Bicomplexes

Bicomplexes for Abelian categories

2017.05.02

2 May 2017

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Chapter 1

Bicomplexes

Let us create the following chain complex of chain complexes of left presentations over $\mathbb{Z}$:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 8 \\
0 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "(2)"'] & \mathbb{Z} \arrow[l, "(2)"'] \arrow[d, "(4)"] & 0 \arrow[l] \arrow[d] & 7 \\
0 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "1 \mapsto \bar{1}"'] & \mathbb{Z} \arrow[l, "(1)"'] \arrow[d, "1 \mapsto \bar{1}"] & 0 \arrow[l] \arrow[d] & 6 \\
0 \arrow[d] & \mathbb{Z}/\langle 2 \rangle \arrow[l] \arrow[d] & \mathbb{Z}/\langle 4 \rangle \arrow[l, "\bar{1} \mapsto \bar{1}"] \arrow[d] & 0 \arrow[l] \arrow[d] & 5 \\
0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 4 \\
\mathbf{0} & C_9 \arrow[l, Leftarrow] & C_{10} \arrow[l, Leftarrow, "\phi"'] & \mathbf{0} \arrow[l, Leftarrow] &
\end{tikzcd}
$$

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> lp_cat := CategoryOfHomalgLeftModules( ZZ );
intrinsic Category of left presentations of Z with ambient objects
gap> chains_lp_cat := ChainComplexCategory( lp_cat );
Chain complexes category over intrinsic Category
of left presentations of Z with ambient objects
gap> chains_chains_lp_cat := ChainComplexCategory( chains_lp_cat );
Chain complexes category over chain complexes category over
intrinsic Category of left presentations of Z with ambient objects
gap> Bicomplexes_cat := AsCategoryOfBicomplexes( chains_chains_lp_cat );
Chain complexes category over chain complexes category over
```

```
intrinsic Category of left presentations of Z with ambient objects
as bicomplexes
gap> F1 := HomalgFreeLeftModule( 1, ZZ );
<A free left module of rank 1 on a free generator>
gap> d7 := HomalgMap( HomalgMatrix( "[ [ 4 ] ]", 1, 1, ZZ ), F1, F1 );
<An endo"morphism" of a left module>
gap> d6 := CokernelProjection( d7 );
<An epimorphism of left modules>
gap> C10 := ChainComplex( [ d6, d7 ], 6 );
<A bounded object in chain complexes category over intrinsic
Category of left presentations of Z with ambient objects with
active lower bound 4 and active upper bound 8>
gap> t7 := HomalgMap( HomalgMatrix( "[ [ 2 ] ]", 1, 1, ZZ ), F1, F1 );
<An endo"morphism" of a left module>
gap> t6 := CokernelProjection( t7 );
<An epimorphism of left modules>
gap> C9 := ChainComplex( [ t6, t7 ], 6 );
<A bounded object in chain complexes category over intrinsic
Category of left presentations of Z with ambient objects with
active lower bound 4 and active upper bound 8>
gap> phi5 := HomalgMap( HomalgIdentityMatrix( 1, ZZ ), C10[ 5 ], C9[ 5 ] );
<A "homomorphism" of left modules>
gap> phi6 := HomalgMap( HomalgIdentityMatrix( 1, ZZ ), F1, F1 );
<An endo"morphism" of a left module>
gap> phi7 := HomalgMap( 2 * HomalgIdentityMatrix( 1, ZZ ), F1, F1 );
<An endo"morphism" of a left module>
gap> phi := ChainMorphism( C10, C9, [ phi5, phi6, phi7 ], 5 );
<A bounded morphism in chain complexes category over intrinsic
Category of left presentations of Z with ambient objects with
active lower bound 4 and active upper bound 8>
gap> C := ChainComplex( [ phi ], 10 );
<A bounded object in chain complexes category over chain complexes
category over intrinsic Category of left presentations of Z with
ambient objects with active lower bound 8 and active upper bound 11>
```

Now we compute its associated homological bicomplex B and the total complex of B:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 8 \\
0 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "(-2)"'] & \mathbb{Z} \arrow[l, "(2)"'] \arrow[d, "(4)"] & 0 \arrow[l] \arrow[d] & 7 \\
0 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "1 \mapsto \bar{1}"'] & \mathbb{Z} \arrow[l, "(1)"'] \arrow[d, "1 \mapsto \bar{1}"] & 0 \arrow[l] \arrow[d] & 6 \\
0 \arrow[d] & \mathbb{Z}/\langle 2 \rangle \arrow[l] \arrow[d] & \mathbb{Z}/\langle 4 \rangle \arrow[l, "\bar{1} \mapsto \bar{1}"] \arrow[d] & 0 \arrow[l] \arrow[d] & 5 \\
0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 4 \\
8 & 9 & 10 & 11 &
\end{tikzcd}
$$

Example

```
gap> B := HomologicalBicomplex( C );
<A homological bicomplex in intrinsic Category of left presentations
of Z with ambient objects concentrated in window [ 8 .. 11 ] x [ 4 .. 8 ]>
gap> Display( VerticalDifferentialAt( B, 9, 7 ) );
[ [  -2 ] ]

the map is currently represented by the above 1 x 1 matrix
gap> T := TotalComplex( B );
<A bounded object in chain complexes category over intrinsic Category
of left presentations of Z with ambient objects with active lower
bound 13 and active upper bound 18>
gap> T[ 13 ];
<A zero left module>
gap> T[ 14 ];
<A cyclic torsion left module presented by 1 relation
 for a cyclic generator>
gap> T[ 15 ];
<A non-torsion left module presented by 1 relation for 2 generators>
gap> T[ 16 ];
<A free left module of rank 2 on free generators>
gap> T[ 17 ];
<A free left module of rank 1 on a free generator>
gap> T[ 18 ];
<A zero left module>
gap> Display( T^16 );
[ [  -2,   0 ],
  [   1,   1 ] ]

the map is currently represented by the above 2 x 2 matrix
```

```
gap> IsExact( T );
true
gap> T;
<A cyclic, bounded object in chain complexes category over intrinsic
Category of left presentations of Z with ambient objects with active
lower bound 13 and active upper bound 18>
gap> Display( T, 13, 18 );

