

# UGALY

## Universal Groups Acting Locally

v2.0

1 June 2020

**Khalil Hannouch**

**Stephan Tornier**

**Khalil Hannouch**

Email: [khalil.hannouch@newcastle.edu.au](mailto:khalil.hannouch@newcastle.edu.au)

Homepage: <https://www.newcastle.edu.au/profile/khalil-hannouch>

**Stephan Tornier**

Email: [stephan.tornier@newcastle.edu.au](mailto:stephan.tornier@newcastle.edu.au)

Homepage: <https://www.newcastle.edu.au/profile/stephan-tornier>

## Abstract

UGALY (Universal Groups Acting Locally) is a GAP package that provides methods to create, analyse and find local actions of universal groups acting on locally finite regular trees, following Burger-Mozes and Tournier.

## Copyright

UGALY is free software; you can redistribute it and/or modify it under the terms of the [GNU General Public License](#) as published by the Free Software Foundation; either version 3 of the License, or (at your option) any later version.

## Acknowledgements

The second author owes thanks to Marc Burger and George Willis for their support and acknowledges contributions from the SNSF Doc.Mobility fellowship 172120 and the ARC Discovery Project DP120100996 to the development of an early version of this codebase. In its present form, the development of UGALY was made possible by the ARC Laureate Fellowship FL170100032 and the ARC DECRA Fellowship DE210100180.

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
2.1	Local actions . . . . .	5
2.2	Finite balls . . . . .	8
2.3	Addresses and leaves . . . . .	8
<b>3</b>	<b>Compatibility</b>	<b>11</b>
3.1	The compatibility condition (C) . . . . .	11
3.2	Compatible elements . . . . .	11
3.3	Compatible subgroups . . . . .	13
<b>4</b>	<b>Examples</b>	<b>16</b>
4.1	Discrete groups . . . . .	16
4.2	Maximal extensions . . . . .	19
4.3	Normal subgroups and partitions . . . . .	20
4.4	Abelian quotients . . . . .	21
4.5	Semidirect products . . . . .	23
<b>5</b>	<b>Discreteness</b>	<b>26</b>
5.1	The discreteness condition (D) . . . . .	26
5.2	Discreteness . . . . .	26
5.3	Cocycles . . . . .	27
	<b>References</b>	<b>29</b>
	<b>Index</b>	<b>30</b>

# Chapter 1

## Introduction

Let  $\Omega$  be a set of cardinality  $d \in \mathbb{N}_{\geq 3}$  and let  $T_d = (V, E)$  be the  $d$ -regular tree. We follow Serre's graph theory notation [Ser80]. Given a subgroup  $H$  of the automorphism group  $\text{Aut}(T_d)$  of  $T_d$ , and a vertex  $x \in V$ , the stabilizer  $H_x$  of  $x$  in  $H$  induces a permutation group on the set  $E(x) := \{e \in E \mid o(e) = x\}$  of edges issuing from  $x$ . We say that  $H$  is locally "P" if for every  $x \in V$  said permutation group satisfies the property "P", e.g. being transitive, semiprimitive, quasiprimitive or 2-transitive. In [BM00], Burger-Mozes develop a remarkable structure theory of closed, non-discrete, locally quasiprimitive subgroups of  $\text{Aut}(T_d)$ , which resembles the theory of semisimple Lie groups. They complement this structure theory with a particularly accessible class of subgroups of  $\text{Aut}(T_d)$  with prescribed local action: Given  $F \leq \text{Sym}(\Omega)$  their universal group  $U(F)$  is closed in  $\text{Aut}(T_d)$ , vertex-transitive, compactly generated and locally permutation isomorphic to  $F$ . It is discrete if and only if  $F$  is semiregular. When  $F$  is transitive,  $U(F)$  is maximal up to conjugation among vertex-transitive subgroups of  $\text{Aut}(T_d)$  that are locally permutation isomorphic to  $F$ , hence *universal*.

This construction was generalized by the second author in [Tor20]: In the spirit of  $k$ -closures of groups acting on trees developed in [BEW15], we generalize the universal group construction by prescribing the local action on balls of a given radius  $k \in \mathbb{N}$ , the Burger-Mozes construction corresponding to the case  $k = 1$ . Fix a tree  $B_{d,k}$  which is isomorphic to a ball of radius  $k$  in the labelled tree  $T_d$  and let  $l_x^k : B(x, k) \rightarrow B_{d,k}$  ( $x \in V$ ) be the unique label-respecting isomorphism. Then

$$\sigma_k : \text{Aut}(T_d) \times V \rightarrow \text{Aut}(B_{d,k}), (g, x) \rightarrow l_{gx}^k \circ g \circ (l_x^k)^{-1}$$

captures the  $k$ -local action of  $g$  at the vertex  $x \in V$ .

With this we can make the following definition: Let  $F \leq \text{Aut}(B_{d,k})$ . Define

$$U_k(F) := \{g \in \text{Aut}(T_d) \mid \forall x \in V : \sigma_k(g, x) \in F\}.$$

While  $U_k(F)$  is always closed, vertex-transitive and compactly generated, other properties of  $U(F)$  do *not* carry over. Foremost, the group  $U_k(F)$  need not be locally action isomorphic to  $F$  and we say that  $F \leq \text{Aut}(B_{d,k})$  satisfies condition (C) if it is. This can be viewed as an interchangeability condition on neighbouring local actions, see Section 3.1. There is also a discreteness condition (D) on  $F \leq \text{Aut}(B_{d,k})$  in terms of certain stabilizers in  $F$  under which  $U_k(F)$  is discrete, see Section 5.1.

UGALY provides methods to create, analyse and find local actions  $F \leq \text{Aut}(B_{d,k})$  that satisfy condition (C) and/or (D), including the constructions  $\Gamma$ ,  $\Delta$ ,  $\Phi$ ,  $\Sigma$ , and  $\Pi$  developed in [Tor20]. This package was developed within the [Zero-Dimensional Symmetry Research Group](#) in the [School of Mathematical and Physical Sciences](#) at [The University of Newcastle](#) as part of a project course taken by the first author, supervised by the second author.

## Chapter 2

# Preliminaries

We recall the following notation from the Introduction which is essential throughout this manual, cf. [Tor20]. Let  $\Omega$  be a set of cardinality  $d \in \mathbb{N}_{\geq 3}$  and let  $T_d = (V, E)$  denote the  $d$ -regular tree, following the graph theory notation in [Ser80]. A *labelling*  $l$  of  $T_d$  is a map  $l : E \rightarrow \Omega$  such that for every  $x \in V$  the restriction  $l_x : E(x) \rightarrow \Omega$ ,  $e \mapsto l(e)$  is a bijection, and  $l(e) = l(\bar{e})$  for all  $e \in E$ . For every  $k \in \mathbb{N}$ , fix a tree  $B_{d,k}$  which is isomorphic to a ball of radius  $k$  around a vertex in  $T_d$  and carry over the labelling of  $T_d$  to  $B_{d,k}$  via the chosen isomorphism. We denote the center of  $B_{d,k}$  by  $b$ .

For every  $x \in V$  there is a unique, label-respecting isomorphism  $l_x^k : B(x, k) \rightarrow B_{d,k}$ . We define the  $k$ -local action  $\sigma_k(g, x) \in \text{Aut}(B_{d,k})$  of an automorphism  $g \in \text{Aut}(T_d)$  at a vertex  $x \in V$  via the map

$$\sigma_k : \text{Aut}(T_d) \times V \rightarrow \text{Aut}(B_{d,k}), \sigma_k(g, x) \mapsto \sigma_k(g, x) := l_{gx}^k \circ g \circ (l_x^k)^{-1}.$$

### 2.1 Local actions

In this package, a local action  $F \leq \text{Aut}(B_{d,k})$  are handled as objects of the category `IsLocalAction` (??) and have several attributes and properties introduced throughout this manual. Most importantly, a local action always stores the degree  $d$  and the radius  $k$  of the ball  $B_{d,k}$  that it acts on.

