1.

$$p(t) = 5t^3 - 3t^2 + 7t - 2 \Longrightarrow p(t) = -2 + t(7 + t(-3 + 5t))$$

2.
(a)

$$P_{n-1}(t) = a_0 + a_1x + a_2^2 + ...a_{n-1}x^{n-1}$$

Using Horner's method

$$P_{n-1}(t) = a_0 + x(a_1 + x(a_2 + ...x(a_{n-2} + xa_{n-1})...)))$$

so we can easily find that it needs to multiply for $(n-1)$ times
(b)

$$P(x) = y_1l_1(x) + y_2l_2(x) + \ldots + y_nl_n(x)$$

it is also like this

$$P_{n-1}(t) = y_1\frac{(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_2)(x_1-x_3)...(x_1-x_n)} + ...y_n\frac{(x-x_1)(x-x_2)...(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})}$$

(the number should be written in t , sorry my bad)
because only the molecular is unknown, so we can write it in such a copy

$$y = (x-x_2)(x-x_3)...(x-x_n) + ...(x-x_1)(x-x_2)...(x-x_{n-1})$$

so we can easily say that it needs $n^n times$ of multiplications
(c)

$$P_{n-1}(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + ... + a_{n-2}(x-x_0)(x-x_1)...(x-x_{n-2})$$

Using Horner's method again

$$P_{n-1}(t) = a_0 + (t-t_0)(a_1 + (t-t_1)(a_2 + ...(t-t_{n-3}(a_{n-2} + (t-t_{n-2})a_{n-1})...)))$$

So as problem #a we can know that it needs $(n-1)$ times of multiplications
3.
(a)
In general, there is a unique polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

Writing down the Vandermonde system for this data gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix}$$

Solving this system by Gaussian elimination yields the solution $\tilde{a} = (5, 2, 3, 1)$so that the interpolating polynomial is

$$P(x) = 5 + 2x + 3x^2 + x^3$$

(b)
Using Lagrange interpolation

$$P(x) = y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$+y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

we plug in the data, we can also get the answer

$$P(x) = 11 * \frac{(x - 2)(x - 3)(x - 4)}{(1 - 2)(1 - 3)(1 - 4)} + 29 * \frac{(x - 1)(x - 3)(x - 4)}{(2 - 1)(2 - 3)(2 - 4)}$$

$$+65 * \frac{(x - 1)(x - 2)(x - 4)}{(3 - 1)(3 - 2)(3 - 4)} + 125 * \frac{(x - 1)(x - 2)(x - 3)}{(4 - 1)(4 - 2)(4 - 3)}$$

then after we calculate the answer we can get $P(x) = 5 + 2x + 3x^2 + x^3$
It is the same as the answer in (a)
(c)
The matrix way:
For the given data, this system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{bmatrix} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix}$$

whose solution is $\tilde{a} = (11, 18, 9, 1)$. Thus, the interpolating polynomial is

$$P(x) = 11 + (x - 1) * 18 + (x - 1)(x - 2) * 9 + (x - 1)(x - 2)(x - 3) = 5 + 2x + 3x^2 + x^3$$

The devided methods way:
we need to calculate $\triangle, \triangle^2 \triangle^3$for this question.
$\triangle = \frac{29-11}{2-1} = 18$, $\triangle_2 = \frac{65-29}{1} = 36$
$\triangle^2 = \frac{36-18}{2} = 9$, $\triangle_2^2 = \frac{60-36}{2} = 12$
$\triangle^3 = \frac{12-9}{3} = 1$
we plug in the numbers again we can get the asnwer again

$$P(x) = 5 + 2x + 3x^2 + x^3$$

I am not sure if it is correct because I did not see resource about this part, and I learned by myself about this part online

The incremental interpolation way:

$$y(x) = c_0 N_0(x) + c_1 N_1(x) + c_2 N_2(x) + c_3 N_3(x)$$

$$c_n = \frac{y(t_n) - y_{n-1}(t)}{\prod_{j=0}^{n-1}(t_n - t_j)}$$

Then we get

$$N_n(t) = \prod_{j=0}^{n-1}(x - x_j)$$

$$M_n(t) = c_k N_k(t)$$

Then we try getting $y_0(t), y_1(t), y_2(t)$,to finally get the answer $y(t)$

$$y_0(t) = 11$$

$$y_1(t) = y_0(t) + M_1(t) = 11 + \frac{29 - 11}{1}(t - 1) = -7 + 18t$$

$$y_2(t) = y_1(t) + M_2(t)$$

Then

$$y(t) = y_2(t) + M_3(t) = y_1(t) + M_2(t) + M_3(t) = 5 + 2t + 3t^2 + t^3$$

Again as the part above, I am not sure if it is correct because I did not see resource about this part, and I learned by myself about this part online

The answers are all the same as the answer in (a)

4.

(a)

We have known that

$$\pi(t) = (t - t_1)(t - t_2)...(t - t_n)$$

then we try calculating the derivative of equation

$$\pi^{'}(t) = 1 * (t - t_2)...(t - t_n) + 1 * (t - t_1)(t - t_3)....(t - t_n) +$$

$$.....1 * (t - t_1)...(t - t_j - 1)(t - t_j + 1)...(t - t_n) + 1 * (t - t_1)...(t - t_{n-1})$$

$$\pi'(t_j) = 0 + 0... + (t_j - t_1)...(t_j - t_j - 1)(t_j - t_j + 1)...(t_j - t_n) + 0 + 0 + ...$$

which is the answer we need to prove
(b)
Using Lagrange interpolant

$$l_j(t) = \frac{(t - t_1)(t - t_2)...(t - t_j - 1)(t - t_j + 1)...(t - t_n)}{(t_j - t_1)(t_j - t_2)...(t_j - t_j - 1)(t_j - t_j + 1)...(t_j - t_n)}$$

$$\Longrightarrow = \frac{1}{t - t_j} * \frac{(t - t_1)(t - t_2)...(t - t_j - 1)(t - t_j)(t - t_j + 1)...(t - t_n)}{(t_j - t_1)(t_j - t_2)...(t_j - t_j - 1)(t_j - t_j + 1)...(t_j - t_n)}$$

$$\Longrightarrow = \frac{1}{t - t_j} * \frac{\pi(t)}{\pi'(t_j)} = \frac{\pi(t)}{(t - t_j)\pi'(t_j)}$$