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Hw 1
CS323

1. (a). $\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \frac{a x_k^2 - c}{2a x_k + b} \Longrightarrow A = \frac{aA^2 - c}{2aA + b} \Rightarrow 2aA^2 + Ab = aA^2 - c$

$\Longrightarrow aA^2 + bA + c = 0.$

we can see that A is the solution of $ax^2+bx+c=0.$

(b). $\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \frac{3x_k^2 + a}{4x_k} \Longrightarrow A = \frac{3A^2+a}{4A} \Longrightarrow 4A^2 = 3A^2 + a \Longrightarrow$

$A^2 = a \Longrightarrow A^2 - a = 0$

A is the solution of $x^2 - a = 0.$

2. (a). $e^x = x+2 \Longrightarrow x = e^x - 2$

so we get $x_{k+1} = e^{x_k} - 2$ for iteration

we set $f(x) = x_{k+1} = e^{x_k} - 2$

$f'(x) = e^{x_k}$

we know if $x_k > 0$, $|e^{x_k}|$ would > 1 $(|f'(x)| > 1)$

to make it converge

it needs $|e^{x_k}| < 1$, then $x_k < 0$

So when $x_k < 0$, it's convergence and effective.

(b). Again we set $f(x) = x_{k+1} = \frac{x_k}{1+x_k^2}$

$f'(x) = \frac{1-x_k^2}{(1+x_k^2)^2}$

To make it convergence $|f'(x)|$ must < 1

which means $\left|\frac{1-x_k^2}{(1+x_k^2)^2}\right| < 1$

$\Longrightarrow |1-x_k^2| < (1+x_k^2)^2$

$\Longrightarrow x_k > 1$ or $x_k < -1$

so when $x_k \in (-\infty, -1) \cap (1, \infty)$, it's convergence and effective

3. (a). to get $\sqrt[3]{a}$, we can get the hint from $x^3 - a = 0$ which the answ is $\sqrt[3]{a}$

$f(x)$ is $x^3 - a$

$f'(x) = 3x^2$

we get in newton's rule:

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$= x_k - \frac{x_k^3 - a}{3x_k^2}$

(b). To get $\log a$ we can set

$e^x - a = 0$

which is $f(x) = e^x - a$

$f'(x) = e^x$

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Use Newton's rule to get the answer:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{e^x - a}{e^x}$$

(c), To get $\arctan(x)$ we can set

$$\tan(x) - a = 0.$$

$$f(x) = \tan(x) - a$$

$$f'(x) = \tan^2(x) + 1$$

Use Newton's rule to get the answer:

$$x_{k+1} = x_k - \frac{\tan(x) - a}{\tan^2(x) + 1}$$

4. (a). To solve $f(x)=0$

we can set $y=f(x)$

The tangent line of it is

$y = f'(x_0)(x-x_0)+f(x_0)$

we get $f(x)=y=0$, and to clearly see it, we set

$x_0 = x_k$, $x = x_{k+1}$

$0 = f'(x_k)(x_{k+1}-x_k)+f(x_k)$

$\therefore$ $f(x)=0$ is the solution which makes method converge

(b). we know $0 = f'(x_k)(x_{k+1}-x_k)+f(x_k)$ from (a).

$\Rightarrow x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

It's the same as newton's method.

so $g'(x) = \dfrac{f(x_k)f''(x_k)}{f'(x_k)^2}$

when assuming — $f(x_k)=0$

— $f'(x_k)\neq 0$

f'' is bounded near x_k.

$\lim_{x\to x_k} g'(x) = \dfrac{f(x_k)f''(x_k)}{f'(x_k)^2} = 0.$

c). x' is a root of $f(x)=0$, $|x_k - x'| = \varepsilon < 1$

We use Taylor's expansion:

$0 = f(x') = f(x_k - \varepsilon) = f'(x_k)(x'-x_k) + \dfrac{f''(\xi)}{2}(x'-x_k)^2$

From $0 = f'(x_k)(x_{k+1}-x_k)+f(x_k)$

and the equation above

we get $e_n = x' - x_k$, $e_{n+1} = x' - x_{k+1}$

$\Rightarrow e_{n+1} = \dfrac{-f''(\varepsilon)}{2f'(x_k)}$

so $e_{n+1} \approx e_n^2$

It's a super linear convergence

(or called it quadratic)

The first version I did for 4a 4b, not sure which one is correct so I upload both of them. After I discussed with my friends, I thought this might be incorrect.

~~$x_k - \frac{f(x_k)}{f'(x_k)} = x_{k+1}$~~

4.(a). $\hat{x}_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ (same to Newton's rule)

$$x_{k+1} = \frac{x_k + \hat{x}_{k+1}}{2} = \frac{2x_k - \frac{f(x_k)}{f'(x_k)}}{2}$$

$$\Longrightarrow 2x_{k+1} - 2x_k = -\frac{f(x_k)}{f'(x_k)}$$

$$2(x_{k+1} - x_k)\cdot f'(x_k) + f(x_k) = 0.$$

y is 0 for it, so it converge to the solution of $f(x) = 0$.

(b). $2(x_{k+1} - x_k)\cdot f'(x_k) + f(x_k) = 0.$

$$\Longrightarrow 2x_k - \frac{f(x_k)}{f'(x_k)} = 2x_{k+1}$$

Close to the same version of Newton's rule $\Longleftarrow$ $x_{k+1} = x_k - \frac{f(x_k)}{2f'(x_k)}$

— $f(x_1) = 0$

— $f'(x_1) \neq 0$

— $f''(x_1)$ bounded near x

Then $g'(x) = (x_{k+1})' = \frac{3f'(x)^2 + f(x)f''(x)}{4f'(x)^2}$

Then. $\lim_{x \to x_1} g'(x) = 0$