#### 2.1.1 IsLocalAction (for IsPermGroup)

▷ `IsLocalAction(arg)` (filter)

**Returns:** true or false

Local actions  $F \leq \text{Aut}(B_{d,k})$  are stored together with their degree (see `LocalActionDegree` (??)), radius (see `LocalActionRadius` (??)) and other attributes in this category.

Example

```
gap> G:=WreathProduct(SymmetricGroup(2),SymmetricGroup(3));
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(G);
false
gap> H:=AutBall(3,2);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(H);
true
gap> K:=LocalAction(3,2,G);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
```

```
gap> IsLocalAction(K);
true
```

### 2.1.2 LocalAction (for IsInt, IsInt, IsPermGroup)

▷ LocalAction( $d, k, F$ ) (operation)

**Returns:** the regular rooted tree group  $G$  as an object of the category IsLocalAction (??), checking that  $F$  is indeed a subgroup of  $\text{Aut}(B_{d,k})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}_0$  and a group  $F \leq \text{Aut}(B_{d,k})$ .

Example

```
gap> G:=WreathProduct(SymmetricGroup(2),SymmetricGroup(3));
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(G);
false
gap> G:=LocalAction(3,2,G);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(G);
```

### 2.1.3 LocalActionNC (for IsInt, IsInt, IsPermGroup)

▷ LocalActionNC( $d, k, F$ ) (operation)

**Returns:** the regular rooted tree group  $G$  as an object of the category IsLocalAction (??), without checking that  $F$  is indeed a subgroup of  $\text{Aut}(B_{d,k})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}_0$  and a group  $F \leq \text{Aut}(B_{d,k})$ .

### 2.1.4 LocalActionDegree (for IsLocalAction)

▷ LocalActionDegree( $F$ ) (attribute)

**Returns:** the degree  $d$  of the ball  $B_{d,k}$  that  $F$  is acting on.

The argument of this attribute is a local action  $F \leq \text{Aut}(B_{d,k})$  (see IsLocalAction (??)).

Example

```
gap> F:=PHI(4,AlternatingGroup(4));
Group([ (1,5,7)(2,4,8)(3,6,9)(10,11,12), (1,2,3)(4,7,10)(5,9,11)(6,8,12),
(1,2,3), (4,5,6), (7,8,9), (10,11,12) ])
gap> LocalActionDegree(F);
4
```

### 2.1.5 LocalActionRadius (for IsLocalAction)

▷ LocalActionRadius( $F$ ) (attribute)

**Returns:** the radius  $k$  of the ball  $B_{d,k}$  that  $F$  is acting on.

The argument of this attribute is a local action  $F \leq \text{Aut}(B_{d,k})$  (see IsLocalAction (??)).

Example

```
gap> F:=PHI(4,AlternatingGroup(4));
Group([ (1,5,7)(2,4,8)(3,6,9)(10,11,12), (1,2,3)(4,7,10)(5,9,11)(6,8,12),
(1,2,3), (4,5,6), (7,8,9), (10,11,12) ])
gap> LocalActionRadius(F);
2
```

### 2.1.6 LocalAction (for $r, d, k, \text{aut}, \text{addr}$ )

▷ `LocalAction( $r, d, k, \text{aut}, \text{addr}$ )` (operation)

**Returns:** the  $r$ -local action  $\sigma_r(\text{aut}, \text{addr})$  of the automorphism  $\text{aut}$  of  $B_{d,k}$  at the vertex represented by the address  $\text{addr}$ .

The arguments of this method are a radius  $r$ , a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , an automorphism  $\text{aut}$  of  $B_{d,k}$ , and an address  $\text{addr}$ .

Example

```
gap> a:=(1,3,5)(2,4,6);; a in AutBall(3,2);
true
gap> LocalAction(2,3,2,a,[]);
(1,3,5)(2,4,6)
gap> LocalAction(1,3,2,a,[]);
(1,2,3)
gap> LocalAction(1,3,2,a,[1]);
(1,2)
```

Example

```
gap> b:=Random(AutBall(3,4));
(1,20,4,17,2,19,3,18)(5,22,8,23,6,21,7,24)(9,10)(13,16,14,15)
gap> LocalAction(2,3,4,b,[3,1]);
(1,4)(2,3)
gap> LocalAction(3,3,4,b,[3,1]);
Error, the sum of input argument r=3 and the length of input argument
addr=[ 3, 1 ] must not exceed input argument k=4
```

### 2.1.7 Projection (for $F, r$ )

▷ `Projection( $F, r$ )` (operation)

**Returns:** the restriction of the projection map  $\text{Aut}(B_{d,k}) \rightarrow \text{Aut}(B_{d,r})$  to  $F$ .

The arguments of this method are a local action  $F \leq \text{Aut}(B_{d,k})$ , and a projection radius  $r \leq k$ .

Example

```
gap> F:=GAMMA(4,3,SymmetricGroup(3));
Group([ (1,16,19)(2,15,20)(3,13,18)(4,14,17)(5,10,23)(6,9,24)(7,12,22)
(8,11,21), (1,9)(2,10)(3,12)(4,11)(5,15)(6,16)(7,13)(8,14)(17,21)(18,22)
(19,24)(20,23) ])
gap> pr:=Projection(F,2);
<action homomorphism>
gap> a:=Random(F);; Image(pr,a);
(1,4,5)(2,3,6)
```

### 2.1.8 ImageOfProjection

▷ `ImageOfProjection( $F, r$ )` (function)

**Returns:** the local action  $\sigma_r(F, b) \leq \text{Aut}(B_{d,r})$ .

The arguments of this method are a local action  $F \leq \text{Aut}(B_{d,k})$ , and a projection radius  $r \leq k$ . This method uses `LocalAction` (2.1.6) on generators rather than `Projection` (2.1.7) on the group to compute the image.

## Example

```
gap> AutBall(3,2);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> ImageOfProjection(AutBall(3,2),1);
Group([ (), (), (1,2,3), (1,2) ])
```

## 2.2 Finite balls

The automorphism groups of the finite labelled balls  $B_{d,k}$  lie at the center of this package. The method `AutBall` (2.2.1) produces these automorphism groups as iterated wreath products. The result is a permutation group on the set of leaves of  $B_{d,k}$ .

### 2.2.1 AutBall

▷ `AutBall( $d, k$ )` (function)

**Returns:** the local action  $\text{Aut}(B_{d,k})$  as a permutation group of the  $d \cdot (d-1)^{k-1}$  leaves of  $B_{d,k}$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$  and a radius  $k \in \mathbb{N}_0$ .

## Example

```
gap> G:=AutBall(3,2);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> Size(G);
48
```

## 2.3 Addresses and leaves

The vertices at distance  $n$  from the center  $b$  of  $B_{d,k}$  are addressed as elements of the set

$$\Omega^{(n)} := \{(\omega_1, \dots, \omega_n) \in \Omega^n \mid \forall l \in \{1, \dots, n-1\} : \omega_l \neq \omega_{l+1}\},$$

i.e. as lists of length  $n$  of elements from  $[1..d]$  such that no two consecutive entries are equal. They are ordered according to the lexicographic order on  $\Omega^{(n)}$ . The center  $b$  itself is addressed by the empty list  $[]$ . Note that the leaves of  $B_{d,k}$  correspond to elements of  $\Omega^{(k)}$ .

