

UGALY

Universal Groups Acting Locally

v2.0

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Khalil Hannouch

Stephan Tornier

Khalil Hannouch

Email: khalil.hannouch@newcastle.edu.au

Homepage: <https://www.newcastle.edu.au/profile/khalil-hannouch>

Stephan Tornier

Email: stephan.tornier@newcastle.edu.au

Homepage: <https://www.newcastle.edu.au/profile/stephan-tornier>

Abstract

UGALY (Universal Groups Acting Locally) is a GAP package that provides methods to create, analyse and find local actions of universal groups acting on locally finite regular trees, following Burger-Mozes and Tournier.

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Chapter 1

Introduction

Let Ω be a set of cardinality $d \in \mathbb{N}_{\geq 3}$ and let $T_d = (V, E)$ be the d -regular tree. We follow Serre's graph theory notation [Ser80]. Given a subgroup H of the automorphism group $\text{Aut}(T_d)$ of T_d , and a vertex $x \in V$, the stabilizer H_x of x in H induces a permutation group on the set $E(x) := \{e \in E \mid o(e) = x\}$ of edges issuing from x . We say that H is locally "P" if for every $x \in V$ said permutation group satisfies the property "P", e.g. being transitive, semiprimitive, quasiprimitive or 2-transitive. In [BM00], Burger-Mozes develop a remarkable structure theory of closed, non-discrete, locally quasiprimitive subgroups of $\text{Aut}(T_d)$, which resembles the theory of semisimple Lie groups. They complement this structure theory with a particularly accessible class of subgroups of $\text{Aut}(T_d)$ with prescribed local action: Given $F \leq \text{Sym}(\Omega)$ their universal group $U(F)$ is closed in $\text{Aut}(T_d)$, vertex-transitive, compactly generated and locally permutation isomorphic to F . It is discrete if and only if F is semiregular. When F is transitive, $U(F)$ is maximal up to conjugation among vertex-transitive subgroups of $\text{Aut}(T_d)$ that are locally permutation isomorphic to F , hence *universal*.

This construction was generalized by the second author in [Tor20]: In the spirit of k -closures of groups acting on trees developed in [BEW15], we generalize the universal group construction by prescribing the local action on balls of a given radius $k \in \mathbb{N}$, the Burger-Mozes construction corresponding to the case $k = 1$. Fix a tree $B_{d,k}$ which is isomorphic to a ball of radius k in the labelled tree T_d and let $l_x^k : B(x, k) \rightarrow B_{d,k}$ ($x \in V$) be the unique label-respecting isomorphism. Then

$$\sigma_k : \text{Aut}(T_d) \times V \rightarrow \text{Aut}(B_{d,k}), (g, x) \rightarrow l_{gx}^k \circ g \circ (l_x^k)^{-1}$$

captures the k -local action of g at the vertex $x \in V$.

With this we can make the following definition: Let $F \leq \text{Aut}(B_{d,k})$. Define

$$U_k(F) := \{g \in \text{Aut}(T_d) \mid \forall x \in V : \sigma_k(g, x) \in F\}.$$

While $U_k(F)$ is always closed, vertex-transitive and compactly generated, other properties of $U(F)$ do *not* carry over. Foremost, the group $U_k(F)$ need not be locally action isomorphic to F and we say that $F \leq \text{Aut}(B_{d,k})$ satisfies condition (C) if it is. This can be viewed as an interchangeability condition on neighbouring local actions, see Section 3.1. There is also a discreteness condition (D) on $F \leq \text{Aut}(B_{d,k})$ in terms of certain stabilizers in F under which $U_k(F)$ is discrete, see Section 5.1.

UGALY provides methods to create, analyse and find local actions $F \leq \text{Aut}(B_{d,k})$ that satisfy condition (C) and/or (D), including the constructions Γ , Δ , Φ , Σ , and Π developed in [Tor20]. This package was developed within the [Zero-Dimensional Symmetry Research Group](#) in the [School of Mathematical and Physical Sciences](#) at [The University of Newcastle](#) as part of a project course taken by the first author, supervised by the second author.

1.1 Purpose

UGALY serves both a research and an educational purpose. It consolidates a rudimentary codebase that was developed by the second author in the course of research undertaken towards the article [Tor20]. This codebase had been tremendously beneficial in achieving the results of [Tor20] in the first place and so there has always been a desire to make it available to a wider audience.

From a research perspective, UGALY introduces computational methods to the world of locally compact groups. Due to the Cayley-Abels graph construction [KM08], groups acting on trees form a particularly significant class of totally disconnected locally compact groups. Burger-Mozes universal groups [BM00] and their generalisations $U_k(F)$, where $F \leq \text{Aut}(B_{d,k})$ satisfies the compatibility condition (C), are among the most accessible of these groups and form a significant subclass: in fact, due to [Tor20, Corollary 4.32], the locally transitive, generalised universal groups are exactly the closed, locally transitive subgroups of $\text{Aut}(T_d)$ that contain an inversion of order 2 and satisfy one of the independence properties (P_k) (see [BEW15]) that generalise Tits' independence property (P), see [Tit70]. UGALY provides the means to generate a library of all these groups in terms of their k -local action, and we envision to add such a library in a future version of this package. In the case $k = 1$ of classical Burger-Mozes groups, it coincides with the library of finite transitive permutation groups `TransGrp`. As another example, we obtain the following six groups for $(d, k) = (3, 2)$.

Example

```
gap> A3:=LocalAction(3,1,TransitiveGroup(3,1));
A3
gap> S3:=LocalAction(3,1,TransitiveGroup(3,2));
S3
gap> A3_extn:=ConjugacyClassRepsCompatibleGroupsWithProjection(2,A3);
[ Group([ (1,4,5)(2,3,6) ]) ]
gap> S3_extn:=ConjugacyClassRepsCompatibleGroupsWithProjection(2,S3);
[ Group([ (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (1,2)(3,4)(5,6), (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5,4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5)(4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (5,6), (3,5,4,6) ]) ]
```

These groups have been identified to stem from certain general constructions much like there are families of finite transitive groups. See Chapter 4 for the functions appearing below.

Example

```
gap> PHI(A3)=A3_extn[1];
true
gap> GAMMA(3,S3)=S3_extn[1];
true
gap> DELTA(3,S3)=S3_extn[2];
false
gap> IsConjugate(AutBall(3,2),DELTA(3,S3),S3_extn[2]);
true
gap> rho:=SignHomomorphism(S3);
gap> PI(2,3,S3,rho,[0,1])=S3_extn[3];
true
gap> PI(2,3,S3,rho,[1])=S3_extn[4];
true
gap> PHI(S3)=S3_extn[5];
true
```

UGALY may also be a useful tool in the context of the Weiss conjecture [Wei78] which in particular states that there are only finitely many conjugacy classes of discrete, vertex-transitive and locally primitive subgroup of $\text{Aut}(T_d)$. When such a group contains an inversion of order 2, it can be written as a universal group $U_k(F)$, where $F \leq \text{Aut}(B_{d,k})$ satisfies both the compatibility condition (C) and the discreteness condition (D), due to [Tor20, Corollary 4.38]. Therefore, UGALY can be used to construct explicit examples of groups relevant to the Weiss conjecture. Their structure as well as patterns in their appearance may provide more insight into the conjecture and suggest directions of research. At the very least, UGALY provides lower bounds on their numbers. For example ($d = 4$):

Example

```
gap> A4:=LocalAction(4,1,PrimitiveGroup(4,1));;
gap> S4:=LocalAction(4,1,PrimitiveGroup(4,2));;
gap> A4_extn:=ConjugacyClassRepsCompatibleGroupsWithProjection(2,A4);;
gap> Size(A4_extn); Size(Filtered(A4_extn,SatisfiesD));
5
2
gap> S4_extn:=ConjugacyClassRepsCompatibleGroupsWithProjection(2,S4);;
gap> Size(S4_extn); Size(Filtered(S4_extn,SatisfiesD));
13
3
```

