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Hw6
CS206

1. (a). length 3: $\{\{H, H, H\}\}$

length 4: $\{(T, H, H, H)\}$

length 5: $\{(T, T, H, H, H)\}$, $\{(H, T, H, H, H)\}$

length 6: $\{(T, T, T, H, H, H)\}$, $\{(T, H, T, H, H, H)\}$, $\{(H, H, T, H, H, H)\}$, $\{(H, T, T, H, H, H)\}$

(b). ~~To get $x \cdot y = 2^n$~~

$P(X=x) = \begin{cases} pq^{x-1} & ; x \in N, x \geq 1 \\ 0 & \end{cases}$

$P(A_1) = (\frac{1}{2}) \cdot (\frac{1}{2})^2 = \frac{1}{8}$

$P(A_2) = (\frac{1}{2}) \cdot (\frac{1}{2})^{4-1} = \frac{1}{16}$

$P(A_3) = (\frac{1}{2}) \cdot (\frac{1}{2})^{5-1} = \frac{1}{32}$

$P(A_4) = (\frac{1}{2}) \cdot (\frac{1}{2})^{6-1} = \frac{1}{64}$

(c). $E(X) = \sum_{x=1}^{\infty} x \, P[X=x]$

$= \sum_{x=1}^{\infty} x \cdot p q^{x-1}$

$= p(1 + 2q + 3q^2 + \cdots)$

$= p(1-q)^{-2}$

$= \frac{p}{p^2} = \frac{1}{p}$

$E(X) = \frac{1}{1/2} = 2$

$E(x+1) = 2+1 = 3$

$E(x+2) = 2+2 = 4$

$E(x+3) = 2+3 = 5$

which we find

$E(x+3) = E(x) + E(x+1)$

$\therefore \boxed{E(X) = E(X-3) + E(X-2)}$

2. (a). To get $x \cdot y = 2^n (x_L \cdot y_L) + 2^{n/2} \cdot (x_L \cdot y_R + x_R \cdot y_L) + (x_R \cdot y_R)$

we need to multiply 4 times: $x_L y_L$, $x_L y_R$, $x_R y_L$, $x_R y_R$

so $T(n) = 4 \cdot T(\frac{n}{2}) + O(n)$

$\because 4 > 2^1$

$\therefore T(n) = O(n^{\log_b a})$

$\log_b a = \log_2 4 = 2$

so $\boxed{T(n) = O(n^2)}$

(b). Here we get $x \cdot y = 2^n (x_L \cdot y_L) + 2^{n/2} \cdot [(x_L + x_R) \cdot (y_L + y_R) - (x_L \cdot y_L) - (x_R \cdot y_R)] + (x_R \cdot y_R)$

we need to multiply 3 times now: $x_L y_L$, $x_R y_R$, $(x_L + x_R) \cdot (y_L + y_R)$

so $T(n) = 3 \cdot T(\frac{n}{2}) + O(n)$

$\because 3 > 2^1$

$\therefore T(n) = O(n^{\log_b a})$

$\log_b a = \log_2 3$

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then $\boxed{T(n) = O(n^{\log_2 3})}$ this method works better than the method in (a).
$\log_2 3 < 2$, so

3. We set $B(x) = a$
~~2nd~~ next ~~year~~ month: $B(x+1) = a+b$ (b here means the growth of branches of the $(n-1)$th ~~year~~ month.
Third ~~year~~ month: $B(x+2) = a+b+a$ (it grows a branches because they need two months)
the growth of branches is for the nth ~~year~~ month,

$$B(x+2) = B(x+1) + B(x)$$
so
$$B(x) = B(x-1) + B(x-2)$$

$$\boxed{B(x) = \begin{cases} \cancel{1 \quad \text{if } x=1} \\ 1 & \text{if } x=2 \\ B(x-1)+B(x-2) & \text{if } x \geq 3 \end{cases}}$$

$$f(a) = \sum_{x=1}^{\infty} B(x)\, a^{x-1}$$
$$a f(a) = \sum_{x=1}^{\infty} B(x)\, a^{x} = \sum_{x=2}^{\infty} B(x-1)\, a^{x-1}$$
$$a^2 f(a) = \sum_{x=1}^{\infty} B(x)\, a^{x+1} = \sum_{x=3} B(x-2)\, a^{x-1}$$
$$f(a) - a f(a) - a^2 f(a)$$
$$= B(1) + B(2)a + \sum_{x=3}^{\infty} B(x)\, a^{x-1} - B(1)a - \sum_{x=3}^{\infty} B(x-1)\, a^{x-1} - \sum_{x=3}^{\infty} B(x-2)\, a^{x-1}$$
$$= 1 + a - a + \sum_{x=3}^{\infty} 0 \cdot a^{x-1}$$
$$= 1$$
$$\because f(a) - a f(a) - a^2 f(a) = 1$$
$$f(a) = \frac{1}{1-a-a^2}$$
$$= \frac{1}{\sqrt{5}}\left(\frac{\varphi_1}{1-\varphi_1 a} - \frac{\varphi_2}{1-\varphi_2 a}\right)$$
$$\boxed{\varphi_1 = \frac{1+\sqrt{5}}{2}, \quad \varphi_2 = \frac{1-\sqrt{5}}{2}}$$

Using geometric series we get:
$$f(a) = \sum_{x=1}^{\infty} \left(\frac{\varphi_1^x - \varphi_2^x}{\sqrt{5}}\right) a^{x-1}$$
$$B(x) = \boxed{\frac{\varphi_1^x - \varphi_2^x}{\sqrt{5}}}$$