### 2.3.1 BallAddresses

▷ `BallAddresses( $d, k$ )` (function)

**Returns:** a list of all addresses of vertices in  $B_{d,k}$  in ascending order with respect to length, lexicographically ordered within each level. See `AddressOfLeaf` (2.3.3) and `LeafOfAddress` (2.3.4) for the correspondence between the leaves of  $B_{d,k}$  and addresses of length  $k$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$  and a radius  $k \in \mathbb{N}_0$ .

## Example

```
gap> BallAddresses(3,1);
[ [], [ 1 ], [ 2 ], [ 3 ] ]
gap> BallAddresses(3,2);
[ [], [ 1 ], [ 2 ], [ 3 ], [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 2, 3 ],
[ 3, 1 ], [ 3, 2 ] ]
```



### 2.3.2 LeafAddresses

▷ `LeafAddresses( $d$ ,  $k$ )`

(function)

**Returns:** a list of addresses of the leaves of  $B_{d,k}$  in lexicographic order.

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$  and a radius  $k \in \mathbb{N}_0$ .

Example

```
gap> LeafAddresses(3,2);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 2, 3 ], [ 3, 1 ], [ 3, 2 ] ]
```

### 2.3.3 AddressOfLeaf

▷ `AddressOfLeaf( $d$ ,  $k$ ,  $lf$ )`

(function)

**Returns:** the address of the leaf  $lf$  of  $B_{d,k}$  with respect to the lexicographic order.

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , and a leaf  $lf$  of  $B_{d,k}$ .

Example

```
gap> AddressOfLeaf(3,2,1);
[ 1, 2 ]
gap> AddressOfLeaf(3,3,1);
[ 1, 2, 1 ]
```

### 2.3.4 LeafOfAddress

▷ `LeafOfAddress( $d$ ,  $k$ ,  $addr$ )`

(function)

**Returns:** the smallest leaf (integer) whose address has  $addr$  as a prefix.

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , and an address  $addr$ .

Example

```
gap> LeafOfAddress(3,2,[1,2]);
1
gap> LeafOfAddress(3,2,[3]);
5
gap> LeafOfAddress(3,2,[]);
1
```

### 2.3.5 ImageAddress

▷ `ImageAddress( $d$ ,  $k$ ,  $aut$ ,  $addr$ )`

(function)

**Returns:** the address of the image of the vertex represented by the address  $addr$  under the automorphism  $aut$  of  $B_{d,k}$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , an automorphism  $aut$  of  $B_{d,k}$ , and an address  $addr$ .

Example

```
gap> ImageAddress(3,2,(1,2),[1,2]);
[ 1, 3 ]
gap> ImageAddress(3,2,(1,2),[1]);
[ 1 ]
```

### 2.3.6 ComposeAddresses

▷ `ComposeAddresses(addr1, addr2)` (function)

**Returns:** the concatenation of the addresses *addr1* and *addr2* with reduction as per [Tor20, Section 3.2].

The arguments of this method are two addresses *addr1* and *addr2*.

Example

```
gap> ComposeAddresses([1,3],[2,1]);  
[ 1, 3, 2, 1 ]  
gap> ComposeAddresses([1,3,2],[2,1]);  
[ 1, 3, 1 ]
```

## Chapter 3

# Compatibility

### 3.1 The compatibility condition (C)

A subgroup  $F \leq \text{Aut}(B_{d,k})$  satisfies the compatibility condition (C) if and only if  $U_k(F)$  is locally action isomorphic to  $F$ , see [Tor20, Proposition 3.8]. The term *compatibility* comes from the following translation of this condition into properties of the  $(k-1)$ -local actions of elements of  $F$ : The group  $F$  satisfies (C) if and only if

$$\forall \alpha \in F \forall \omega \in \Omega \exists \beta \in F : \sigma_{k-1}(\alpha, b) = \sigma_{k-1}(\beta, b_\omega), \sigma_{k-1}(\alpha, b_\omega) = \sigma_{k-1}(\beta, b).$$

### 3.2 Compatible elements

This section is concerned with testing compatibility of two given elements (see `AreCompatibleBallElements` (3.2.1)) and finding an/all elements that is/are compatible with a given one (see `CompatibleBallElement` (3.2.2), `CompatibilitySet` (3.2.3)).

#### 3.2.1 AreCompatibleBallElements

▷ `AreCompatibleBallElements(d, k, aut1, aut2, dir)` (function)

**Returns:** true if `aut1` and `aut2` are compatible with each other in direction `dir`, and false otherwise.

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , two automorphisms `aut1`, `aut2`  $\in \text{Aut}(B_{d,k})$ , and a direction `dir`  $\in [1..d]$ .

Example

```
gap> AreCompatibleBallElements(3,1,(1,2),(1,2,3),1);
true
gap> AreCompatibleBallElements(3,1,(1,2),(1,2,3),2);
false
```

Example

```
gap> a:=(1,3,5)(2,4,6);; a in AutBall(3,2);
true
gap> LocalAction(1,3,2,a,[]); LocalAction(1,3,2,a,[1]);
(1,2,3)
```

```

(1,2)
gap> b:=(1,4)(2,3);; b in AutBall(3,2);
true
gap> LocalAction(1,3,2,b,[]); LocalAction(1,3,2,b,[1]);
(1,2)
(1,2,3)

gap> AreCompatibleBallElements(3,2,a,b,1);
true
gap> AreCompatibleBallElements(3,2,a,b,3);
false

```

### 3.2.2 CompatibleBallElement

▷ `CompatibleBallElement( $F$ ,  $aut$ ,  $dir$ )` (function)

**Returns:** an element of  $F$  that is compatible with  $aut$  in direction  $dir$  if one exists, and fail otherwise.

The arguments of this method are a local action  $F \leq \text{Aut}(B_{d,k})$ , an element  $aut \in F$ , and a direction  $dir \in [1..d]$ .

Example

```

gap> a:=Random(AutBall(5,1)); dir:=Random([1..5]);
(1,3,2,5)
4
gap> CompatibleBallElement(AutBall(5,1),a,dir);
(1,3,2,5)

```

Example

```

gap> a:=(1,3,5)(2,4,6);; a in AutBall(3,2);
true
gap> CompatibleBallElement(AutBall(3,2),a,1);
(1,4,2,3)

```

### 3.2.3 CompatibilitySet

▷ `CompatibilitySet( $F$ ,  $aut$ ,  $dir$ )` (operation)

▷ `CompatibilitySet( $F$ ,  $aut$ ,  $dirs$ )` (operation)

**for the arguments  $F$ ,  $aut$ ,  $dir$**

Returns: the list of elements of  $F$  that are compatible with  $aut$  in direction  $dir$ .

The arguments of this method are a local action  $F$  of  $\leq \text{Aut}(B_{d,k})$ , an automorphism  $aut \in F$ , and a direction  $dir \in [1..d]$ .

**for the arguments  $F$ ,  $aut$ ,  $dirs$**

Returns: the list of elements of  $F$  that are compatible with  $aut$  in all directions of  $dirs$ .

The arguments of this method are a local action  $F$  of  $\leq \text{Aut}(B_{d,k})$ , an automorphism  $aut \in F$ , and a sublist of directions  $dirs \subseteq [1..d]$ .

## Example

```

gap> F:=LocalAction(4,1,TransitiveGroup(4,3));
D(4)
gap> G:=LocalAction(4,1,SymmetricGroup(4));
Sym( [ 1 .. 4 ] )
gap> CompatibilitySet(G,aut,1);
RightCoset(Sym( [ 2 .. 4 ] ),(1,3))
gap> CompatibilitySet(F,aut,1);
RightCoset(Group([ (2,4) ]),(1,3))
gap> CompatibilitySet(F,aut,[1,3]);
RightCoset(Group([ (2,4) ]),(1,3))
gap> CompatibilitySet(F,aut,[1,2]);
RightCoset(Group(()),(1,3))

```

### 3.2.4 AssembleAutomorphism

▷ `AssembleAutomorphism( $d$ ,  $k$ ,  $auts$ )` (function)

**Returns:** the automorphism  $(aut, (auts[i])_{i=1}^d)$  of  $B_{d,k+1}$ , where  $aut$  is implicit in  $(auts[i])_{i=1}^d$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , and a list  $auts$  of  $d$  automorphisms  $(auts[i])_{i=1}^d$  of  $B_{d,k}$  which comes from an element  $(aut, (auts[i])_{i=1}^d)$  of  $Aut(B_{d,k+1})$ .

