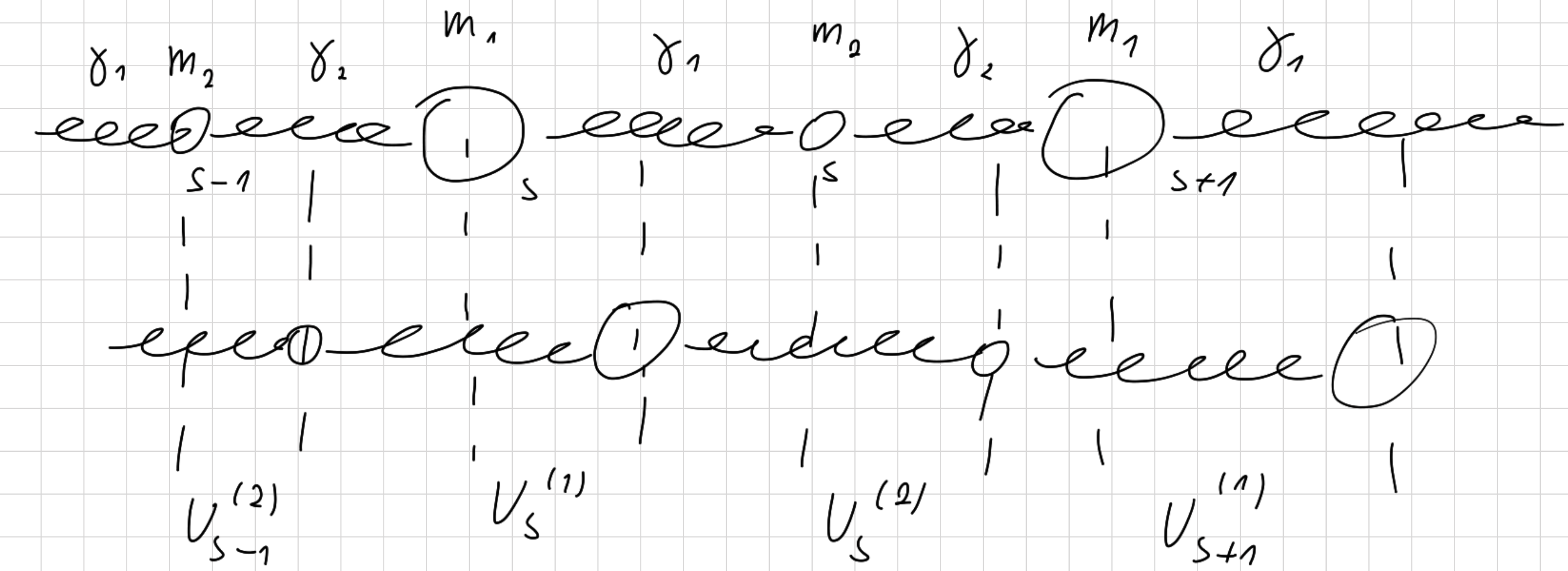

m_i – masy atomów

γ_i – stałe sprężystości

$U_s^{(i)}$ – wychylenia atomów

Zakładamy falę płaską.

$$U_s^{(i)} = U_0 \exp[i(k\underbrace{as}_{x} - \omega t)]$$

$$\frac{dU_s^{(i)}}{dt} = -i\omega U_0 \exp[i(kx - \omega t)]$$

$$\frac{d^2U_s^{(i)}}{dt^2} = -\omega^2 U_0 \exp[i(kx - \omega t)] = -\omega^2 U_s^{(i)}$$

$$U_{s-1}^{(i)} = U_0 \exp[i(kas - ka - \omega t)] = U_s^{(i)} \exp(-ika)$$

$$U_{s+1}^{(i)} = U_s^{(i)} \exp(ika)$$

z prawa Hooke'a i 2 zas. dynamiki

$$F = -\gamma \Delta x \quad , F = ma$$

stąd:

$$m_1 \frac{d^2U_s^{(1)}}{dt^2} = -\gamma_2 (U_s^{(1)} - U_{s-1}^{(2)}) - (-\gamma_1 (U_s^{(2)} - U_s^{(1)}))$$

$$m_2 \frac{d^2U_s^{(2)}}{dt^2} = -\gamma_1 (U_s^{(2)} - U_s^{(1)}) - (-\gamma_2 (U_{s+1}^{(1)} - U_s^{(2)}))$$