-----------------------------------------------------------------
In index 13

Object is
0

Differential is
(an empty 0 x 0 matrix)

the map is currently represented by the above 0 x 0 matrix

-----------------------------------------------------------------
In index 14

Object is
Z/< 2 >

Differential is
(an empty 1 x 0 matrix)

the map is currently represented by the above 1 x 0 matrix

-----------------------------------------------------------------
In index 15

Object is
[ [  0,  4 ] ]

Cokernel of the map

Z^(1x1) --> Z^(1x2),

currently represented by the above matrix

Differential is
[ [  -1 ],
  [   1 ] ]

the map is currently represented by the above 2 x 1 matrix

-----------------------------------------------------------------
In index 16

Object is
Z^(1 x 2)
```

```
Differential is
[ [  -2,   0 ],
  [   1,   1 ] ]

the map is currently represented by the above 2 x 2 matrix

----------------------------------------------------------------
In index 17

Object is
Z^(1 x 1)

Differential is
[ [  2,  4 ] ]

the map is currently represented by the above 1 x 2 matrix

----------------------------------------------------------------
In index 18

Object is
0

Differential is
(an empty 0 x 1 matrix)

the map is currently represented by the above 0 x 1 matrix
```

1.1 Categories

1.1.1 IsCapCategoryBicomplexCell (for IsCapCategoryCell andIsAttributeStoringRep)

▷ `IsCapCategoryBicomplexCell(`*arg*`)` (filter)

Returns: `true` or `false`

The GAP category of cells in a CAP category of bicomplexes.

1.1.2 IsCapCategoryBicomplexObject (for IsCapCategoryBicomplexCell andIsCapCategoryObject)

▷ `IsCapCategoryBicomplexObject(`*arg*`)` (filter)

Returns: `true` or `false`

The GAP category of bicomplex objects in a CAP category of bicomplexes.

1.1.3 IsCapCategoryHomologicalBicomplexObject (for IsCapCategoryBicomplexObject)

▷ `IsCapCategoryHomologicalBicomplexObject(`*arg*`)` (filter)

Returns: `true` or `false`

The GAP category of homological bicomplex objects in a CAP category of bicomplexes.

1.1.4 IsCapCategoryCohomologicalBicomplexObject (for IsCapCategoryBicomplexObject)

▷ `IsCapCategoryCohomologicalBicomplexObject(arg)` (filter)

Returns: `true` or `false`

The GAP category of cohomological bicomplex objects in a CAP category of bicomplexes.

1.1.5 IsCapCategoryBicomplexMorphism (for IsCapCategoryBicomplexCell andIsCapCategoryMorphism)

▷ `IsCapCategoryBicomplexMorphism(arg)` (filter)

Returns: `true` or `false`

The GAP category of bicomplex morphisms in a CAP category of bicomplexes.

1.1.6 IsCapCategoryHomologicalBicomplexMorphism (for IsCapCategoryBicomplexMorphism)

▷ `IsCapCategoryHomologicalBicomplexMorphism(arg)` (filter)

Returns: `true` or `false`

The GAP category of homological bicomplex morphisms in a CAP category of bicomplexes.

1.1.7 IsCapCategoryCohomologicalBicomplexMorphism (for IsCapCategoryBicomplexMorphism)

▷ `IsCapCategoryCohomologicalBicomplexMorphism(arg)` (filter)

Returns: `true` or `false`

The GAP category of cohomological bicomplex morphisms in a CAP category of bicomplexes.

1.2 Constructors

1.2.1 AssociatedBicomplexObject (for IsChainOrCochainComplex)

▷ `AssociatedBicomplexObject(cx)` (attribute)

Returns: a CAP object

Return the bicomplex associated to the complex of complexes *cx*.

1.2.2 AssociatedBicomplexMorphism (for IsChainOrCochainMorphism)

▷ `AssociatedBicomplexMorphism(mu)` (attribute)

Returns: a CAP morphism

Return the morphism of bicomplexes associated to the chain morphism between two complexes of complexes *mu*.

1.2.3 AssociatedBicomplexFunctor (for IsCapFunctor, IsString)

▷ `AssociatedBicomplexFunctor(F, name)` (operation)
▷ `AssociatedBicomplexFunctor(F)` (attribute)

Returns: a CAP functor

1.2.4 AssociatedBicomplex (for IsCapNaturalTransformation, IsString)

▷ `AssociatedBicomplex(eta, name)` (operation)
▷ `AssociatedBicomplex(eta)` (attribute)

Returns: a CAP natural transformation

1.2.5 AsCategoryOfBicomplexes (for IsCapCategory)

▷ `AsCategoryOfBicomplexes(A)` (attribute)

Returns: a CAP category

Return the category of bicomplexes of the Abelian category `A` of complexes of complexes.

1.3 Attributes

1.3.1 UnderlyingCategoryOfComplexesOfComplexes (for IsCapCategory)

▷ `UnderlyingCategoryOfComplexesOfComplexes(Bicx)` (attribute)

The category of double complexes underlying the category of bicomplexes `Bicx`.

1.3.2 UnderlyingCapCategoryCell (for IsObject)

▷ `UnderlyingCapCategoryCell(B)` (attribute)

The complex of complexes underlying the bicomplex `B`.

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