## Example

```

gap> aut:=Random(AutBall(3,2));
(1,2)(3,6)(4,5)
gap> auts:=[];
gap> for i in [1..3] do auts[i]:=CompatibleBallElement(3,2,AutBall(3,2),aut,i); od;
gap> auts;
[ (1,2)(3,5)(4,6), (1,3,5)(2,4,6), (1,5,3)(2,6,4) ]
gap> a:=AssembleAutomorphism(3,2,auts);
(1,3)(2,4)(5,11)(6,12)(7,9)(8,10)
gap> a in AutBall(3,3);
true
gap> LocalAction(2,3,3,a,[]);
(1,2)(3,6)(4,5)

```

## 3.3 Compatible subgroups

Using the methods of Section 3.2, this section provides methods to test groups for the compatibility condition and search for compatible subgroups inside a given group, e.g.  $Aut(B_{d,k})$ , or with a certain image under some projection.

### 3.3.1 MaximalCompatibleSubgroup (for IsLocalAction)

▷ `MaximalCompatibleSubgroup( $F$ )` (attribute)

**Returns:** The local action  $C(F) \leq Aut(B_{d,k})$ , which is the maximal compatible subgroup of  $F$ .

The argument of this attribute is a local action  $F \leq Aut(B_{d,k})$  (see `IsLocalAction(??)`).

## Example

```

gap> F:=LocalAction(3,1,Group((1,2)));
Group([ (1,2) ])
gap> MaximalCompatibleSubgroup(F);

```

```

Group([ (1,2) ])
gap> G:=LocalAction(3,2,Group((1,2)));
Group([ (1,2) ])
gap> MaximalCompatibleSubgroup(G);
Group(())

```

### 3.3.2 SatisfiesC (for IsLocalAction)

▷ `SatisfiesC(F)` (property)

**Returns:** true if  $F$  satisfies the compatibility condition (C), and false otherwise.

The argument of this property is a local action  $F \leq \text{Aut}(B_{d,k})$  (see `IsLocalAction(??)`).

Example

```

gap> D:=DELTA(3,SymmetricGroup(3));
Group([ (1,3,6)(2,4,5), (1,3)(2,4), (1,2)(3,4)(5,6) ])
gap> SatisfiesC(D);
true

```

### 3.3.3 CompatibleSubgroups

▷ `CompatibleSubgroups(F)` (function)

**Returns:** the list of all compatible subgroups of  $F$ .

The argument of this method is a local action  $F \leq \text{Aut}(B_{d,k})$ . This method calls `AllSubgroups` on  $F$  and is therefore slow. Use for instructional purposes on small examples only, and use `ConjugacyClassRepsCompatibleSubgroups(??)` or `ConjugacyClassRepsCompatibleSubgroupsWithProjection(??)` for computations.

Example

```

gap> G:=GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> list:=CompatibleSubgroups(G);
[ Group(()), Group([ (1,2)(3,5)(4,6) ]), Group([ (1,3)(2,4)(5,6) ]),
  Group([ (1,6)(2,5)(3,4) ]), Group([ (1,4,5)(2,3,6) ]), Group([ (1,4,5)
    (2,3,6), (1,3)(2,4)(5,6) ]) ]
gap> Size(list);
6
gap> Size(AllSubgroups(SymmetricGroup(3)));
6

```

### 3.3.4 ConjugacyClassRepsCompatibleSubgroups (for IsLocalAction)

▷ `ConjugacyClassRepsCompatibleSubgroups(F)` (attribute)

**Returns:** a list of compatible representatives of conjugacy classes of  $F$  that contain a compatible subgroup.

The argument of this method is a local action  $F$  of  $\text{Aut}(B_{d,k})$ .

Example

```

gap> ConjugacyClassRepsCompatibleSubgroups(AutBall(3,2));
[ Group(()), Group([ (1,2)(3,5)(4,6) ]), Group([ (1,4,5)(2,3,6) ]),
  Group([ (3,5)(4,6), (1,2) ]), Group([ (1,2)(3,5)(4,6), (1,3,6)
    (2,4,5) ]), Group([ (3,5)(4,6), (1,3,5)(2,4,6), (1,2)(3,4)(5,6) ]),

```

```

Group([ (1,2)(3,5)(4,6), (1,3,5)(2,4,6), (1,2)(5,6), (1,2)(3,4) ]),
Group([ (3,5)(4,6), (1,3,5)(2,4,6), (1,2)(5,6), (1,2)(3,4) ]),
Group([ (5,6), (3,4), (1,2), (1,3,5)(2,4,6), (3,5)(4,6) ]) ]

```

### 3.3.5 ConjugacyClassRepsCompatibleGroupsWithProjection

▷ `ConjugacyClassRepsCompatibleGroupsWithProjection(l, F)` (function)

**Returns:** a list of compatible representatives of conjugacy classes of  $\text{Aut}(B_{d,l})$  that contain a compatible group which projects to  $F \leq \text{Aut}(B_{d,r})$ .

The arguments of this method are a radius  $l \in \mathbb{N}$ , and a local action  $F \leq \text{Aut}(B_{d,k})$  for some  $k \leq l$ .

Example

```

gap> S3:=LocalAction(3,1,SymmetricGroup(3));
Sym( [ 1 .. 3 ] )
gap> ConjugacyClassRepsCompatibleGroupsWithProjection(2,S3);
[ Group([ (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (1,2)(3,4)(5,6), (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5,4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5)(4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (5,6), (3,5,4,6) ]) ]
gap> A3:=LocalAction(3,1,AlternatingGroup(3));
Alt( [ 1 .. 3 ] )
gap> ConjugacyClassRepsCompatibleGroupsWithProjection(2,A3);
[ Group([ (1,4,5)(2,3,6) ]) ]

```

Example

```

gap> F:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(F);;
gap> H1:=PI(2,3,F,rho,[0,1]);;
gap> H2:=PI(2,3,F,rho,[1]);;
gap> Size(ConjugacyClassRepsCompatibleGroupsWithProjection(3,H1));
2
gap> Size(ConjugacyClassRepsCompatibleGroupsWithProjection(3,H2));
4

```

# Chapter 4

## Examples

Several classes of examples of subgroups of  $\text{Aut}(B_{d,k})$  that satisfy (C) and or (D) are constructed in [Tor20] and implemented in this section. For a given permutation group  $F \leq S_d$ , there are always the three local actions  $\Gamma(F)$ ,  $\Delta(F)$  and  $\Phi(F)$  on  $\text{Aut}(B_{d,2})$  that project onto  $F$ . For some  $F$ , these are all distinct and yield all universal groups that have  $F$  as their 1-local action, see [Tor20, Theorem 3.32]. More examples arise in particular when either point stabilizers in  $F$  are not simple,  $F$  preserves a partition, or  $F$  is not perfect.

### 4.1 Discrete groups

Here, we implement the local actions  $\Gamma(F), \Delta(F) \leq \text{Aut}(B_{d,2})$ , both of which satisfy both (C) and (D), see [Tor20, Section 3.4.1].

#### 4.1.1 gamma

- ▷ `gamma(d, a)` (operation)
- ▷ `gamma(l, d, a)` (operation)
- ▷ `gamma(l, d, s, addr)` (operation)
- ▷ `gamma(d, k, aut, z)` (operation)

**for the arguments  $d, a$**

Returns: the automorphism  $\gamma(a) = (a, (a)_{\omega \in \Omega}) \in \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$  and a permutation  $a \in S_d$ .