Every subgroup $F \leq \text{Aut}(B_{d,k})$ which satisfies both the compatibility condition (C) and the discreteness condition (D) admits an involutive compatibility cocycle (see [Tor20, Section 3.2.2]), i.e. a map $z : F \times \{1, \dots, d\} \rightarrow F$ which satisfies certain properties reflecting the discreteness of the group $U_k(F)$. It is intriguing that some groups $F \leq \text{Aut}(B_{d,k})$ with these properties stem from groups $F' \leq \text{Aut}(B_{d,k-1})$ that satisfy (C), admit an involutive compatibility cocycle z but do not satisfy (D), in the sense that $F = \Gamma_z(F')$ (see [Tor20, Proposition 3.26]), whereas others do not. For example, in the case $d = 3$, five of the seven conjugacy classes of discrete, vertex-transitive and locally primitive subgroups of $\text{Aut}(T_3)$ come from generalised universal groups. Of these five, three arise from groups F' as above while the remaining two do not, see [Tor20, Example 4.39].

Example

```
gap> S3:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(S3);;
gap> H:=PI(2,3,S3,rho,[1]);;
gap> [SatisfiesC(H), SatisfiesD(H), not InvolutiveCompatibilityCocycle(H)=fail];
[ true, false, true ]
gap> grps:=ConjugacyClassRepsCompatibleGroupsWithProjection(3,H);; Size(grps);
4
gap> Size(Filtered(grps,SatisfiesD));
2
gap> z:=InvolutiveCompatibilityCocycle(H);;
gap> Size(Intersection(GAMMA(H,z)^AutBall(3,3),grps));
1
```

The number of different involutive compatibility cocycles that a group $F \leq \text{Aut}(B_{d,k})$ may admit is also mysterious, including in the case $k = 1$.

Example

```
gap> grps:=AllTransitiveGroups(NrMovedPoints,4);
[ C(4) = 4, E(4) = 2[x]2, D(4), A4, S4 ]
gap> Apply(grps,H->Size(AllInvolutiveCompatibilityCocycles(LocalAction(4,1,H))));
gap> grps;
[ 1, 1, 8, 28, 256 ]
```

From an educational point of view, we envision that UGALY could be used to enhance the learning experience of students in the area of groups acting on trees. The class of generalised universal groups forms an ideal framework for this purpose. For example, to internalise the widely used concept of local actions it may be helpful to take a 2-local action in the form of an automorphism of $B_{3,2}$ (a rotation in the case below), decompose it into its 1-local actions and recover the automorphism from them.

Example

```
gap> aut:=Random(AutBall(3,2));
(1,5,3,2,6,4)
gap> aut_center:=LocalAction(1,3,2,aut,[]);
(1,3,2)
gap> aut_1:=LocalAction(1,3,2,aut,[1]);
(1,3,2)
gap> aut_2:=LocalAction(1,3,2,aut,[2]);
(1,3,2)
gap> aut_3:=LocalAction(1,3,2,aut,[3]);
(2,3)
gap> AssembleAutomorphism(3,1,[aut_1,aut_2,aut_3]);
(1,5,3,2,6,4)
```

The computationally inclined student may also benefit from verifying existing theorems using UGALY. For example, one way to phrase a part of Tutte's work [Tut47] [Tut59] is to say that there are only three conjugacy classes of discrete, locally transitive subgroups of $\text{Aut}(T_3)$ that contain an inversion of order 2 and are P_2 -closed. Due to [Tor20, Corollary 4.38], this can be verified by checking that among all locally transitive subgroups of $\text{Aut}(B_{3,2})$ which satisfy the compatibility condition (C), only three also satisfy the discreteness condition (D).

Example

```
gap> A3:=LocalAction(3,1,TransitiveGroup(3,1));;
gap> S3:=LocalAction(3,1,TransitiveGroup(3,2));;
gap> A3_extn:=ConjugacyClassRepsCompatibleGroupsWithProjection(2,A3);
[ Group([ (1,4,5)(2,3,6) ]) ]
gap> S3_extn:=ConjugacyClassRepsCompatibleGroupsWithProjection(2,S3);
[ Group([ (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (1,2)(3,4)(5,6), (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5,4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5)(4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (5,6), (3,5,4,6) ]) ]
gap> Apply(A3_extn,SatisfiesD); A3_extn;
[ true ]
gap> Apply(S3_extn,SatisfiesD); S3_extn;
[ true, true, false, false, false ]
```

It may also be instructive to generate involutive compatibility cocycles computationally and check parts of the axioms manually.

Example

```
gap> S3:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(S3);;
gap> H:=PI(2,3,S3,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(H);;
gap> a:=Random(H); Image(z,[Image(z,[a,1]),1]);
(3,4)(5,6)
(3,4)(5,6)
```

Chapter 2

Preliminaries

We recall the following notation from the Introduction which is essential throughout this manual, cf. [Tor20]. Let Ω be a set of cardinality $d \in \mathbb{N}_{\geq 3}$ and let $T_d = (V, E)$ denote the d -regular tree, following the graph theory notation in [Ser80]. A *labelling* l of T_d is a map $l : E \rightarrow \Omega$ such that for every $x \in V$ the restriction $l_x : E(x) \rightarrow \Omega$, $e \mapsto l(e)$ is a bijection, and $l(e) = l(\bar{e})$ for all $e \in E$. For every $k \in \mathbb{N}$, fix a tree $B_{d,k}$ which is isomorphic to a ball of radius k around a vertex in T_d and carry over the labelling of T_d to $B_{d,k}$ via the chosen isomorphism. We denote the center of $B_{d,k}$ by b .

For every $x \in V$ there is a unique, label-respecting isomorphism $l_x^k : B(x, k) \rightarrow B_{d,k}$. We define the k -local action $\sigma_k(g, x) \in \text{Aut}(B_{d,k})$ of an automorphism $g \in \text{Aut}(T_d)$ at a vertex $x \in V$ via the map

$$\sigma_k : \text{Aut}(T_d) \times V \rightarrow \text{Aut}(B_{d,k}), \sigma_k(g, x) \mapsto \sigma_k(g, x) := l_{gx}^k \circ g \circ (l_x^k)^{-1}.$$

2.1 Local actions

In this package, local actions $F \leq \text{Aut}(B_{d,k})$ are handled as objects of the category `IsLocalAction` (??) and have several attributes and properties introduced throughout this manual. Most importantly, a local action always stores the degree d and the radius k of the ball $B_{d,k}$ that it acts on.

2.1.1 IsLocalAction (for IsPermGroup)

▷ `IsLocalAction(arg)` (filter)

Returns: true or false

Local actions $F \leq \text{Aut}(B_{d,k})$ are stored together with their degree (see `LocalActionDegree` (??)), radius (see `LocalActionRadius` (??)) as well as other attributes and properties in this category.

Example

```
gap> G:=WreathProduct(SymmetricGroup(2),SymmetricGroup(3));
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(G);
false
gap> H:=AutBall(3,2);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(H);
true
gap> K:=LocalAction(3,2,G);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
```



```
gap> IsLocalAction(K);
true
```

2.1.2 LocalAction (for IsInt, IsInt, IsPermGroup)

▷ LocalAction(d, k, F) (operation)

Returns: the regular rooted tree group G as an object of the category IsLocalAction (??), checking that F is indeed a subgroup of $\text{Aut}(B_{d,k})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}_0$ and a group $F \leq \text{Aut}(B_{d,k})$.