Podstawiając

$$\begin{cases} m_1\left(-\omega^2 U_s^{(1)}\right) = \gamma_1\left(U_s^2 - U_s^1\right) - \gamma_2\left(U_s^1 - U_s^2 \exp(-ika)\right) \\ m_2\left(-\omega^2 U_s^2\right) = \gamma_2\left(U_s^1 \exp(ika) - U_s^2\right) - \gamma_1\left(U_s^2 - U_s^1\right) \end{cases}$$

$$\begin{cases} -\omega^2 m_1 U_s^1 = \gamma_1 U_s^2 - \gamma_1 U_s^1 - \gamma_2 U_s^1 + \gamma_2 \exp(-ika) U_s^2 \\ -\omega^2 m_2 U_s^2 = \gamma_2 U_s^1 \exp(ika) - \gamma_2 U_s^2 - \gamma_1 U_s^2 + \gamma_1 U_s^1 \end{cases}$$

$$\begin{cases} \left[\gamma_1 + \gamma_2 \exp(-ika)\right] U_s^2 + \left(\omega^2 m_1 - \gamma_1 - \gamma_2\right) U_s^1 = 0 \\ \left[\gamma_2 \exp(ika) + \gamma_1\right] U_s^1 + \left(\omega^2 m_2 - \gamma_1 - \gamma_2\right) U_s^2 = 0 \end{cases}$$

$$\begin{bmatrix} \omega^2 m_1 - \gamma_1 - \gamma_2 & \gamma_1 + \gamma_2 \exp(-ika) \\ \gamma_2 \exp(ika) + \gamma_1 & \omega^2 m_2 - \gamma_1 - \gamma_2 \end{bmatrix} \begin{bmatrix} U_s^1 \\ U_s^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Szukamy takiej ω, że wyznacznik $= 0$

$$\left(\omega^2 m_1 - \gamma_1 - \gamma_2\right)\left(\omega^2 m_2 - \gamma_1 - \gamma_2\right) - \left(\gamma_1 + \gamma_2 \exp(-ika)\right)\left(\gamma_2 \exp(ika) + \gamma_1\right) = 0$$

$$m_1 m_2 \omega^4 + (-\gamma_1 - \gamma_2)\omega^2 m_2 + (-\gamma_1 - \gamma_2) m_1 \omega^2 + (-\gamma_1 - \gamma_2)^2 +$$
$$- \left(\gamma_1 + \gamma_2 \exp(-ika)\right)\left(\gamma_2 \exp(ika) + \gamma_1\right) = 0$$

$$z = \omega^2$$

$$m_1 m_2 z^2 + (-\gamma_1 - \gamma_2)(m_1 + m_2) z + (-\gamma_1 - \gamma_2)^2 - \left(\gamma_1 + \gamma_2 \exp(-ika)\right)\left(\gamma_2 \exp(ika) + \gamma_1\right) = 0$$

$$m_1 m_2 z^2 + (-\gamma_1 - \gamma_2)(m_1 + m_2) z + \cancel{\gamma_1^2} + 2\gamma_1\gamma_2 + \cancel{\gamma_2^2} - \left(\gamma_1\gamma_2\left[\exp(ika) + \exp(-ika)\right] + \cancel{\gamma_1^2} + \cancel{\gamma_2^2}\right) = 0$$

$$\Delta = b^2 - 4ac$$

$$z = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\Delta = (\gamma_1 + \gamma_2)^2 (m_1 + m_2)^2 - 4 m_1 m_2 \left(2\gamma_1\gamma_2 - \gamma_1\gamma_2 \cdot 2\cos(ka)\right)$$

$$\Delta = (\gamma_1 + \gamma_2)^2 (m_1 + m_2)^2 - 8 m_1 m_2 \gamma_1 \gamma_2 \left(1 - \cos(ka)\right)$$

$$z = \frac{(\gamma_1 + \gamma_2)(m_1 + m_2) \pm \sqrt{(\gamma_1 + \gamma_2)^2 (m_1 + m_2)^2 - 8 m_1 m_2 \gamma_1 \gamma_2 \left(1 - \cos(ka)\right)}}{2 m_1 m_2}$$

$\omega = \pm\sqrt{z}$ ← odrzucamy wartości ujemne

ostatecznie

$$\omega = \sqrt{\frac{(\gamma_1 + \gamma_2)(m_1 + m_2) \pm \sqrt{(\gamma_1 + \gamma_2)^2 (m_1 + m_2)^2 - 8 m_1 m_2 \gamma_1 \gamma_2 \left(1 - \cos(ka)\right)}}{2 m_1 m_2}}$$