**for the arguments  $l, d, a$**

Returns: the automorphism  $\gamma^l(a) \in \text{Aut}(B_{d,l})$  all of whose 1-local actions are given by  $a$ .

The arguments of this method are a radius  $l \in \mathbb{N}$ , a degree  $d \in \mathbb{N}_{\geq 3}$  and a permutation  $a \in S_d$ .

**for the arguments  $l, d, s, addr$**

Returns: the automorphism of  $B_{d,k}$  whose 1-local actions are given by  $s$  at vertices whose address has `addr` as a prefix and are trivial elsewhere.

The arguments of this method are a radius  $l \in \mathbb{N}$ , a degree  $d \in \mathbb{N}_{\geq 3}$ , a permutation  $s \in S_d$  and an address `addr` of a vertex in  $B_{d,l}$  whose last entry is fixed by  $s$ .



**for the arguments**  $d, k, aut, z$

Returns: the automorphism  $\gamma_z(aut) = (aut, (z(aut, \omega))_{\omega \in \Omega}) \in \text{Aut}(B_{d,k+1})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , an automorphism  $aut$  of  $B_{d,k}$ , and an involutive compatibility cocycle  $z$  of a subgroup of  $\text{Aut}(B_{d,k})$  that contains  $aut$  (see `InvolutiveCompatibilityCocycle(??)`).

Example

```
gap> gamma(3, (1,2));
(1,3)(2,4)(5,6)
```

Example

```
gap> gamma(2,3, (1,2));
(1,3)(2,4)(5,6)
gap> gamma(3,3, (1,2));
(1,5)(2,6)(3,8)(4,7)(9,11)(10,12)
```

Example

```
gap> gamma(3,3, (1,2), [1,3]);
(3,4)
gap> gamma(3,3, (1,2), []);
(1,5)(2,6)(3,8)(4,7)(9,11)(10,12)
```

Example

```
gap> S3:=SymmetricGroup(3);;
gap> z1:=AllInvolutiveCompatibilityCocycles(3,1,S3)[1];;
gap> gamma(3,1, (1,2), z1);
(1,4)(2,3)(5,6)
gap> z3:=AllInvolutiveCompatibilityCocycles(3,1,S3)[3];;
gap> gamma(3,1, (1,2), z3);
(1,3)(2,4)(5,6)
```

#### 4.1.2 GAMMA

- ▷ `GAMMA( $d, F$ )` (operation)
- ▷ `GAMMA( $l, d, F$ )` (operation)
- ▷ `GAMMA( $F, z$ )` (operation)

**for the arguments**  $d, F$

Returns: the local action  $\Gamma(F) = \{(a, (a)_\omega) \mid a \in F\} \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , and a subgroup  $F$  of  $S_d$ .

**for the arguments**  $l, d, F$

Returns: the group  $\Gamma^l(F) \leq \text{Aut}(B_{d,l})$ .

The arguments of this method are a radius  $l \in \mathbb{N}$ , a degree  $d \in \mathbb{N}_{\geq 3}$ , and a subgroup  $F$  of  $S_d$ .

**for the arguments**  $d, k, F, z$

Returns: the group  $\Gamma_z(F) = \{(a, (z(a, \omega))_{\omega \in \Omega}) \mid a \in F\} \leq \text{Aut}(B_{d,k+1})$ .

The arguments of this method are a local action  $F \leq \text{Aut}(B_{d,k})$  and an involutive compatibility cocycle  $z$  of  $F$  (see `InvolutiveCompatibilityCocycle(??)`).

Example

```
gap> F:=TransitiveGroup(4,3);;
gap> GAMMA(4,F);
Group([ (1,5,9,10)(2,6,7,11)(3,4,8,12), (1,8)(2,7)(3,9)(4,5)(10,12) ])
```

Example

```
gap> GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> GAMMA(2,3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> GAMMA(3,3,SymmetricGroup(3));
Group([ (1,8,10)(2,7,9)(3,5,12)(4,6,11), (1,5)(2,6)(3,8)(4,7)(9,11)(10,12) ])
```

Example

```
gap> F:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(F);;
gap> H:=PI(2,3,F,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(3,2,H);;
gap> GAMMA(H,z);
Group([ (), (), (1,8,9)(2,7,10)(3,5,11)(4,6,12), (1,8,9)(2,7,10)(3,5,11)(4,6,12),
(1,7,3,5)(2,8,4,6)(9,11,10,12) ])
```

### 4.1.3 DELTA

- ▷ DELTA( $d, F$ ) (operation)
- ▷ DELTA( $d, F, C$ ) (operation)

**for the arguments  $d, F$**

Returns: the group  $\Delta(F) \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , and a *transitive* subgroup  $F$  of  $S_d$ .

**for the arguments  $d, F, C$**

Returns: the group  $\Delta(F, C) \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a *transitive* subgroup  $F$  of  $S_d$ , and a central subgroup  $C$  of the stabilizer  $F_1$  of 1 in  $F$ .

Example

```
gap> F:=SymmetricGroup(3);;
gap> D:=DELTA(3,F);
Group([ (1,3,6)(2,4,5), (1,3)(2,4), (1,2)(3,4)(5,6) ])
gap> F1:=Stabilizer(F,1);;
gap> D1:=DELTA(3,F,F1);
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6), (1,2)(3,4)(5,6) ])
gap> D=D1;
false
gap> G:=AutBall(3,2);;
gap> D^G=D1^G;
true
```

Example

```
gap> F:=PrimitiveGroup(5,3);
AGL(1, 5)
```

```

gap> F1:=Stabilizer(F,1);
Group([ (2,3,4,5) ])
gap> C:=Group((2,4)(3,5));
Group([ (2,4)(3,5) ])
gap> Index(F1,C);
2
gap> Index(DELTA(5,F,F1),DELTA(5,F,C));
2

```

## 4.2 Maximal extensions

For any  $F \leq \text{Aut}(B_{d,k})$  that satisfies (C), the group  $\Phi(F) \leq \text{Aut}(B_{d,k+1})$  is the maximal extension of  $F$  that satisfies (C) as well. It stems from the action of  $U_k(F)$  on balls of radius  $k+1$  in  $T_d$ .

### 4.2.1 PHI

- ▷  $\text{PHI}(d, F)$  (operation)
- ▷  $\text{PHI}(F)$  (operation)
- ▷  $\text{PHI}(1, F)$  (operation)

**for the arguments  $d, F$**

Returns: the group  $\Phi(F) = \{(a, (a_\omega)_\omega) \mid a \in F, \forall \omega \in \Omega : a_\omega \in C_F(a, \omega)\} \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$  and a permutation group  $F \leq S_d$ .

**for the arguments  $d, k, F$**

Returns: the group  $\Phi_k(F) = \{(a, (a_\omega)_\omega) \mid a \in F, \forall \omega \in \Omega : a_\omega \in C_F(a, \omega)\} \leq \text{Aut}(B_{d,k+1})$ .

The argument of this method is a local action  $F \leq \text{Aut}(B_{d,k})$ .

**for the arguments  $1, d, k, F$**

Returns: the group  $\Phi^l(F) = \Phi_{l-1} \circ \dots \circ \Phi_{k+1} \circ \Phi_k(F) \leq \text{Aut}(B_{d,l})$ .

The arguments of this method are a radius  $1 \in \mathbb{N}$  and a local action  $F \leq \text{Aut}(B_{d,k})$ .