Example

```
gap> G:=WreathProduct(SymmetricGroup(2),SymmetricGroup(3));
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(G);
false
gap> G:=LocalAction(3,2,G);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> IsLocalAction(G);
```

2.1.3 LocalActionNC (for IsInt, IsInt, IsPermGroup)

▷ LocalActionNC(d, k, F) (operation)

Returns: the regular rooted tree group G as an object of the category IsLocalAction (??), without checking that F is indeed a subgroup of $\text{Aut}(B_{d,k})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}_0$ and a group $F \leq \text{Aut}(B_{d,k})$.

2.1.4 LocalActionDegree (for IsLocalAction)

▷ LocalActionDegree(F) (attribute)

Returns: the degree d of the ball $B_{d,k}$ that F is acting on.

The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see IsLocalAction (??)).

Example

```
gap> F:=PHI(4,AlternatingGroup(4));
Group([ (1,5,7)(2,4,8)(3,6,9)(10,11,12), (1,2,3)(4,7,10)(5,9,11)(6,8,12),
(1,2,3), (4,5,6), (7,8,9), (10,11,12) ])
gap> LocalActionDegree(F);
4
```

2.1.5 LocalActionRadius (for IsLocalAction)

▷ LocalActionRadius(F) (attribute)

Returns: the radius k of the ball $B_{d,k}$ that F is acting on.

The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see IsLocalAction (??)).

Example

```
gap> F:=PHI(4,AlternatingGroup(4));
Group([ (1,5,7)(2,4,8)(3,6,9)(10,11,12), (1,2,3)(4,7,10)(5,9,11)(6,8,12),
(1,2,3), (4,5,6), (7,8,9), (10,11,12) ])
gap> LocalActionRadius(F);
2
```

2.1.6 LocalAction (for $r, d, k, \text{aut}, \text{addr}$)

▷ `LocalAction($r, d, k, \text{aut}, \text{addr}$)` (operation)

Returns: the r -local action $\sigma_r(\text{aut}, \text{addr})$ of the automorphism aut of $B_{d,k}$ at the vertex represented by the address addr .

The arguments of this method are a radius r , a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, an automorphism aut of $B_{d,k}$, and an address addr .

Example

```
gap> a:=(1,3,5)(2,4,6);; a in AutBall(3,2);
true
gap> LocalAction(2,3,2,a,[]);
(1,3,5)(2,4,6)
gap> LocalAction(1,3,2,a,[]);
(1,2,3)
gap> LocalAction(1,3,2,a,[1]);
(1,2)
```

Example

```
gap> b:=Random(AutBall(3,4));
(1,20,4,17,2,19,3,18)(5,22,8,23,6,21,7,24)(9,10)(13,16,14,15)
gap> LocalAction(2,3,4,b,[3,1]);
(1,4)(2,3)
gap> LocalAction(3,3,4,b,[3,1]);
Error, the sum of input argument r=3 and the length of input argument
addr=[ 3, 1 ] must not exceed input argument k=4
```

2.1.7 Projection (for F, r)

▷ `Projection(F, r)` (operation)

Returns: the restriction of the projection map $\text{Aut}(B_{d,k}) \rightarrow \text{Aut}(B_{d,r})$ to F .

The arguments of this method are a local action $F \leq \text{Aut}(B_{d,k})$, and a projection radius $r \leq k$.

Example

```
gap> F:=GAMMA(4,3,SymmetricGroup(3));
Group([ (1,16,19)(2,15,20)(3,13,18)(4,14,17)(5,10,23)(6,9,24)(7,12,22)
(8,11,21), (1,9)(2,10)(3,12)(4,11)(5,15)(6,16)(7,13)(8,14)(17,21)(18,22)
(19,24)(20,23) ])
gap> pr:=Projection(F,2);
<action homomorphism>
gap> a:=Random(F);; Image(pr,a);
(1,4,5)(2,3,6)
```

2.1.8 ImageOfProjection

▷ `ImageOfProjection(F, r)` (function)

Returns: the local action $\sigma_r(F, b) \leq \text{Aut}(B_{d,r})$.

The arguments of this method are a local action $F \leq \text{Aut}(B_{d,k})$, and a projection radius $r \leq k$. This method uses `LocalAction` (2.1.6) on generators rather than `Projection` (2.1.7) on the group to compute the image.

Example

```
gap> AutBall(3,2);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> ImageOfProjection(AutBall(3,2),1);
Group([ (), (), (1,2,3), (1,2) ])
```

2.2 Finite balls

The automorphism groups of the finite labelled balls $B_{d,k}$ lie at the center of this package. The method `AutBall` (2.2.1) produces these automorphism groups as iterated wreath products. The result is a permutation group on the set of leaves of $B_{d,k}$.

2.2.1 AutBall

▷ `AutBall(d, k)` (function)

Returns: the local action $\text{Aut}(B_{d,k})$ as a permutation group of the $d \cdot (d-1)^{k-1}$ leaves of $B_{d,k}$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$ and a radius $k \in \mathbb{N}_0$.

Example

```
gap> G:=AutBall(3,2);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6), (1,3)(2,4) ])
gap> Size(G);
48
```

2.3 Addresses and leaves

The vertices at distance n from the center b of $B_{d,k}$ are addressed as elements of the set

$$\Omega^{(n)} := \{(\omega_1, \dots, \omega_n) \in \Omega^n \mid \forall l \in \{1, \dots, n-1\} : \omega_l \neq \omega_{l+1}\},$$

i.e. as lists of length n of elements from $[1..d]$ such that no two consecutive entries are equal. They are ordered according to the lexicographic order on $\Omega^{(n)}$. The center b itself is addressed by the empty list $[]$. Note that the leaves of $B_{d,k}$ correspond to elements of $\Omega^{(k)}$.

2.3.1 BallAddresses

▷ `BallAddresses(d, k)` (function)

Returns: a list of all addresses of vertices in $B_{d,k}$ in ascending order with respect to length, lexicographically ordered within each level. See `AddressOfLeaf` (2.3.3) and `LeafOfAddress` (2.3.4) for the correspondence between the leaves of $B_{d,k}$ and addresses of length k .

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$ and a radius $k \in \mathbb{N}_0$.

Example

```
gap> BallAddresses(3,1);
[ [], [ 1 ], [ 2 ], [ 3 ] ]
gap> BallAddresses(3,2);
[ [], [ 1 ], [ 2 ], [ 3 ], [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 2, 3 ],
[ 3, 1 ], [ 3, 2 ] ]
```

2.3.2 LeafAddresses

▷ `LeafAddresses(d , k)`

(function)

Returns: a list of addresses of the leaves of $B_{d,k}$ in lexicographic order.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$ and a radius $k \in \mathbb{N}_0$.

Example

```
gap> LeafAddresses(3,2);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 2, 3 ], [ 3, 1 ], [ 3, 2 ] ]
```

2.3.3 AddressOfLeaf

▷ `AddressOfLeaf(d , k , lf)`

(function)

Returns: the address of the leaf lf of $B_{d,k}$ with respect to the lexicographic order.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, and a leaf lf of $B_{d,k}$.

Example

```
gap> AddressOfLeaf(3,2,1);
[ 1, 2 ]
gap> AddressOfLeaf(3,3,1);
[ 1, 2, 1 ]
```

2.3.4 LeafOfAddress

▷ `LeafOfAddress(d , k , $addr$)`

(function)

Returns: the smallest leaf (integer) whose address has $addr$ as a prefix.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, and an address $addr$.

Example

```
gap> LeafOfAddress(3,2,[1,2]);
1
gap> LeafOfAddress(3,2,[3]);
5
gap> LeafOfAddress(3,2,[]);
1
```

2.3.5 ImageAddress

▷ `ImageAddress(d , k , aut , $addr$)`

(function)

Returns: the address of the image of the vertex represented by the address $addr$ under the automorphism aut of $B_{d,k}$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, an automorphism aut of $B_{d,k}$, and an address $addr$.

