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CS 206
Hw 3

1. (a). We know that penny = 1 cent, so we can write the series as

$$x^0 + x^1 + x^2 + x^3 \cdots x^n = \boxed{\frac{1}{1-x}}$$ (we learned in class)

(b). for nickel we know nickel = 5 cents, so

$$x^0 + x^5 + x^{10} + \cdots x^n$$

we use $z = x^5$, and plug in

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z} = \boxed{\frac{1}{1-x^5}}$$

(c). This time we have penny and nickel together, so

$$(1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)$$

$$= \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x^5}\right)$$

$$= \boxed{\frac{1}{(1-x)(1-x^5)}}$$

(d). This time we add dime (10 cents), quarters (25 cents), half-dollar (50 cents)

As we write above, we can get it easily

$$\boxed{\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}}$$

(e). To find the ways, we just need to get the coefficient of x^{50} (which is when $n = 50$).

In the bottom equation

$$(1 + x^1 + x^2 + \cdots x^{50})(1 + x^5 + \cdots x^{50})(1 + x^{10} + \cdots x^{50}) + (1 + x^{25} + x^{50})$$

$$+ (1 + x^{50})$$

Just find the coefficient of the list above.

2. (a). So we can write the equation:

$$x^3 + x^4 + x^5 + \cdots x^n = x^3(1 + x + x^2 + \cdots x^{n-3})$$

$$= x^3\left(\frac{1}{1-x}\right) = \boxed{\frac{x^3}{1-x}}$$

(b). we can easily write it: ~~write~~

$$\boxed{1 + x + x^2}$$ (at most 2, so it's close at 2)

(c). when it's 0, it's $x^0 = 1$

when it's 2, it's x^2

so the answer is $x^0 + x^2 = \boxed{1 + x^2}$

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(d). we can write it as the equation:

$$1 + x^4 + x^8 + \cdots x^n$$

we use z replace x^4, plug in z

$$1 + z + z^2 + \cdots = \frac{1}{1-z} = \boxed{\frac{1}{1-x^4}}$$

(e). It's just the multiply of the answer above.

$$\left(\frac{x^3}{1-x}\right)(1+x+x^2)(1+x^2)\left(\frac{1}{1-x^4}\right)$$

$$= \frac{x^3(1+x+x^2)\cancel{(1+x^2)}}{(1-x)^2(1+x)\cancel{(1+x^2)}}$$

$$= \frac{x^3(1+x+x^2)}{(1-x)^2(1+x)}$$

(f) From plazza, I know we should use partial fraction decomposition.

$$\text{so } \frac{x^3(1+x+x^2)}{(1-x)^2(1+x)} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

after calculating I get

$$A = -\frac{1}{4},\ B = \frac{21}{4},\ C = \frac{3}{2}$$

$$\frac{-\frac{1}{4}}{1+x} + \frac{\frac{21}{4}}{1-x} + \frac{\frac{3}{2}}{(1-x)^2}$$

$$= \frac{21}{4}\binom{n+1-1}{n} + \frac{3}{2}\binom{n+2-1}{n} + \left(-\frac{1}{4}\right)\cdot(-1)^{n-3}\binom{n+1-1}{n}$$

$\Downarrow$ get from online

$$= \frac{21}{4} + \frac{3}{2}n - \frac{1}{4}(-1)^{n-3}$$

(I'm not sure if it's correct, but I try to solve it for a long time, and search it online for the coefficient of $\left(\frac{1}{1+x}\right)$

~~3. (a).~~

3. (a). we get $P(x)$ by multiply the generating functions of all kinds of animals.