Example

```

gap> PHI(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6), (1,2), (3,4), (5,6) ])
gap> last=AutBall(3,2);
true
gap> PHI(3,AlternatingGroup(3));
Group([ (1,4,5)(2,3,6) ])
gap> last=GAMMA(3,AlternatingGroup(3));
true

```

Example

```

gap> S3:=LocalAction(3,1,SymmetricGroup(3));;
gap> groups:=ConjugacyClassRepsCompatibleSubgroupsWithProjection(2,S3);
[ Group([ (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (1,2)(3,4)(5,6), (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5,4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5)(4,6) ]),

```

```

Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (5,6), (3,5,4,6) ]) ]
gap> for G in groups do Print(Size(G),",",Size(PHI(G)),"\n"); od;
6,6
12,12
24,192
24,192
48,3072

```

#### Example

```

gap> PHI(3,LocalAction(4,1,SymmetricGroup(4)));
<permutation group with 34 generators>
gap> last=AutBall(4,3);
true

```

#### Example

```

ap> rho:=SignHomomorphism(SymmetricGroup(3));;
gap> F:=PI(2,3,SymmetricGroup(3),rho,[1]);; Size(F);
24
gap> P:=PHI(4,F);; Size(P);
12288
gap> IsSubgroup(AutBall(3,4),P);
true
gap> SatisfiesC(P);
true

```

## 4.3 Normal subgroups and partitions

When point stabilizers in  $F \leq S_d$  are not simple, or  $F$  preserves a partition, more universal groups can be constructed as follows.

### 4.3.1 PHI

- ▷  $\text{PHI}(d, F, N)$  (operation)
- ▷  $\text{PHI}(d, F, P)$  (operation)
- ▷  $\text{PHI}(F, P)$  (operation)

**for the arguments  $d, F, N$**

Returns: the group  $\Phi(F, N) \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a *transitive* permutation group  $F \leq S_d$  and a normal subgroup  $N$  of the stabilizer  $F_1$  of 1 in  $F$ .

**for the arguments  $d, F, P$**

Returns: the group  $\Phi(F, P) = \{(a, (a_\omega)_\omega) \mid a \in F, a_\omega \in C_F(a, \omega) \text{ constant w.r.t. } P\} \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$  and a permutation group  $F \leq S_d$  and a partition  $P$  of  $[1 \dots d]$  preserved by  $F$ .

**for the arguments  $d, k, F, P$**

Returns: the group  $\Phi_k(F, P) = \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F, \alpha_\omega \in C_F(\alpha, \omega) \text{ constant w.r.t. } P\} \leq \text{Aut}(B_{d,k+1})$ .

The arguments of this method are a local action  $F \leq \text{Aut}(B_{d,k})$  and a partition  $P$  of  $[1..d]$  preserved by  $\pi F \leq S_d$ . This method assumes that all compatibility sets with respect to a partition element are non-empty and that all compatibility sets of the identity with respect to a partition element are non-trivial.

Example

```
gap> F:=SymmetricGroup(4);;
gap> F1:=Stabilizer(P,1);
Sym( [ 2 .. 4 ] )
gap> grps:=NormalSubgroups(F1);
[ Sym( [ 2 .. 4 ] ), Alt( [ 2 .. 4 ] ), Group(()) ]
gap> N:=grps[2];
Alt( [ 2 .. 4 ] )
gap> PHI(4,F,N);
Group([ (1,5,9,10)(2,6,7,11)(3,4,8,12), (1,4)(2,5)(3,6)(7,8)(10,11),
(1,2,3) ])
gap> Index(F1,N);
2
gap> Index(PHI(4,F,F1),PHI(4,F,N));
16
```

Example

```
gap> F:=TransitiveGroup(4,3);
D(4)
gap> P:=Blocks(F,[1..4]);
[ [ 1, 3 ], [ 2, 4 ] ]
gap> G:=PHI(4,F,P);
Group([ (1,5,9,10)(2,6,7,11)(3,4,8,12), (1,8)(2,7)(3,9)(4,5)(10,12), (1,3)
(8,9), (4,5)(10,12) ])
gap> aut:=Random(G);
(1,5,9,10)(2,6,7,11)(3,4,8,12)
gap> LocalAction(1,4,2,a,[1]); LocalAction(1,4,2,a,[3]);
(1,2,3,4)
(1,2,3,4)
gap> LocalAction(1,4,2,a,[2]); LocalAction(1,4,2,a,[4]);
(1,4)(2,3)
(1,4)(2,3)
```

Example

```
gap> H:=TransitiveGroup(4,3);
D(4)
gap> P:=Blocks(H,[1..4]);
[ [ 1, 3 ], [ 2, 4 ] ]
gap> F:=PHI(4,H,P);;
gap> G:=PHI(F,P);
<permutation group with 5 generators>
gap> SatisfiesC(G);
true
```

## 4.4 Abelian quotients

When a permutation group  $F \leq S_d$  is not perfect, i.e. it admits an abelian quotient  $\rho : F \twoheadrightarrow A$ , more universal groups can be constructed by imposing restrictions of the form  $\prod_{r \in R} \prod_{x \in S(b,r)} \rho(\sigma_1(\alpha, x)) = 1$  on elements  $\alpha \in \Phi^k(F) \leq \text{Aut}(B_{d,k})$ .

#### 4.4.1 SignHomomorphism

▷ `SignHomomorphism( $F$ )` (function)

**Returns:** the sign homomorphism from  $F$  to  $S_2$ .

The argument of this method is a permutation group  $F \leq S_d$ . This method can be used as an example for the argument *rho* in the methods `SpheresProduct` (4.4.3) and `PI` (4.4.4).

Example

```
gap> F:=SymmetricGroup(3);;
gap> sign:=SignHomomorphism(F);
MappingByFunction( Sym( [ 1 .. 3 ] ), Sym( [ 1 .. 2 ] ), function( g ) ... end )
gap> Image(sign,(2,3));
(1,2)
gap> Image(sign,(1,2,3));
()
```

#### 4.4.2 AbelianizationHomomorphism

▷ `AbelianizationHomomorphism( $F$ )` (function)

**Returns:** the homomorphism from  $F$  to  $F/[F, F]$ .

The argument of this method is a permutation group  $F \leq S_d$ . This method can be used as an example for the argument *rho* in the methods `SpheresProduct` (4.4.3) and `PI` (4.4.4).

Example

```
gap> F:=PrimitiveGroup(5,3);
AGL(1, 5)
gap> ab:=AbelianizationHomomorphism(PrimitiveGroup(5,3));
[ (2,3,4,5), (1,2,3,5,4) ] -> [ f1, <identity> of ... ]
gap> Elements(Range(ab));
[ <identity> of ..., f1, f2, f1*f2 ]
gap> StructureDescription(Range(ab));
"C4"
```

#### 4.4.3 SpheresProduct

▷ `SpheresProduct( $d, k, aut, rho, R$ )` (function)

**Returns:** the product  $\prod_{r \in R} \prod_{x \in S(b,r)} rho(\sigma_1(aut, x)) \in \text{im}(rho)$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a radius  $k \in \mathbb{N}$ , an automorphism *aut* of  $B_{d,k}$  all of whose 1-local actions are in the domain of the homomorphism *rho* from a subgroup of  $S_d$  to an abelian group, and a sublist *R* of  $[0..k-1]$ . This method is used in the implementation of `PI` (4.4.4).