Example

```
gap> ImageAddress(3,2,(1,2),[1,2]);
[ 1, 3 ]
gap> ImageAddress(3,2,(1,2),[1]);
[ 1 ]
```

2.3.6 ComposeAddresses

▷ `ComposeAddresses(addr1, addr2)` (function)

Returns: the concatenation of the addresses *addr1* and *addr2* with reduction as per [Tor20, Section 3.2].

The arguments of this method are two addresses *addr1* and *addr2*.

Example

```
gap> ComposeAddresses([1,3],[2,1]);  
[ 1, 3, 2, 1 ]  
gap> ComposeAddresses([1,3,2],[2,1]);  
[ 1, 3, 1 ]
```

Chapter 3

Compatibility

3.1 The compatibility condition (C)

A subgroup $F \leq \text{Aut}(B_{d,k})$ satisfies the compatibility condition (C) if and only if $U_k(F)$ is locally action isomorphic to F , see [Tor20, Proposition 3.8]. The term *compatibility* comes from the following translation of this condition into properties of the $(k-1)$ -local actions of elements of F : The group F satisfies (C) if and only if

$$\forall \alpha \in F \forall \omega \in \Omega \exists \beta \in F : \sigma_{k-1}(\alpha, b) = \sigma_{k-1}(\beta, b_\omega), \sigma_{k-1}(\alpha, b_\omega) = \sigma_{k-1}(\beta, b).$$

3.2 Compatible elements

This section is concerned with testing compatibility of two given elements (see `AreCompatibleBallElements` (3.2.1)) and finding an/all elements that is/are compatible with a given one (see `CompatibleBallElement` (3.2.2), `CompatibilitySet` (3.2.3)).

3.2.1 AreCompatibleBallElements

▷ `AreCompatibleBallElements(d, k, aut1, aut2, dir)` (function)

Returns: true if `aut1` and `aut2` are compatible with each other in direction `dir`, and false otherwise.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, two automorphisms `aut1`, `aut2` $\in \text{Aut}(B_{d,k})$, and a direction `dir` $\in [1..d]$.

Example

```
gap> AreCompatibleBallElements(3,1,(1,2),(1,2,3),1);
true
gap> AreCompatibleBallElements(3,1,(1,2),(1,2,3),2);
false
```

Example

```
gap> a:=(1,3,5)(2,4,6);; a in AutBall(3,2);
true
gap> LocalAction(1,3,2,a,[]); LocalAction(1,3,2,a,[1]);
(1,2,3)
```

```

(1,2)
gap> b:=(1,4)(2,3);; b in AutBall(3,2);
true
gap> LocalAction(1,3,2,b,[]); LocalAction(1,3,2,b,[1]);
(1,2)
(1,2,3)

gap> AreCompatibleBallElements(3,2,a,b,1);
true
gap> AreCompatibleBallElements(3,2,a,b,3);
false

```

3.2.2 CompatibleBallElement

▷ `CompatibleBallElement(F , aut , dir)` (function)

Returns: an element of F that is compatible with aut in direction dir if one exists, and fail otherwise.

The arguments of this method are a local action $F \leq \text{Aut}(B_{d,k})$, an element $aut \in F$, and a direction $dir \in [1..d]$.

Example

```

gap> a:=Random(AutBall(5,1)); dir:=Random([1..5]);
(1,3,2,5)
4
gap> CompatibleBallElement(AutBall(5,1),a,dir);
(1,3,2,5)

```

Example

```

gap> a:=(1,3,5)(2,4,6);; a in AutBall(3,2);
true
gap> CompatibleBallElement(AutBall(3,2),a,1);
(1,4,2,3)

```

3.2.3 CompatibilitySet

▷ `CompatibilitySet(F , aut , dir)` (operation)

▷ `CompatibilitySet(F , aut , $dirs$)` (operation)

for the arguments F , aut , dir

Returns: the list of elements of F that are compatible with aut in direction dir .

The arguments of this method are a local action F of $\leq \text{Aut}(B_{d,k})$, an automorphism $aut \in F$, and a direction $dir \in [1..d]$.

for the arguments F , aut , $dirs$

Returns: the list of elements of F that are compatible with aut in all directions of $dirs$.

The arguments of this method are a local action F of $\leq \text{Aut}(B_{d,k})$, an automorphism $aut \in F$, and a sublist of directions $dirs \subseteq [1..d]$.

Example

```

gap> F:=LocalAction(4,1,TransitiveGroup(4,3));
D(4)
gap> G:=LocalAction(4,1,SymmetricGroup(4));
Sym( [ 1 .. 4 ] )
gap> CompatibilitySet(G,aut,1);
RightCoset(Sym( [ 2 .. 4 ] ),(1,3))
gap> CompatibilitySet(F,aut,1);
RightCoset(Group([ (2,4) ]),(1,3))
gap> CompatibilitySet(F,aut,[1,3]);
RightCoset(Group([ (2,4) ]),(1,3))
gap> CompatibilitySet(F,aut,[1,2]);
RightCoset(Group(()),(1,3))

```

3.2.4 AssembleAutomorphism

- ▷ `AssembleAutomorphism(d , k , $auts$)` (function)
Returns: the automorphism $(aut, (auts[i])_{i=1}^d)$ of $B_{d,k+1}$, where aut is implicit in $(auts[i])_{i=1}^d$.
The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, and a list $auts$ of d automorphisms $(auts[i])_{i=1}^d$ of $B_{d,k}$ which comes from an element $(aut, (auts[i])_{i=1}^d)$ of $\text{Aut}(B_{d,k+1})$.

Example

```

gap> aut:=Random(AutBall(3,2));
(1,2)(3,6)(4,5)
gap> auts:=[];
gap> for i in [1..3] do auts[i]:=CompatibleBallElement(3,2,AutBall(3,2),aut,i); od;
gap> auts;
[ (1,2)(3,5)(4,6), (1,3,5)(2,4,6), (1,5,3)(2,6,4) ]
gap> a:=AssembleAutomorphism(3,2,auts);
(1,3)(2,4)(5,11)(6,12)(7,9)(8,10)
gap> a in AutBall(3,3);
true
gap> LocalAction(2,3,3,a,[]);
(1,2)(3,6)(4,5)

```

3.3 Compatible subgroups

Using the methods of Section 3.2, this section provides methods to test groups for the compatibility condition and search for compatible subgroups inside a given group, e.g. $\text{Aut}(B_{d,k})$, or with a certain image under some projection.

3.3.1 MaximalCompatibleSubgroup (for IsLocalAction)

- ▷ `MaximalCompatibleSubgroup(F)` (attribute)
Returns: The local action $C(F) \leq \text{Aut}(B_{d,k})$, which is the maximal compatible subgroup of F .
The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see `IsLocalAction(??)`).

Example

```

gap> F:=LocalAction(3,1,Group((1,2)));
Group([ (1,2) ])
gap> MaximalCompatibleSubgroup(F);

```



```

Group([ (1,2) ])
gap> G:=LocalAction(3,2,Group((1,2)));
Group([ (1,2) ])
gap> MaximalCompatibleSubgroup(G);
Group(())

```

3.3.2 SatisfiesC (for IsLocalAction)

▷ `SatisfiesC(F)` (property)

Returns: true if F satisfies the compatibility condition (C), and false otherwise.

The argument of this property is a local action $F \leq \text{Aut}(B_{d,k})$ (see `IsLocalAction(??)`).

Example

```

gap> D:=DELTA(3,SymmetricGroup(3));
Group([ (1,3,6)(2,4,5), (1,3)(2,4), (1,2)(3,4)(5,6) ])
gap> SatisfiesC(D);
true

```

3.3.3 CompatibleSubgroups

▷ `CompatibleSubgroups(F)` (function)

Returns: the list of all compatible subgroups of F .