$$P(x) = (x^2 + x^4 + \cdots)(1+x)(x^2 + x^3 + \cdots)(4x^2 + 8x^2 + \cdots)$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$P(x) = \left(\frac{x^2}{1-x^2}\right)(1+x)\left(\frac{x^2}{1-x}\right)\left(\frac{4x^2}{1-2x}\right)$$

$$= \frac{x^2\cancel{(1+x)}x^2\cdot 4x^2}{\cancel{(1+x)}(1-x)^2(1-2x)}$$

$$= \frac{4x^6}{(1-x)^2(1-2x)}$$

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(b). Again like 2(f). we use partial function decomposition

$$\frac{4x^6}{(1-x)^2(1-2x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x}$$

After calculating we get

$$A = -16, \quad B = -4, \quad C = -\frac{1}{4}$$

so it's $\frac{-16}{1-x} + \frac{-4}{(1-x)^2} + \frac{-\frac{1}{4}}{1-2x}$

$$= (-16) \cdot \binom{n+1-1}{n} + (-4) \cdot \binom{n+2-1}{n} + \left(-\frac{1}{4}\right)\binom{n+1-1}{n} 2^n$$

$$\cancel{= -16 + (-4)}$$

$$= \boxed{-16 - 4n - \frac{1}{4} \cdot 2^n}$$

4. (a). We can know there is no x^0 in wrap of strings because in [s] (a wrap of string), there's at least one "[ ]"

so we can set it as $P(x) = P_1 x + P_2 x^2 + \cdots$

And in $P(x)$, there's always one more left/right bracket than $C(x)$

so $C_n = P_{n+1}$

$(C_0 = P_1, \ C_1 = P_2)$

$$\therefore P(x) = P_1 x + P_2 x^2 + \cdots$$

$$= C_0 x + C_1 x^2 + \cdots$$

$$= x(C_0 + C_1 x + C_2 x^2 + \cdots)$$

$$= x[C(x)]$$

(b). Because all the brackets with good count ends up with 0, as we know in question.

When it becomes 0, next one will be a new wrap, so that's why with a good count, there's a unique sequence of strings

(c). $C = 1 + xC + (xC)^2 + \cdots$

$$C - 1 = xC + (xC)^2 + \cdots$$

$$C - 1 = xC(1 + xC + (xC)^2 + \cdots)$$

$$C - 1 = xC^2$$

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$c - xc^2 = 1$

$c(1 - xc) = 1$

$c = \frac{1}{1-xc}$

$c - xc^2 = 1$

$xc^2 + 1 - c = 0.$

use quadratic equation, we know

$c = \frac{1 \pm \sqrt{1-4x}}{2x}$

we know $D(x) = 2x\,C(x)$

$D(x) = 2x c_0 + 2x^2 c_1 + \cdots$

$C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$

we can see $d_1 = 2c_0$, we know $d_{n+1} = 2c_n$

$c_n = \frac{d_{n+1}}{2}$

(d). $D(x) = 2x\,C(x)$

$= 2x \cdot \left(\frac{1 \pm \sqrt{1-4x}}{2x}\right)$

$= 1 \pm \sqrt{1-4x}$

because $c_0 = 1$, so $D(x) = 1 - \sqrt{1-4x}$

(e). get from hint we know

$D^{(n)}(0) = (2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1 \cdot 2^n$

$D^{(n)}(0) = n!\,d_n \cancel{d_{n+1}x + d_{n+2}x^2 + \cdots}$

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(f). from $d_n = \frac{(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1}{n!} 2^n$ we know

$$d_{n+1} = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{(n+1)!} 2^{n+1}$$

from $C_n = \frac{d_{n+1}}{2}$

we know $C_n = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{(n+1)!} 2^n$

$$= \frac{2n\cdot(2n-1)(2n-2)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}{2n(2n-2)(2n-4)\cdots 4\cdot 2\cdot (n+1)\cdot n!} 2^n.$$

$$= \frac{2n!}{\cancel{2}^n\cdot(n!)\cdot(n+1)\cdot(n!)} \cancel{2}^n$$

$$= \left(\frac{1}{n+1}\right)\left(\frac{2n!}{n!\,n!}\right)$$

$$= \frac{1}{n+1}\binom{2n}{n}$$