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Hw 5
CS 206.

1. For the reason $E(X_i) = p$, $E(Y_i) = np$, they are not 0.
so we can't say $var(X_i Y_i) = var(X_i) + var(Y_i)$

$Var(X_i) = p(1-p)$, $var(Y_i) = np(1-p)$

$Var(X_i Y_i) = Var(X_i) + Var(Y_i) + Var(X_i)(E(Y_i))^2 + Var(Y_i)(E(X_i))^2$

$Var(Z) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i Y_i)$
$= \frac{1}{n^2} \cdot n \cdot (p + np + n^2p^3(1-p) + np^3(1-p))$
$= \frac{p}{n} + p + np^3(1-p) + p^3(1-p)$

2. $p =$ P(chose fair) P(head on fair) + P(biased coin) P(head on biased)
$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}$
$= \frac{1}{4} + \frac{1}{8}$
$= \frac{3}{8}$

$E(X) = \frac{1}{p}$ (expectation of $\underbrace{\text{the number flips for 1st head}}_{X}$)
$= \boxed{\frac{8}{3}}$

$Var(X) = \frac{1-p}{p^2}$
$= \frac{1-\frac{3}{8}}{(\frac{3}{8})^2}$
$= \boxed{\frac{40}{9}}$

3. $P(T=n) = P(n_1=0)P(n_2=0)P(n_3=0)\cdots P(n_{n-1}=0)$
$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\left(\frac{n-3}{n-2}\right)\cdots\frac{1}{n-k+1}$
$= \frac{1}{n}$

$E(T) = \sum_{n=1}^{n} n\,P(T=n) = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \boxed{\frac{n+1}{2}}$

$Var[T] = E(T^2) - E^2(T)$
$= \sum_{n=1}^{n} n^2 P(T=n) - \left(\frac{n+1}{2}\right)^2$
$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$

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$Var[T] = \boxed{\frac{n^2-1}{12}}$

4. (a). To get probability of getting at least 3 spades, we can use $1 - P(X \leq 2)$

$P(X=0) = \dfrac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}}$ → 0 from spade, → 5 from others. → total probability

so $P(X \geq 3) = 1 - P(X \leq 2)$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \dfrac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}} - \dfrac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} - \dfrac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}$

$= 1 - 0.22 - 0.41 - 0.27$

$= \boxed{0.1}$

(b). Markov's theorem: $P(R \geq x) \leq \dfrac{E(R)}{x}$

which here is $P(X \geq 3) \leq \dfrac{E(X)}{3}$

so we need to get $E(X)$ first,

$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) \cdots 5 \times P(X=5)$

$= 1.25$

so it's $P(X \geq 3) \leq \dfrac{1.25}{3}$

$\leq \boxed{\frac{5}{12}}$

(c). Chebyshev's theorem: $P(|R - E(R)| \geq x) \leq \dfrac{Var(R)}{x^2}$

which here is $P(|R - 1.25| \geq 1.75) \leq \dfrac{Var(R)}{(1.75)^2}$

so we need to get $Var(X)$ first,

$Var(X) = E(X^2) - E(X)^2$

$= [0^2 \cdot P(X=0) + 1 \cdot P(X=1) \cdots + 5 \cdot P(X=5)] - (1.25)^2$

$= \cancel{2.41 - 1.25^2}\ 2.4265 - 1.25^2$

$= 2.4265 - 1.5625$

$= 0.864$

$P(|R - 1.25| \geq 1.75) \leq \dfrac{0.864}{(1.75)^2}$

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$$P(|R-1.25| \geq 1.75) \leq \boxed{0.28}$$

5. we can set $Y = X+100$

and we know X is strictly > -100, so $Y > 0$

$E[Y] = E[X+100] = E[X] + 100 = -60+100 = 40$

Using markov's theorem,

$$P(X \geq -20) = P(Y \geq 80) \leq \frac{E(Y)}{80} = \frac{40}{80} = \boxed{\frac{1}{2}}$$

6. Chebyshev's theorem:

$$P(|X-350| \geq 50) \leq \frac{Var(X)}{50^2}$$

so we need to get $Var(X)$ first,

$$E(X') = \sum_{k=1}^{6} kP(X_k = n)$$

$$= \frac{1}{6} \sum_{k=1}^{6} k$$

$$= \frac{1}{6} \left(\frac{6 \cdot (6+1)}{2} \right)$$

$$= \frac{7}{2}$$

$$Var(X') = E[X'^2] - E[X']^2$$

$$= \sum_{k=1}^{6} k^2 P(X_k = n) - \left(\frac{7}{2}\right)^2$$

$$= \frac{1}{6} \cdot \left(\frac{6 \cdot (6+1)(12+1)}{6} \right) - \left(\frac{7}{2}\right)^2$$

$$= \frac{35}{12}$$

$$Var(X) = Var(X') \cdot 100$$

$$= \frac{875}{3}$$

$$P(|X-350| \geq 50) \leq \frac{\frac{875}{3}}{50^2} = \boxed{\frac{7}{60}}$$

7. $Var(X-Y) = E[(X-Y)^2] - [E(X-Y)]^2$

$= E(X^2+Y^2-2XY) - [E(X) - E(Y)]^2$

$= E(X^2+Y^2-2XY) - [E(X)^2 + E(Y)^2 - E(2XY)]$

$= E(X^2) + E(Y^2) - 2E(XY) - E(X)^2 - E(Y)^2 + 2E(XY)$

$= [E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2]$

$= Var(X) + Var(Y)$