The argument of this method is a local action $F \leq \text{Aut}(B_{d,k})$. This method calls `AllSubgroups` on F and is therefore slow. Use for instructional purposes on small examples only, and use `ConjugacyClassRepsCompatibleSubgroups(??)` or `ConjugacyClassRepsCompatibleSubgroupsWithProjection(??)` for computations.

Example

```

gap> G:=GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> list:=CompatibleSubgroups(G);
[ Group(()), Group([ (1,2)(3,5)(4,6) ]), Group([ (1,3)(2,4)(5,6) ]),
  Group([ (1,6)(2,5)(3,4) ]), Group([ (1,4,5)(2,3,6) ]), Group([ (1,4,5)
    (2,3,6), (1,3)(2,4)(5,6) ]) ]
gap> Size(list);
6
gap> Size(AllSubgroups(SymmetricGroup(3)));
6

```

3.3.4 ConjugacyClassRepsCompatibleSubgroups (for IsLocalAction)

▷ `ConjugacyClassRepsCompatibleSubgroups(F)` (attribute)

Returns: a list of compatible representatives of conjugacy classes of F that contain a compatible subgroup.

The argument of this method is a local action F of $\text{Aut}(B_{d,k})$.

Example

```

gap> ConjugacyClassRepsCompatibleSubgroups(AutBall(3,2));
[ Group(()), Group([ (1,2)(3,5)(4,6) ]), Group([ (1,4,5)(2,3,6) ]),
  Group([ (3,5)(4,6), (1,2) ]), Group([ (1,2)(3,5)(4,6), (1,3,6)
    (2,4,5) ]), Group([ (3,5)(4,6), (1,3,5)(2,4,6), (1,2)(3,4)(5,6) ]),

```

```

Group([ (1,2)(3,5)(4,6), (1,3,5)(2,4,6), (1,2)(5,6), (1,2)(3,4) ]),
Group([ (3,5)(4,6), (1,3,5)(2,4,6), (1,2)(5,6), (1,2)(3,4) ]),
Group([ (5,6), (3,4), (1,2), (1,3,5)(2,4,6), (3,5)(4,6) ]) ]

```

3.3.5 ConjugacyClassRepsCompatibleGroupsWithProjection

▷ `ConjugacyClassRepsCompatibleGroupsWithProjection(l, F)` (function)

Returns: a list of compatible representatives of conjugacy classes of $\text{Aut}(B_{d,l})$ that contain a compatible group which projects to $F \leq \text{Aut}(B_{d,r})$.

The arguments of this method are a radius $l \in \mathbb{N}$, and a local action $F \leq \text{Aut}(B_{d,k})$ for some $k \leq l$.

Example

```

gap> S3:=LocalAction(3,1,SymmetricGroup(3));
Sym( [ 1 .. 3 ] )
gap> ConjugacyClassRepsCompatibleGroupsWithProjection(2,S3);
[ Group([ (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (1,2)(3,4)(5,6), (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5,4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5)(4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (5,6), (3,5,4,6) ]) ]
gap> A3:=LocalAction(3,1,AlternatingGroup(3));
Alt( [ 1 .. 3 ] )
gap> ConjugacyClassRepsCompatibleGroupsWithProjection(2,A3);
[ Group([ (1,4,5)(2,3,6) ]) ]

```

Example

```

gap> F:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(F);;
gap> H1:=PI(2,3,F,rho,[0,1]);;
gap> H2:=PI(2,3,F,rho,[1]);;
gap> Size(ConjugacyClassRepsCompatibleGroupsWithProjection(3,H1));
2
gap> Size(ConjugacyClassRepsCompatibleGroupsWithProjection(3,H2));
4

```

Chapter 4

Examples

Several classes of examples of subgroups of $\text{Aut}(B_{d,k})$ that satisfy (C) and or (D) are constructed in [Tor20] and implemented in this section. For a given permutation group $F \leq S_d$, there are always the three local actions $\Gamma(F)$, $\Delta(F)$ and $\Phi(F)$ on $\text{Aut}(B_{d,2})$ that project onto F . For some F , these are all distinct and yield all universal groups that have F as their 1-local action, see [Tor20, Theorem 3.32]. More examples arise in particular when either point stabilizers in F are not simple, F preserves a partition, or F is not perfect.

4.1 Discrete groups

Here, we implement the local actions $\Gamma(F), \Delta(F) \leq \text{Aut}(B_{d,2})$, both of which satisfy both (C) and (D), see [Tor20, Section 3.4.1].

4.1.1 gamma

- ▷ `gamma(d, a)` (operation)
- ▷ `gamma(l, d, a)` (operation)
- ▷ `gamma(l, d, s, addr)` (operation)
- ▷ `gamma(d, k, aut, z)` (operation)

for the arguments d, a

Returns: the automorphism $\gamma(a) = (a, (a)_{\omega \in \Omega}) \in \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$ and a permutation $a \in S_d$.

for the arguments l, d, a

Returns: the automorphism $\gamma^l(a) \in \text{Aut}(B_{d,l})$ all of whose 1-local actions are given by a .

The arguments of this method are a radius $l \in \mathbb{N}$, a degree $d \in \mathbb{N}_{\geq 3}$ and a permutation $a \in S_d$.

for the arguments $l, d, s, addr$

Returns: the automorphism of $B_{d,k}$ whose 1-local actions are given by s at vertices whose address has `addr` as a prefix and are trivial elsewhere.

The arguments of this method are a radius $l \in \mathbb{N}$, a degree $d \in \mathbb{N}_{\geq 3}$, a permutation $s \in S_d$ and an address `addr` of a vertex in $B_{d,l}$ whose last entry is fixed by s .

for the arguments d, k, aut, z

Returns: the automorphism $\gamma_z(aut) = (aut, (z(aut, \omega))_{\omega \in \Omega}) \in \text{Aut}(B_{d,k+1})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, an automorphism aut of $B_{d,k}$, and an involutive compatibility cocycle z of a subgroup of $\text{Aut}(B_{d,k})$ that contains aut (see `InvolutiveCompatibilityCocycle` (??)).

Example

```
gap> gamma(3, (1,2));
(1,3) (2,4) (5,6)
```

Example

```
gap> gamma(2,3, (1,2));
(1,3) (2,4) (5,6)
gap> gamma(3,3, (1,2));
(1,5) (2,6) (3,8) (4,7) (9,11) (10,12)
```

Example

```
gap> gamma(3,3, (1,2), [1,3]);
(3,4)
gap> gamma(3,3, (1,2), []);
(1,5) (2,6) (3,8) (4,7) (9,11) (10,12)
```

Example

```
gap> S3:=SymmetricGroup(3);;
gap> z1:=AllInvolutiveCompatibilityCocycles(3,1,S3)[1];;
gap> gamma(3,1, (1,2), z1);
(1,4) (2,3) (5,6)
gap> z3:=AllInvolutiveCompatibilityCocycles(3,1,S3)[3];;
gap> gamma(3,1, (1,2), z3);
(1,3) (2,4) (5,6)
```

4.1.2 GAMMA

- ▷ `GAMMA(d, F)` (operation)
- ▷ `GAMMA(1, d, F)` (operation)
- ▷ `GAMMA(F, z)` (operation)

for the arguments d, F

Returns: the local action $\Gamma(F) = \{(a, (a)_\omega) \mid a \in F\} \leq \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, and a subgroup F of S_d .

for the arguments $1, d, F$

Returns: the group $\Gamma^1(F) \leq \text{Aut}(B_{d,1})$.