Example

```
gap> rho:=SignHomomorphism(SymmetricGroup(3));;
gap> SpheresProduct(3,2,gamma(2,3,(1,2)),rho,[0]);
(1,2)
gap> SpheresProduct(3,2,gamma(2,3,(1,2)),rho,[0,1]);
()
```

Example

```
gap> F:=PrimitiveGroup(5,3);
AGL(1, 5)
gap> rho:=AbelianizationHomomorphism(F);;
gap> Elements(Range(rho));
```

```

[ <identity> of ..., f1, f2, f1*f2 ]
gap> StructureDescription(Range(rho));
"C4"
gap> aut:=Random(F);
(1,2,4,5)
gap> SpheresProduct(5,3,gamma(3,5,aut),rho,[2]);
<identity> of ...
gap> SpheresProduct(5,3,gamma(3,5,aut),rho,[1,2]);
f1*f2
gap> SpheresProduct(5,3,gamma(3,5,aut),rho,[0,1,2]);
f2

```

#### 4.4.4 PI

▷  $\text{PI}(l, d, F, \text{rho}, R)$  (function)

**Returns:** the group  $\Pi^l(F, \text{rho}, R) = \{\alpha \in \Phi^l(F) \mid \prod_{r \in R} \prod_{x \in S(b,r)} \text{rho}(\sigma_1(\alpha, x)) = 1\} \leq \text{Aut}(B_{d,l})$ .

The arguments of this method are a degree  $l \in \mathbb{N}_{\geq 2}$ , a radius  $d \in \mathbb{N}_{\geq 3}$ , a permutation group  $F \leq S_d$ , a homomorphism  $\rho$  from  $F$  to an abelian group that is surjective on every point stabilizer in  $F$ , and a non-empty, non-zero subset  $R$  of  $[0..l-1]$  that contains  $l-1$ .

Example

```

gap> F:=LocalAction(5,1,PrimitiveGroup(5,3));
AGL(1, 5)
gap> rho1:=AbelianizationHomomorphism(F);;
gap> rho2:=SignHomomorphism(F);;
gap> PI(3,5,F,rho1,[0,1,2]);
<permutation group with 4 generators>
gap> Index(PHI(3,F),last);
4
gap> PI(3,5,F,rho2,[0,1,2]);
<permutation group with 6 generators>
gap> Index(PHI(3,F),last);
2

```

## 4.5 Semidirect products

When a subgroup  $F \leq \text{Aut}(B_{d,k})$  satisfies (C) and admits an involutive compatibility cocycle  $z$  (which is automatic when  $k = 1$ ) one can characterise the kernels  $K \leq \Phi_k(F) \cap \ker(\pi_k)$  that fit into a  $z$ -split exact sequence  $1 \rightarrow K \rightarrow \Sigma(F, K) \rightarrow F \rightarrow 1$  for some subgroup  $\Sigma(F, K) \leq \text{Aut}(B_{d,k+1})$  that satisfies (C). This characterisation is implemented in this section.

### 4.5.1 CompatibleKernels

▷  $\text{CompatibleKernels}(d, F)$  (operation)

▷  $\text{CompatibleKernels}(F, z)$  (operation)

for the arguments  $d, F$

Returns: the list of kernels  $K \leq \prod_{\omega \in \Omega} F_\omega \cong \ker \pi \leq \text{Aut}(B_{d,2})$  that are preserved by the action  $F \curvearrowright \prod_{\omega \in \Omega} F_\omega$ ,  $a \cdot (a_\omega)_\omega := (aa_{a^{-1}\omega}a^{-1})_\omega$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , and a permutation group  $F \leq S_d$ . The kernels output by this method are compatible with  $F$  with respect to the standard cocycle (see `InvolutiveCompatibilityCocycle (??)`) and can be used in the method `SIGMA (4.5.2)`.

**for the arguments**  $d, k, F, z$

Returns: the list of kernels  $K \leq \Phi_k(F) \cap \ker(\pi_k) \leq \text{Aut}(B_{d,k+1})$  that are normalized by  $\Gamma_z(F)$  and such that for all  $k \in K$  and  $\omega \in \Omega$  there is  $k_\omega \in K$  with  $\text{pr}_\omega k_\omega = z(\text{pr}_\omega k, \omega)^{-1}$ .

The arguments of this method are a local action  $F \leq \text{Aut}(B_{d,k})$  that satisfies (C) and an involutive compatibility cocycle  $z$  of  $F$  (see `InvolutiveCompatibilityCocycle (??)`). It can be used in the method `SIGMA (4.5.2)`.

Example

```
gap> CompatibleKernels(3, SymmetricGroup(3));
[ Group(), Group([ (1,2)(3,4)(5,6) ]), Group([ (3,4)(5,6), (1,2)(5,6) ]),
  Group([ (5,6), (3,4), (1,2) ]) ]
```

Example

```
gap> P:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(P);;
gap> F:=PI(2,3,P,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(F);;
[ Group(), Group([ (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) ]),
  Group([ (1,2)(3,4)(5,6)(7,8), (5,6)(7,8)(9,10)(11,12) ]),
  Group([ (5,6)(7,8), (1,2)(3,4), (9,10)(11,12) ]) ]
```

## 4.5.2 SIGMA

- ▷ `SIGMA(d, F, K)` (operation)
- ▷ `SIGMA(F, K, z)` (operation)

**for the arguments**  $d, F, K$

Returns: the semidirect product  $\Sigma(F, K) \leq \text{Aut}(B_{d,2})$ .

The arguments of this method are a degree  $d \in \mathbb{N}_{\geq 3}$ , a subgroup  $F$  of  $S_d$  and a compatible kernel  $K$  for  $F$  (see `CompatibleKernels (4.5.1)`).

**for the arguments**  $d, k, F, K, z$

Returns: the semidirect product  $\Sigma_z(F, K) \leq \text{Aut}(B_{d,k+1})$ .

The arguments of this method are a local action  $F$  of  $\text{Aut}(B_{d,k})$  that satisfies (C) and a kernel  $K$  that is compatible for  $F$  with respect to the involutive compatibility cocycle  $z$  (see `InvolutiveCompatibilityCocycle (??)` and `CompatibleKernels (4.5.1)`) of  $F$ .

Example

```
gap> S3:=SymmetricGroup(3);;
gap> kernels:=CompatibleKernels(3,S3);
[ Group(), Group([ (1,2)(3,4)(5,6) ]), Group([ (3,4)(5,6), (1,2)(5,6) ]),
  Group([ (5,6), (3,4), (1,2) ]) ]
gap> for K in kernels do Print(Size(SIGMA(3,S3,K)),"\n"); od;
6
12
24
48
```



## Example

```
gap> P:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(P);;
gap> F:=PI(2,3,P,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(F);;
gap> kernels:=CompatibleKernels(F,z);
[ Group(), Group([ (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) ]),
  Group([ (1,2)(3,4)(5,6)(7,8), (5,6)(7,8)(9,10)(11,12) ]),
  Group([ (5,6)(7,8), (1,2)(3,4), (9,10)(11,12) ]) ]
gap> for K in kernels do Print(Size(SIGMA(F,K,z)),"\n"); od;
24
48
96
192
```

# Chapter 5

## Discreteness

This chapter contains functions that are related to the discreteness property (D) presented in Proposition 3.12 of [Tor20].

### 5.1 The discreteness condition (D)

Said proposition shows that for a given  $F \leq \text{Aut}(B_{d,k})$  the group  $U_k(F)$  is discrete if and only if the maximal compatible subgroup  $C(F)$  of  $F$  satisfies condition (D):

$$\forall \omega \in \Omega : F_{T_\omega} = \{\text{id}\},$$

where  $T_\omega$  is the  $k-1$ -neighbourhood of the the edge  $(b, b_\omega)$  inside  $B_{d,k}$ . In other words,  $F$  satisfies (D) if and only if the compatibility set  $C_F(\text{id}, \omega) = \{\text{id}\}$ . We distinguish between  $F$  satisfying condition (D) and  $U_k(F)$  being discrete with the methods `SatisfiesD(??)` and `IsDiscrete(??)` below.

