1.(a)

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \frac{ax_k^2 - c}{2ax_k + b} \Longrightarrow A = \frac{aA^2 - c}{2aA + b} \Longrightarrow 2aA^2 + Ab = aA^2 - c \Longrightarrow aA^2 + bA + c = 0$$

//we can see that A is the solution of $ax^2 + bx + c = 0$
(b)

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \frac{3x_k^2 + a}{4x_k} \Longrightarrow A = \frac{3A^2 + a}{4A} \Longrightarrow A^2 = a \Longrightarrow A^2 - a = 0$$

//A is the solution of $x^2 - a = 0$
2. (a)

$$e^x = x + 2 \Longrightarrow x = e^x - 2$$

//so we get $x_{k+1} = e^{x_k} - 2$ for iteration
//we set

$$f(x) = x_{k+1} = e^{x_k} - 2$$

$$f'(x) = e^{x_k}$$

//we know if $x_k > 0, |e^{x_k}|$ would > 1 ($|f'(x)| > 1$)
//to make it converge, it needs $|e^{x_k}| < 1$, then $x_k < 0$
//so when $x_k < 0$, it's convergence and effective
(b)
Again we set

$$f(x) = x_{k+1} = \frac{x_k}{1 + x_k^2} \Longrightarrow f'(x) = \frac{1 - x_k^2}{(1 + x_k^2)^2}$$

//to make it convergence

$$|f'(x)| < 1$$

//which means

$$|\frac{1 - x_k^2}{(1 + x_k^2)^2}| < 1 \Longrightarrow |1 - x_k^2| < (1 + x_k^2)^2 \Longrightarrow x_k > 1 or x_k < -1$$

//so when $x_k \epsilon (-\infty, -1) \bigcap (1, \infty)$, it's convergence and effective
3. (a)
//to get $\sqrt[3]{a}$, we can get the hint from $x^3 - a = 0$ which the answer is $\sqrt[3]{a}$

$$f(x) = x^3 - a$$

$$f'(x) = 3x^2$$

//we get in Newton's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - a}{3x_k^2}$$

(b)
//to get $\log a$ we can set

$$e^x - a = 0$$

$$f(x) = e^x - a$$

$$f'(x) = e^x$$

//use Newton's method to get the answer:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{e^x - a}{e^x}$$

(c)
//to get $\arctan(x)$ we can set

$$\tan(x) - a = 0$$

$$f(x) = \tan(x) - a$$

$$f'(x) = \tan^2(x) + 1$$

//use Newton's method to get the answer:

$$x_{k+1} = x_k - \frac{\tan(x) - a}{\tan^2 + 1}$$

4. (a)
//to solve $f(x) = 0$
//we can set

$$y = f(x)$$

//the tangent line of it is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

//we get $f(x) = y = 0$, and to clearly see it, we set $x_0 = x_k, x = x_{k+1}$

$$0 = f'(x_k)(x_{k+1} - x_k) + f(x_k)$$

//so $f(x) = 0$ is the solution which makes method converge
(b)
//we know $0 = f'(x_k)(x_{k+1} - x_k) + f(x_k)$ from (a)

$$\Longrightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

//It is the same as Newton's method
//so

$$g'(x) = \frac{f(x_k)f''(x_k)}{f'(x_k)^2}$$

//when assuming

$$f(x_k) = 0$$

$$f'(x_k) \neq 0$$

f" is bounded near x_k

$$\lim_{x \to x_k} g'(x) = \frac{f(x_k)f''(x_k)}{f'(x_k)^2} = 0$$

(c)
//x' is a root of $f(x) = 0$, $|x_k - x'| = \delta < 1$
//we use Taylor's expansion:

$$0 = f(x') = f(x_k - \delta) = f'(x_k)(x' - x_k) + \frac{f''(\delta)}{2}(x' - x_n)^2$$

//from $0 = f'(x_k)(x_{k+1} - x_k) + f(x_k)$ and the equation above
//we get $e_n = x' - x_k$,$e_{n+1} = x' - x_{k+1}$

$$\Longrightarrow e_{n+1} = \frac{-f''(\delta)}{2f'(x_k)}$$

//so

$$e_{n+1} \approx e_n^2$$

//It's a superlinear convergence (or called it quadratic)
,
Another version:
(I did another version of this problem, and I am not sure which one is correct, the above one is the one I did after discussing with my friends)
4. (a)

$$\hat{x}_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

(same to Newton's method)

$$x_{k+1} = \frac{x_k + \hat{x}_{k+1}}{2} = \frac{2x_k - \frac{f(x_k)}{f'(x_k)}}{2}$$

$$\implies 2x_{k+1} - 2x_k = -\frac{f(x_k)}{f'(x_k)}$$

$$2(x_{k+1} - x_k) \times f'(x_k) + f(x_k) = 0$$

//y is 0 for it, so it converge to the solution of $f(x) = 0$
(b)

$$2(x_{k+1} - x_k) \times f'(x_k) + f(x_k) = 0$$

$$\implies 2x_k - \frac{f(x_k)}{f'(x_k)} = 2x_k + 1$$

$$x_{k+1} = x_k - \frac{f(x_k)}{2f'(x_k)}$$

(it is close to the version of Newton's method)