The arguments of this method are a radius $1 \in \mathbb{N}$, a degree $d \in \mathbb{N}_{\geq 3}$, and a subgroup F of S_d .

for the arguments d, k, F, z

Returns: the group $\Gamma_z(F) = \{(a, (z(a, \omega))_{\omega \in \Omega}) \mid a \in F\} \leq \text{Aut}(B_{d,k+1})$.

The arguments of this method are a local action $F \leq \text{Aut}(B_{d,k})$ and an involutive compatibility cocycle z of F (see `InvolutiveCompatibilityCocycle` (??)).

Example

```
gap> F:=TransitiveGroup(4,3);;
gap> GAMMA(4,F);
Group([ (1,5,9,10)(2,6,7,11)(3,4,8,12), (1,8)(2,7)(3,9)(4,5)(10,12) ])
```

Example

```
gap> GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> GAMMA(2,3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> GAMMA(3,3,SymmetricGroup(3));
Group([ (1,8,10)(2,7,9)(3,5,12)(4,6,11), (1,5)(2,6)(3,8)(4,7)(9,11)(10,12) ])
```

Example

```
gap> F:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(F);;
gap> H:=PI(2,3,F,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(3,2,H);;
gap> GAMMA(H,z);
Group([ (), (), (1,8,9)(2,7,10)(3,5,11)(4,6,12), (1,8,9)(2,7,10)(3,5,11)(4,6,12),
(1,7,3,5)(2,8,4,6)(9,11,10,12) ])
```

4.1.3 DELTA

- ▷ DELTA(d, F) (operation)
- ▷ DELTA(d, F, C) (operation)

for the arguments d, F

Returns: the group $\Delta(F) \leq \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, and a *transitive* subgroup F of S_d .

for the arguments d, F, C

Returns: the group $\Delta(F, C) \leq \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a *transitive* subgroup F of S_d , and a central subgroup C of the stabilizer F_1 of 1 in F .

Example

```
gap> F:=SymmetricGroup(3);;
gap> D:=DELTA(3,F);
Group([ (1,3,6)(2,4,5), (1,3)(2,4), (1,2)(3,4)(5,6) ])
gap> F1:=Stabilizer(F,1);;
gap> D1:=DELTA(3,F,F1);
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6), (1,2)(3,4)(5,6) ])
gap> D=D1;
false
gap> G:=AutBall(3,2);;
gap> D^G=D1^G;
true
```

Example

```
gap> F:=PrimitiveGroup(5,3);
AGL(1, 5)
```

```

gap> F1:=Stabilizer(F,1);
Group([ (2,3,4,5) ])
gap> C:=Group((2,4)(3,5));
Group([ (2,4)(3,5) ])
gap> Index(F1,C);
2
gap> Index(DELTA(5,F,F1),DELTA(5,F,C));
2

```

4.2 Maximal extensions

For any $F \leq \text{Aut}(B_{d,k})$ that satisfies (C), the group $\Phi(F) \leq \text{Aut}(B_{d,k+1})$ is the maximal extension of F that satisfies (C) as well. It stems from the action of $U_k(F)$ on balls of radius $k+1$ in T_d .

4.2.1 PHI

- ▷ $\text{PHI}(F)$ (operation)
- ▷ $\text{PHI}(l, F)$ (operation)

for the arguments d, k, F

Returns: the group $\Phi_k(F) = \{(a, (a_\omega)_\omega) \mid a \in F, \forall \omega \in \Omega : a_\omega \in C_F(a, \omega)\} \leq \text{Aut}(B_{d,k+1})$.

The argument of this method is a local action $F \leq \text{Aut}(B_{d,k})$.

for the arguments l, d, k, F

Returns: the group $\Phi^l(F) = \Phi_{l-1} \circ \dots \circ \Phi_{k+1} \circ \Phi_k(F) \leq \text{Aut}(B_{d,l})$.

The arguments of this method are a radius $l \in \mathbb{N}$ and a local action $F \leq \text{Aut}(B_{d,k})$.

— Example —

```

gap> PHI(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6), (1,2), (3,4), (5,6) ])
gap> last=AutBall(3,2);
true
gap> PHI(3,AlternatingGroup(3));
Group([ (1,4,5)(2,3,6) ])
gap> last=GAMMA(3,AlternatingGroup(3));
true

```

— Example —

```

gap> S3:=LocalAction(3,1,SymmetricGroup(3));;
gap> groups:=ConjugacyClassRepsCompatibleSubgroupsWithProjection(2,S3);
[ Group([ (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (1,2)(3,4)(5,6), (1,2)(3,5)(4,6), (1,4,5)(2,3,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5,4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (3,5)(4,6) ]),
  Group([ (3,4)(5,6), (1,2)(3,4), (1,4,5)(2,3,6), (5,6), (3,5,4,6) ]) ]
gap> for G in groups do Print(Size(G),",",Size(PHI(G)),"\n"); od;
6,6
12,12
24,192

```

```
24,192
48,3072
```

Example

```
gap> PHI(3,LocalAction(4,1,SymmetricGroup(4)));
<permutation group with 34 generators>
gap> last=AutBall(4,3);
true
```

Example

```
ap> rho:=SignHomomorphism(SymmetricGroup(3));;
gap> F:=PI(2,3,SymmetricGroup(3),rho,[1]);; Size(F);
24
gap> P:=PHI(4,F);; Size(P);
12288
gap> IsSubgroup(AutBall(3,4),P);
true
gap> SatisfiesC(P);
true
```

4.3 Normal subgroups and partitions

When point stabilizers in $F \leq S_d$ are not simple, or F preserves a partition, more universal groups can be constructed as follows.

4.3.1 PHI

- ▷ $\text{PHI}(d, F, N)$ (operation)
- ▷ $\text{PHI}(d, F, P)$ (operation)
- ▷ $\text{PHI}(F, P)$ (operation)

for the arguments d, F, N

Returns: the group $\Phi(F, N) \leq \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a *transitive* permutation group $F \leq S_d$ and a normal subgroup N of the stabilizer F_1 of 1 in F .

for the arguments d, F, P

Returns: the group $\Phi(F, P) = \{(a, (a_\omega)_\omega) \mid a \in F, a_\omega \in C_F(a, \omega) \text{ constant w.r.t. } P\} \leq \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$ and a permutation group $F \leq S_d$ and a partition P of $[1 \dots d]$ preserved by F .

for the arguments d, k, F, P

Returns: the group $\Phi_k(F, P) = \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F, \alpha_\omega \in C_F(\alpha, \omega) \text{ constant w.r.t. } P\} \leq \text{Aut}(B_{d,k+1})$.

The arguments of this method are a local action $F \leq \text{Aut}(B_{d,k})$ and a partition P of $[1 \dots d]$ preserved by $\pi F \leq S_d$. This method assumes that all compatibility sets with respect to a partition element are non-empty and that all compatibility sets of the identity with respect to a partition element are non-trivial.

