1.

(a).

$||x||_\infty = max_{1\leq i\leq n}\{|x_i|\}$

and $||x||_1 = \sum_{i=1}^{n}|x_i|$

we know $max_{1\leq i\leq n}\{|x_i|\} \leq \sum_{i=1}^{n}|x_i$

so we get $||x||_\infty \leq ||x||_1$

$n||x||_\infty = n * max_{1\leq i\leq n}\{|x_i|\} = \sum_{i=1}^{n} max_{1\leq i\leq n}\{|x_i|\}$

and $\sum_{i=1}^{n}|x_i \leq \sum_{i=1}^{n} max_{1\leq i\leq n}\{|x_i|\}$

so $||x||_\infty \leq ||x||_1 \leq n||x||_\infty$

It is also the same for x_2

$||x||_2 = \sqrt{x_1^2 + x_2^2 + ...(max_{1\leq i\leq n}\{|x_i|\})^2 + ...x_n^2} \geq \sqrt{(max_{1\leq i\leq n}\{|x_i|\})^2} = ||x||_\infty$

$||x||_2 = \sqrt{x_1^2 + x_2^2 + ...(max_{1\leq i\leq n}\{|x_i|\})^2 + ...x_n^2} \leq \sqrt{\sum_{i=1}^{n}(max_{1\leq i\leq n}\{|x_i|\})^2} = \sqrt{n * (||x||_\infty)^2} = \sqrt{n}||x||_\infty$

so$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$

(b).

$||M||_a = max_{||x||_a=1}||Mx||_a$

$||M||_b = max_{||x||_b=1}||Mx||_b$

Also from lecture we know that $||Mx||_a \leq ||M||_a||x||_a$, and $||Mx||_b \leq ||x||_b$

Then

$||M||_a = max_{||x||_a=1}||Mx||_a \leq max_{||x||_a=1}||M||_a||x||_a \leq \frac{1}{c_1}max_{||x||_b=1}||M||_b||x||_b$ (From $c_1||x||_a \leq ||x||_b$, we can get this, and of course $||M_b|| \geq ||M||_a$)

Then we can know $||M||_a \leq \frac{1}{c_1}||M||_b$

so there must be a positive constant d satisfy the equation

Also

$|M||_b = max_{||x||_b=1}||Mx||_b \leq max_{||x||_b=1}||M||_b||x||_b \leq \frac{1}{c_2}max_{||x||_a=1}||M||_a||x||_a$(From $c_2||x||_a \geq ||x||_b$, we can get this)

Then we can know $d_1||M||_a \leq ||M||_b \leq d_2||M||_a$

2.

(a).

$1 * 1 - (1+\varepsilon)(1-\varepsilon) = 1 - 1 + \varepsilon^2 = \varepsilon^2$

(b).

To make determinant equals 0, we need to make $\varepsilon^2 = 0$, which means that $\varepsilon = 0$

(c).

A = LU = $\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

$l_{11} * u_{11} + 0 * 0 = 1$

$l_{11} * u_{12} + 0 * u_{22} = 1 + \varepsilon$

$l_{21} * u_{11} + l_{22} * 0 = 1 - \varepsilon$

$l_{21} * u_{12} + l_{22} * u_{22} = 1$

$l_{11} = 1, l_{21} = 1 - \varepsilon, l_{22} = 1, u_{11} = 1, u_{12} = 1 + \varepsilon, u_{22} = \varepsilon^2$

so $L = \begin{bmatrix} 1 & 0 \\ 1-\varepsilon & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1+\varepsilon \\ 0 & \varepsilon^2 \end{bmatrix}$

(d).

to make it singular, we need to make sure |u| = 0
$|u| = \varepsilon^2 - 0 * (1 + \varepsilon) = \varepsilon^2$
$\varepsilon^2 = 0, \varepsilon = 0$