### 5.2 Discreteness

#### 5.2.1 SatisfiesD (for IsLocalAction)

▷ `SatisfiesD(F)` (property)

**Returns:** true if  $F$  satisfies the discreteness condition (D), and false otherwise.

The argument of this attribute is a local action  $F \leq \text{Aut}(B_{d,k})$  (see `IsLocalAction(??)`).

Example

```
gap> G:=GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> SatisfiesD(G);
true
```

#### 5.2.2 IsDiscrete (for IsLocalAction)

▷ `IsDiscrete(F)` (property)

**Returns:** true if  $U_k(F)$  is discrete, and false otherwise.

The argument of this attribute is a local action  $F \leq \text{Aut}(B_{d,k})$  (see `IsLocalAction(??)`).

Example

```
gap> G:=GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> IsDiscrete(G);
true
```

Example

```
gap> F:=LocalAction(3,2,Group((1,2)));
Group([ (1,2) ])
gap> IsDiscrete(F);
true
gap> SatisfiesD(F);
false
gap> C:=MaximalCompatibleSubgroup(F);
Group(())
gap> SatisfiesD(C);
true
```

## 5.3 Cocycles

Subgroups  $F \leq \text{Aut}(B_{d,k})$  that satisfy both (C) and (D) admit an involutive compatibility cocycle, i.e. a map  $z: F \times \{1, \dots, d\} \rightarrow F$  that satisfies certain properties, see [Tor20, Section 3.2.2]. When  $F$  satisfies just (C), it may still admit an involutive compatibility cocycle. In this case,  $F$  admits an extension  $\Gamma_z(F) \leq \text{Aut}(B_{d,k})$  that satisfies both (C) and (D). Involutive compatibility cocycles can be searched for using `InvolutiveCompatibilityCocycle(??)` and `AllInvolutiveCompatibilityCocycles(??)` below.

### 5.3.1 InvolutiveCompatibilityCocycle (for IsLocalAction)

▷ `InvolutiveCompatibilityCocycle(F)` (attribute)

**Returns:** an involutive compatibility cocycle of  $F$ , which is a mapping  $F \times [1..d] \rightarrow F$  with certain properties, if it exists, and fail otherwise. When  $k = 1$ , the standard cocycle is returned.

The argument of this attribute is a local action  $F \leq \text{Aut}(B_{d,k})$  (see `IsLocalAction(??)`), which is compatible (see `SatisfiesC(??)`).

Example

```
gap> F:=LocalAction(3,1,AlternatingGroup(3));;
gap> InvolutiveCompatibilityCocycle(F);
MappingByFunction( Domain([ [ () , 1 ], [ () , 2 ], [ () , 3 ], [ (1,3,2), 1 ],
  [ (1,3,2), 2 ], [ (1,3,2), 3 ], [ (1,2,3), 1 ], [ (1,2,3), 2 ],
  [ (1,2,3), 3 ] ]), Alt([ 1 .. 3 ]), function( s ) ... end )
gap> a:=Random(AlternatingGroup(3));; dir:=Random([1..3]);;
gap> a; Image(z,[a,dir]);
(1,3,2)
(1,3,2)
```

Example

```
gap> G:=GAMMA(3,AlternatingGroup(3));
Group([ (1,4,5)(2,3,6) ])
gap> InvolutiveCompatibilityCocycle(G);
MappingByFunction( Domain([ [ () , 1 ], [ () , 2 ], [ () , 3 ],
  [ (1,5,4)(2,6,3), 1 ], [ (1,5,4)(2,6,3), 2 ], [ (1,5,4)(2,6,3), 3 ],
```

```

[ (1,4,5)(2,3,6), 1 ], [ (1,4,5)(2,3,6), 2 ], [ (1,4,5)(2,3,6), 3 ]
]), Group([ (1,4,5)(2,3,6) ]), function( s ) ... end )
gap> InvolutiveCompatibilityCocycle(AutBall(3,2));
fail

```

### 5.3.2 AllInvolutiveCompatibilityCocycles (for IsLocalAction)

▷ AllInvolutiveCompatibilityCocycles( $F$ ) (attribute)

**Returns:** the list of all involutive compatibility cocycles of  $F$ .

The argument of this attribute is a local action  $F \leq \text{Aut}(B_{d,k})$  (see IsLocalAction (??)), which is compatible (see SatisfiesC (??)).

Example

```

gap> S3:=LocalAction(3,1,SymmetricGroup(3));
gap> Size(AllInvolutiveCompatibilityCocycles(S3));
4
gap> Size(AllInvolutiveCompatibilityCocycles(GAMMA(3,SymmetricGroup(3))));
1

```

# References

- [BEW15] C. Banks, M. Elder, and G. Willis. Simple groups of automorphisms of trees determined by their actions on finite subtrees. *Journal of Group Theory*, 18(2):235–261, 2015. [4](#)
- [BM00] M. Burger and S. Mozes. Groups acting on trees: from local to global structure. *Publications Mathématiques de l’IHÉS*, 92(1):113–150, 2000. [4](#)
- [Ser80] J. P. Serre. *Trees*. Springer, 1980. [4](#), [5](#)
- [Tor20] S. Tornier. Groups acting on trees with prescribed local action. *arxiv preprint: 2002.09876*, 2020. [4](#), [5](#), [10](#), [11](#), [16](#), [26](#), [27](#)

# Index

- AbelianizationHomomorphism, [22](#)
- AddressOfLeaf, [9](#)
- AllInvolutiveCompatibilityCocycles
  - for IsLocalAction, [28](#)
- AreCompatibleBallElements, [11](#)
- AssembleAutomorphism, [13](#)
- AutBall, [8](#)
- BallAddresses, [8](#)
- CompatibilitySet
  - for F, aut, dir, [12](#)
  - for F, aut, dirs, [12](#)
- CompatibleBallElement, [12](#)
- CompatibleKernels
  - for d, F, [23](#)
  - for F, z, [23](#)
- CompatibleSubgroups, [14](#)
- ComposeAddresses, [10](#)
- ConjugacyClassRepsCompatibleGroups-  
WithProjection, [15](#)
- ConjugacyClassRepsCompatibleSubgroups
  - for IsLocalAction, [14](#)
- DELTA
  - for d, F, [18](#)
  - for d, F, C, [18](#)
- GAMMA
  - for d, F, [17](#)
  - for F, z, [17](#)
  - for l, d, F, [17](#)
- gamma
  - for d, a, [16](#)
  - for d, k, aut, z, [16](#)
  - for l, d, a, [16](#)
  - for l, d, s, addr, [16](#)
- ImageAddress, [9](#)
- ImageOfProjection, [7](#)
- InvolutiveCompatibilityCocycle
  - for IsLocalAction, [27](#)
- IsDiscrete
  - for IsLocalAction, [26](#)
- IsLocalAction
  - for IsPermGroup, [5](#)
- LeafAddresses, [9](#)
- LeafOfAddress, [9](#)
- LocalAction
  - for IsInt, IsInt, IsPermGroup, [6](#)
  - for r, d, k, aut, addr, [7](#)
- LocalActionDegree
  - for IsLocalAction, [6](#)
- LocalActionNC
  - for IsInt, IsInt, IsPermGroup, [6](#)
- LocalActionRadius
  - for IsLocalAction, [6](#)
- MaximalCompatibleSubgroup
  - for IsLocalAction, [13](#)
- PHI
  - for d, F, [19](#)
  - for d, F, N, [20](#)
  - for d, F, P, [20](#)
  - for F, [19](#)
  - for F, P, [20](#)
  - for l, F, [19](#)
- PI, [23](#)
- Projection
  - for F, r, [7](#)
- SatisfiesC
  - for IsLocalAction, [14](#)
- SatisfiesD
  - for IsLocalAction, [26](#)
- SIGMA
  - for d, F, K, [24](#)
  - for F, K, z, [24](#)

SignHomomorphism, [22](#)

SpheresProduct, [22](#)