Example

```

gap> F:=SymmetricGroup(4);;
gap> F1:=Stabilizer(P,1);
Sym( [ 2 .. 4 ] )
gap> grps:=NormalSubgroups(F1);
[ Sym( [ 2 .. 4 ] ), Alt( [ 2 .. 4 ] ), Group(()) ]
gap> N:=grps[2];
Alt( [ 2 .. 4 ] )
gap> PHI(4,F,N);
Group([ (1,5,9,10)(2,6,7,11)(3,4,8,12), (1,4)(2,5)(3,6)(7,8)(10,11),
(1,2,3) ])
gap> Index(F1,N);
2
gap> Index(PHI(4,F,F1),PHI(4,F,N));
16

```

Example

```

gap> F:=TransitiveGroup(4,3);
D(4)
gap> P:=Blocks(F,[1..4]);
[ [ 1, 3 ], [ 2, 4 ] ]
gap> G:=PHI(4,F,P);
Group([ (1,5,9,10)(2,6,7,11)(3,4,8,12), (1,8)(2,7)(3,9)(4,5)(10,12), (1,3)
(8,9), (4,5)(10,12) ])
gap> aut:=Random(G);
(1,5,9,10)(2,6,7,11)(3,4,8,12)
gap> LocalAction(1,4,2,a,[1]); LocalAction(1,4,2,a,[3]);
(1,2,3,4)
(1,2,3,4)
gap> LocalAction(1,4,2,a,[2]); LocalAction(1,4,2,a,[4]);
(1,4)(2,3)
(1,4)(2,3)

```

Example

```

gap> H:=TransitiveGroup(4,3);
D(4)
gap> P:=Blocks(H,[1..4]);
[ [ 1, 3 ], [ 2, 4 ] ]
gap> F:=PHI(4,H,P);;
gap> G:=PHI(F,P);
<permutation group with 5 generators>
gap> SatisfiesC(G);
true

```

4.4 Abelian quotients

When a permutation group $F \leq S_d$ is not perfect, i.e. it admits an abelian quotient $\rho : F \twoheadrightarrow A$, more universal groups can be constructed by imposing restrictions of the form $\prod_{r \in R} \prod_{x \in S(b,r)} \rho(\sigma_1(\alpha, x)) = 1$ on elements $\alpha \in \Phi^k(F) \leq \text{Aut}(B_{d,k})$.

4.4.1 SignHomomorphism

▷ `SignHomomorphism(F)` (function)

Returns: the sign homomorphism from F to S_2 .

The argument of this method is a permutation group $F \leq S_d$. This method can be used as an example for the argument *rho* in the methods `SpheresProduct` (4.4.3) and `PI` (4.4.4).

Example

```
gap> F:=SymmetricGroup(3);;
gap> sign:=SignHomomorphism(F);
MappingByFunction( Sym( [ 1 .. 3 ] ), Sym( [ 1 .. 2 ] ), function( g ) ... end )
gap> Image(sign,(2,3));
(1,2)
gap> Image(sign,(1,2,3));
()
```

4.4.2 AbelianizationHomomorphism

▷ `AbelianizationHomomorphism(F)` (function)

Returns: the homomorphism from F to $F/[F, F]$.

The argument of this method is a permutation group $F \leq S_d$. This method can be used as an example for the argument *rho* in the methods `SpheresProduct` (4.4.3) and `PI` (4.4.4).

Example

```
gap> F:=PrimitiveGroup(5,3);
AGL(1, 5)
gap> ab:=AbelianizationHomomorphism(PrimitiveGroup(5,3));
[ (2,3,4,5), (1,2,3,5,4) ] -> [ f1, <identity> of ... ]
gap> Elements(Range(ab));
[ <identity> of ..., f1, f2, f1*f2 ]
gap> StructureDescription(Range(ab));
"C4"
```

4.4.3 SpheresProduct

▷ `SpheresProduct(d, k, aut, rho, R)` (function)

Returns: the product $\prod_{r \in R} \prod_{x \in S(b,r)} rho(\sigma_1(aut, x)) \in \text{im}(rho)$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a radius $k \in \mathbb{N}$, an automorphism *aut* of $B_{d,k}$ all of whose 1-local actions are in the domain of the homomorphism *rho* from a subgroup of S_d to an abelian group, and a sublist *R* of $[0..k-1]$. This method is used in the implementation of `PI` (4.4.4).

Example

```
gap> rho:=SignHomomorphism(SymmetricGroup(3));;
gap> SpheresProduct(3,2,gamma(2,3,(1,2)),rho,[0]);
(1,2)
gap> SpheresProduct(3,2,gamma(2,3,(1,2)),rho,[0,1]);
()
```

Example

```
gap> F:=PrimitiveGroup(5,3);
AGL(1, 5)
gap> rho:=AbelianizationHomomorphism(F);;
gap> Elements(Range(rho));
```

```

[ <identity> of ..., f1, f2, f1*f2 ]
gap> StructureDescription(Range(rho));
"C4"
gap> aut:=Random(F);
(1,2,4,5)
gap> SpheresProduct(5,3,gamma(3,5,aut),rho,[2]);
<identity> of ...
gap> SpheresProduct(5,3,gamma(3,5,aut),rho,[1,2]);
f1*f2
gap> SpheresProduct(5,3,gamma(3,5,aut),rho,[0,1,2]);
f2

```

4.4.4 PI

▷ $\text{PI}(l, d, F, \text{rho}, R)$ (function)

Returns: the group $\Pi^l(F, \text{rho}, R) = \{\alpha \in \Phi^l(F) \mid \prod_{r \in R} \prod_{x \in S(b,r)} \text{rho}(\sigma_1(\alpha, x)) = 1\} \leq \text{Aut}(B_{d,l})$.

The arguments of this method are a degree $l \in \mathbb{N}_{\geq 2}$, a radius $d \in \mathbb{N}_{\geq 3}$, a permutation group $F \leq S_d$, a homomorphism ρ from F to an abelian group that is surjective on every point stabilizer in F , and a non-empty, non-zero subset R of $[0..l-1]$ that contains $l-1$.

Example

```

gap> F:=LocalAction(5,1,PrimitiveGroup(5,3));
AGL(1, 5)
gap> rho1:=AbelianizationHomomorphism(F);;
gap> rho2:=SignHomomorphism(F);;
gap> PI(3,5,F,rho1,[0,1,2]);
<permutation group with 4 generators>
gap> Index(PHI(3,F),last);
4
gap> PI(3,5,F,rho2,[0,1,2]);
<permutation group with 6 generators>
gap> Index(PHI(3,F),last);
2

```

4.5 Semidirect products

When a subgroup $F \leq \text{Aut}(B_{d,k})$ satisfies (C) and admits an involutive compatibility cocycle z (which is automatic when $k = 1$) one can characterise the kernels $K \leq \Phi_k(F) \cap \ker(\pi_k)$ that fit into a z -split exact sequence $1 \rightarrow K \rightarrow \Sigma(F, K) \rightarrow F \rightarrow 1$ for some subgroup $\Sigma(F, K) \leq \text{Aut}(B_{d,k+1})$ that satisfies (C). This characterisation is implemented in this section.

4.5.1 CompatibleKernels

▷ $\text{CompatibleKernels}(d, F)$ (operation)
 ▷ $\text{CompatibleKernels}(F, z)$ (operation)

for the arguments d, F

Returns: the list of kernels $K \leq \prod_{\omega \in \Omega} F_\omega \cong \ker \pi \leq \text{Aut}(B_{d,2})$ that are preserved by the action $F \curvearrowright \prod_{\omega \in \Omega} F_\omega$, $a \cdot (a_\omega)_\omega := (aa_{a^{-1}\omega}a^{-1})_\omega$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, and a permutation group $F \leq S_d$. The kernels output by this method are compatible with F with respect to the standard cocycle (see `InvolutiveCompatibilityCocycle (??)`) and can be used in the method `SIGMA (4.5.2)`.

for the arguments d, k, F, z

Returns: the list of kernels $K \leq \Phi_k(F) \cap \ker(\pi_k) \leq \text{Aut}(B_{d,k+1})$ that are normalized by $\Gamma_z(F)$ and such that for all $k \in K$ and $\omega \in \Omega$ there is $k_\omega \in K$ with $\text{pr}_\omega k_\omega = z(\text{pr}_\omega k, \omega)^{-1}$.

The arguments of this method are a local action $F \leq \text{Aut}(B_{d,k})$ that satisfies (C) and an involutive compatibility cocycle z of F (see `InvolutiveCompatibilityCocycle (??)`). It can be used in the method `SIGMA (4.5.2)`.

