$  is defined.



# Example

How many endofunctors are there in the category described by the graph of equation 4 ?

## Natural transformations

Once we have functors, we may ask when two given functors are indeed the same. *Sameness* is – again – the preservation of a common structure. Since our structure-preserving machinery are the functions and their properties, we build a new function between functors that preserves mappings.

## Definition of natural transformation

Let  $F, G: A \to B$  be functors. A natural transformation  $\eta$  from F to G (denoted by  $\eta: F \to G$ ) is a function that assigns to each object A a morphism  $\eta_A: F(A) \to G(A)$  in such a way that the square of equation 8 commutes:

$$F(A) \xrightarrow{\eta_A} G(A)$$

$$\forall f : A \to A' \qquad \downarrow_{F(f)} \qquad \downarrow_{G(f)} \qquad (8)$$

$$F(A') \xrightarrow{\eta_{A'}} G(A')$$

Note that  $\eta$  is indeed a *family* of functions, one for each A, so we may also write that  $\eta = \bigcup_A \eta_A$ .



# Last question

Given the endofunctors of the previous example (equation 4), are there natural transformations among them? Which ones? How many are they? What do we *learn* from the natural transformations of those functors, that gives insight on the original category?