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Computing obstruction for the existence of compact Clifford-Klein form

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Abstract

In this package we develop functions for an algorithm designed to find homogeneous spaces of semisimple non-compact Lie groups which do not admit compact Clifford-Klein forms.

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Contents

Chapter 1

Notation

We use the notation and convention for real Lie algebras as is from CoReLG Package, [DFdG14].

Example

```
gap> G:=RealFormById( "E", 7,3);
<Lie algebra of dimension 133 over SqrtField>
gap> rankG:=Dimension(CartanSubalgebra(G));
7
gap> rankRG:=Dimension(CartanSubspace(G));
3
gap> dimG:=Dimension(G);
133
gap> P:=CartanDecomposition( G ).P;
<vector space over SqrtField, with 54 generators>
gap> dimPforG:=Dimension(P);
54
gap> K:=CartanDecomposition( G ).K;
<Lie algebra of dimension 79 over SqrtField>
gap> rankK:= Dimension(CartanSubalgebra(K));
7
gap> dimK:= Dimension(K);
79
```

Classification can be found in Table 9 in [OV90], p. 312-317.

Chapter 2

Obstruction for the existence of compact Clifford-Klein form

In this chapter we describe functions for algorithm from [BJS+].

2.1 Technical functions

2.1.1 NonCompactDimension

▷ `NonCompactDimension(G)` (function)

For a real Lie algebra G constructed by the function *RealFormById* (from [DFdG14]), this function returns the non-compact dimension of G (dimension of a non-compact part in Cartan decomposition of G).

Example
```
gap> G:=RealFormById("E",6,2); # E6(6)
<Lie algebra of dimension 78 over SqrtField>
gap> dG:=NonCompactDimension(G);
42
```

2.1.2 PCoefficients

▷ `PCoefficients(type, rank)` (function)

Let G be a compact connected Lie group of the type *type* and the rank *rank*. Let $\Lambda P_G = \Lambda(y_1,...,y_l)$ be the exterior algebra over the spaces P_G of the primitive elements in $H^*(G)$. Denote the degrees as follows $|y_j| = 2p_j - 1, j = 1,...,l$. This function returns coefficients $p_1,\ldots,p_l$.

Example
```
gap> PCoefficients("D",5);
[ 2, 4, 6, 8, 5 ]
```

2.1.3 PCalculate

▷ `PCalculate(pi, qi)` (function)

Here $pi = \{p_1, \ldots, p_l\}$ and $qi = \{q_1, \ldots, q_m\}$ are sets of coefficients ($l \geq m$). This function returns the polynomial: $P(t) = \prod_{j=m+1}^{l}(1+t^{2p_j-1})\prod_{i=1}^{m}(1-t^{2p_i})/(1-t^{2q_i})$.

Example
```
gap> PCalculate([4,2,3],[2,2]);
t^9+t^5+t^4+1
```

2.1.4 AllZeroDH

▷ `AllZeroDH(type, rank, id)` (function)

Let G^C be a complex Lie algebra of the type `type` and the rank `rank`. Let G be a real form of G^C with the index `id` (see `RealFormsInformation`,[DFdG14]). This function returns the set of degrees of $P(t)$ that have zero coefficients over all permutation (see Section 7 in [BJS+]).

Example
```
gap> AllZeroDH("F",4,2);
[ 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27 ]
```

Chapter 3

Algorithm example

In this chapter we use additionaly functions from the following packages: CoReLG [DFdG14] and SLA [dG]. We will show in detail the split case (for a non-split case you should use algoritm to generate regular subalgebras from [DFdG15]). For example, we take $G = \mathfrak{e}_{6(6)}$ (tuple "E",6,2 in CoReLG notation). We calculate *AllZeroDH* on it.

Example
```
gap> AllZeroDH("E",6,2);
[ 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24, 27,
  28, 29, 30, 31, 32, 35, 36, 37, 38, 39, 40, 41 ]
```

We generate all regular subalgebras of complexification.

Example
```
gap> GC:=SimpleLieAlgebra("E",6,Rationals);;
gap> REG:=RegularSemisimpleSubalgebras(GC);;
gap> L0:=List( REG, SemiSimpleType );
[ "A1", "A1 A1", "A2 A1", "A4", "D5", "A4 A1", "A2 A1 A1", "A2 A1 A2", "A3 A1",
  "A1 A1 A1", "A2", "A3", "A5", "A2 A2", "D4", "A5 A1", "A3 A1 A1", "A1 A1 A1 A1",
  "A2 A2 A2" ]
```

For each subalgebras we take the split real form and calculate its non-compact dimension.

Example
```
gap> L0[4];
"A4"
gap> RealFormsInformation( "A", 4 );

  There are 4 simple real forms with complexification A4
    1 is of type su(5), compact form
    2 - 3 are of type su(p,5-p) with 1 <= p <= 2
    4 is of type sl(5,R)
  Index '0' returns the realification of A4

gap> G:=RealFormById("A",4,4);;
gap> NonCompactDimension( G );
14
```

Number 14 is in output of *AllZeroDH* function, so for $\mathfrak{g} = e_{6(6)}$ and $\mathfrak{h} = \mathfrak{sl}(5, \mathbb{R})$ corresponding homogeneous spaces G/H do not have compact Clifford–Klein forms.

Example

```
gap> L0[5];
"D5"
gap> RealFormsInformation( "D", 5 );

  There are 7 simple real forms with complexification D5
    1 is of type so(10), compact form
    2 - 3 are of type so(2p,10-2p) with 1 <= p <= 2
    4 is of type so*(10)
    5 is of type so(9,1)
    6 - 7 are of type so(2p+1,10-2p-1) with 1 <= p <= 2
  Index '0' returns the realification of D5

gap> G:=RealFormById("D",5,7);;
gap> NonCompactDimension( G );
25
```

Number 25 is not in output of `AllZeroDH` function, so for $\mathfrak{g} = e_{6(6)}$ and $\mathfrak{h} = \mathfrak{so}(5,5)$ our algoritm does not provide a solution to the problem.

References

[BJS$^+$] M. Bocheński, P. Jastrzębski, A. Szczepkowska, A. Tralle, and A. Woike. Semisimple subalgebras in simple lie algebras and a computational approach to the compact clifford-klein forms problem. https://doi.org/10.1080/10586458.2018.1492475. 5, 6

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