Example

```
gap> CompatibleKernels(3,SymmetricGroup(3));
[ Group(()), Group([ (1,2)(3,4)(5,6) ]), Group([ (3,4)(5,6), (1,2)(5,6) ]),
  Group([ (5,6), (3,4), (1,2) ]) ]
```

Example

```
gap> P:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(P);;
gap> F:=PI(2,3,P,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(F);;
[ Group(()), Group([ (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) ]),
  Group([ (1,2)(3,4)(5,6)(7,8), (5,6)(7,8)(9,10)(11,12) ]),
  Group([ (5,6)(7,8), (1,2)(3,4), (9,10)(11,12) ]) ]
```

4.5.2 SIGMA

- ▷ `SIGMA(d, F, K)` (operation)
- ▷ `SIGMA(F, K, z)` (operation)

for the arguments d, F, K

Returns: the semidirect product $\Sigma(F, K) \leq \text{Aut}(B_{d,2})$.

The arguments of this method are a degree $d \in \mathbb{N}_{\geq 3}$, a subgroup F of S_d and a compatible kernel K for F (see `CompatibleKernels (4.5.1)`).

for the arguments d, k, F, K, z

Returns: the semidirect product $\Sigma_z(F, K) \leq \text{Aut}(B_{d,k+1})$.

The arguments of this method are a local action F of $\text{Aut}(B_{d,k})$ that satisfies (C) and a kernel K that is compatible for F with respect to the involutive compatibility cocycle z (see `InvolutiveCompatibilityCocycle (??)` and `CompatibleKernels (4.5.1)`) of F .

Example

```
gap> S3:=SymmetricGroup(3);;
gap> kernels:=CompatibleKernels(3,S3);
[ Group(()), Group([ (1,2)(3,4)(5,6) ]), Group([ (3,4)(5,6), (1,2)(5,6) ]),
  Group([ (5,6), (3,4), (1,2) ]) ]
gap> for K in kernels do Print(Size(SIGMA(3,S3,K)),"\n"); od;
6
12
24
48
```

Example

```
gap> P:=SymmetricGroup(3);;
gap> rho:=SignHomomorphism(P);;
gap> F:=PI(2,3,P,rho,[1]);;
gap> z:=InvolutiveCompatibilityCocycle(F);;
gap> kernels:=CompatibleKernels(F,z);
[ Group(), Group([ (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) ]),
  Group([ (1,2)(3,4)(5,6)(7,8), (5,6)(7,8)(9,10)(11,12) ]),
  Group([ (5,6)(7,8), (1,2)(3,4), (9,10)(11,12) ]) ]
gap> for K in kernels do Print(Size(SIGMA(F,K,z)),"\n"); od;
24
48
96
192
```

Chapter 5

Discreteness

This chapter contains functions that are related to the discreteness property (D) presented in Proposition 3.12 of [Tor20].

5.1 The discreteness condition (D)

Said proposition shows that for a given $F \leq \text{Aut}(B_{d,k})$ the group $U_k(F)$ is discrete if and only if the maximal compatible subgroup $C(F)$ of F satisfies condition (D):

$$\forall \omega \in \Omega : F_{T_\omega} = \{\text{id}\},$$

where T_ω is the $k-1$ -neighbourhood of the the edge (b, b_ω) inside $B_{d,k}$. In other words, F satisfies (D) if and only if the compatibility set $C_F(\text{id}, \omega) = \{\text{id}\}$. We distinguish between F satisfying condition (D) and $U_k(F)$ being discrete with the methods `SatisfiesD(??)` and `IsDiscrete(??)` below.

5.2 Discreteness

5.2.1 SatisfiesD (for IsLocalAction)

▷ `SatisfiesD(F)` (property)

Returns: true if F satisfies the discreteness condition (D), and false otherwise.

The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see `IsLocalAction(??)`).

Example

```
gap> G:=GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> SatisfiesD(G);
true
```

5.2.2 IsDiscrete (for IsLocalAction)

▷ `IsDiscrete(F)` (property)

Returns: true if $U_k(F)$ is discrete, and false otherwise.

The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see `IsLocalAction(??)`).

Example

```
gap> G:=GAMMA(3,SymmetricGroup(3));
Group([ (1,4,5)(2,3,6), (1,3)(2,4)(5,6) ])
gap> IsDiscrete(G);
true
```

Example

```
gap> F:=LocalAction(3,2,Group((1,2)));
Group([ (1,2) ])
gap> IsDiscrete(F);
true
gap> SatisfiesD(F);
false
gap> C:=MaximalCompatibleSubgroup(F);
Group(())
gap> SatisfiesD(C);
true
```

5.3 Cocycles

Subgroups $F \leq \text{Aut}(B_{d,k})$ that satisfy both (C) and (D) admit an involutive compatibility cocycle, i.e. a map $z: F \times \{1, \dots, d\} \rightarrow F$ that satisfies certain properties, see [Tor20, Section 3.2.2]. When F satisfies just (C), it may still admit an involutive compatibility cocycle. In this case, F admits an extension $\Gamma_z(F) \leq \text{Aut}(B_{d,k})$ that satisfies both (C) and (D). Involutive compatibility cocycles can be searched for using `InvolutiveCompatibilityCocycle(??)` and `AllInvolutiveCompatibilityCocycles(??)` below.

5.3.1 InvolutiveCompatibilityCocycle (for IsLocalAction)

▷ `InvolutiveCompatibilityCocycle(F)` (attribute)

Returns: an involutive compatibility cocycle of F , which is a mapping $F \times [1..d] \rightarrow F$ with certain properties, if it exists, and fail otherwise. When $k = 1$, the standard cocycle is returned.

The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see `IsLocalAction(??)`), which is compatible (see `SatisfiesC(??)`).

Example

```
gap> F:=LocalAction(3,1,AlternatingGroup(3));;
gap> InvolutiveCompatibilityCocycle(F);
MappingByFunction( Domain([ [ () , 1 ], [ () , 2 ], [ () , 3 ], [ (1,3,2), 1 ],
  [ (1,3,2), 2 ], [ (1,3,2), 3 ], [ (1,2,3), 1 ], [ (1,2,3), 2 ],
  [ (1,2,3), 3 ] ]), Alt([ 1 .. 3 ]), function( s ) ... end )
gap> a:=Random(AlternatingGroup(3));; dir:=Random([1..3]);;
gap> a; Image(z,[a,dir]);
(1,3,2)
(1,3,2)
```

Example

```
gap> G:=GAMMA(3,AlternatingGroup(3));
Group([ (1,4,5)(2,3,6) ])
gap> InvolutiveCompatibilityCocycle(G);
MappingByFunction( Domain([ [ () , 1 ], [ () , 2 ], [ () , 3 ],
  [ (1,5,4)(2,6,3), 1 ], [ (1,5,4)(2,6,3), 2 ], [ (1,5,4)(2,6,3), 3 ],
```

```

[ (1,4,5)(2,3,6), 1 ], [ (1,4,5)(2,3,6), 2 ], [ (1,4,5)(2,3,6), 3 ]
]), Group([ (1,4,5)(2,3,6) ]), function( s ) ... end )
gap> InvolutiveCompatibilityCocycle(AutBall(3,2));
fail

```

5.3.2 AllInvolutiveCompatibilityCocycles (for IsLocalAction)

▷ AllInvolutiveCompatibilityCocycles(F) (attribute)

Returns: the list of all involutive compatibility cocycles of F .

The argument of this attribute is a local action $F \leq \text{Aut}(B_{d,k})$ (see IsLocalAction (??)), which is compatible (see SatisfiesC (??)).

Example

```

gap> S3:=LocalAction(3,1,SymmetricGroup(3));
gap> Size(AllInvolutiveCompatibilityCocycles(S3));
4
gap> Size(AllInvolutiveCompatibilityCocycles(GAMMA(3,SymmetricGroup(3))));
1

```

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