

numericalsgps, a package for numerical semigroups

(Version 0.98)

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Colophon

This work started when (in 2004) the first author visited the University of Granada in part of a sabbatical year. Since Version 0.96, the package is maintained by the first two authors. Bug reports, suggestions and comments are, of course, welcome. Please use our email addresses to this effect.

If you have benefited from the use of the *numericalsgps* GAP package in your research, please cite it in addition to GAP itself, following the scheme proposed in <http://www.gap-system.org/Contacts/cite.html>.

If you have predominantly used the functions in the Appendix, contributed by other authors, please cite in addition these authors, referring "software implementations available in the GAP package *NumericalSgps*".

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Chapter 1

Introduction

A *numerical semigroup* is a subset of the set \mathbb{N} of nonnegative integers that is closed under addition, contains 0 and whose complement in \mathbb{N} is finite. The smallest positive integer belonging to a numerical semigroup is its *multiplicity*.

Let S be a numerical semigroup and A be a subset of S . We say that A is a *system of generators* of S if $S = \{k_1 a_1 + \cdots + k_n a_n \mid n, k_1, \dots, k_n \in \mathbb{N}, a_1, \dots, a_n \in A\}$. The set A is a *minimal system of generators* of S if no proper subset of A is a system of generators of S .

Every numerical semigroup has a unique minimal system of generators. This is a data that can be used in order to uniquely define a numerical semigroup. Observe that since the complement of a numerical semigroup in the set of nonnegative integers is finite, this implies that the greatest common divisor of the elements of a numerical semigroup is 1, and the same condition must be fulfilled by its minimal system of generators (or by any of its systems of generators).

Given a numerical semigroup S and a nonzero element s in it, one can consider for every integer i ranging from 0 to $s - 1$, the smallest element in S congruent with i modulo s , say $w(i)$ (this element exists since the complement of S in \mathbb{N} is finite). Clearly $w(0) = 0$. The set $\text{Ap}(S, s) = \{w(0), w(1), \dots, w(s - 1)\}$ is called the *Apéry set* of S with respect to s . Note that a nonnegative integer x congruent with i modulo s belongs to S if and only if $w(i) \leq x$. Thus the pair $(s, \text{Ap}(S, s))$ fully determines the numerical semigroup S (and can be used to easily solve the membership problem to S). This set is in fact one of the most powerful tools known for numerical semigroups, and it is used almost everywhere in the computation of components and invariants associated to a numerical semigroup. Usually the element s is taken to be the multiplicity, since in this way the resulting Apéry set is the smallest possible.

A *gap* of a numerical semigroup S is a nonnegative integer not belonging to S . The set of gaps of S is usually denoted by $H(S)$, and clearly determines uniquely S . Note that if x is a gap of S , then so are all the nonnegative integers dividing it. Thus in order to describe S we do not need to know all its gaps, but only those that are maximal with respect to the partial order induced by division in \mathbb{N} . These gaps are called *fundamental gaps*.

The largest nonnegative integer not belonging to a numerical semigroup S is the *Frobenius number* of S . If S is the set of nonnegative integers, then clearly its Frobenius number is -1 , otherwise its Frobenius number coincides with the maximum of the gaps (or fundamental gaps) of S . The Frobenius number plus one is known as the *conductor* of the semigroup. In this package we refer to the elements in the semigroup that are less than or equal to the conductor as *small elements* of the semigroup. Observe that from the definition, if S is a numerical semigroup with Frobenius number f , then $f + \mathbb{N} \setminus \{0\} \subseteq S$. An integer z is a *pseudo-Frobenius number* of S if $z + S \setminus \{0\} \subseteq S$. Thus the

Frobenius number of S is one of its pseudo-Frobenius numbers. The *type* of a numerical semigroup is the cardinality of the set of its pseudo-Frobenius numbers.

The number of numerical semigroups having a given Frobenius number is finite. The elements in this set of numerical semigroups that are maximal with respect to set inclusion are precisely those numerical semigroups that cannot be expressed as intersection of two other numerical semigroups containing them properly, and thus they are known as *irreducible* numerical semigroups. Clearly, every numerical semigroup is the intersection of (finitely many) irreducible numerical semigroups.

A numerical semigroup S with Frobenius number f is *symmetric* if for every integer x , either $x \in S$ or $f - x \in S$. The set of irreducible numerical semigroups with odd Frobenius number coincides with the set of symmetric numerical semigroups. The numerical semigroup S is *pseudo-symmetric* if f is even and for every integer x not equal to $f/2$ either $x \in S$ or $f - x \in S$. The set of irreducible numerical semigroups with even Frobenius number is precisely the set of pseudo-symmetric numerical semigroups. These two classes of numerical semigroups have been widely studied in the literature due to their nice applications in Algebraic Geometry. This is probably one of the main reasons that made people turn their attention on numerical semigroups again in the last decades. Symmetric numerical semigroups can be also characterized as those with type one, and pseudo-symmetric numerical semigroups are those numerical semigroups with type two and such that its pseudo-Frobenius numbers are its Frobenius number and its Frobenius number divided by two.

Another class of numerical semigroups that caught the attention of researchers working on Algebraic Geometry and Commutative Ring Theory is the class of numerical semigroups with maximal embedding dimension. The *embedding dimension* of a numerical semigroup is the cardinality of its minimal system of generators. It can be shown that the embedding dimension is at most the multiplicity of the numerical semigroup. Thus *maximal embedding dimension* numerical semigroups are those numerical semigroups for which their embedding dimension and multiplicity coincide. These numerical semigroups have nice maximal properties, not only (of course) related to their embedding dimension, but also by means of their presentations. Among maximal embedding dimension there are two classes of numerical semigroups that have been studied due to the connections with the equivalence of algebroid branches. A numerical semigroup S is *Arf* if for every $x \geq y \geq z \in S$, then $x + y - z \in S$; and it is *saturated* if the following condition holds: if $s, s_1, \dots, s_r \in S$ are such that $s_i \leq s$ for all $i \in \{1, \dots, r\}$ and $z_1, \dots, z_r \in \mathbb{Z}$ are such that $z_1 s_1 + \dots + z_r s_r \geq 0$, then $s + z_1 s_1 + \dots + z_r s_r \in S$.

If we look carefully inside the set of fundamental gaps of a numerical semigroup, we see that there are some fulfilling the condition that if they are added to the given numerical semigroup, then the resulting set is again a numerical semigroup. These elements are called *special gaps* of the numerical semigroup. A numerical semigroup other than the set of nonnegative integers is irreducible if and only if it has only a special gap.

The inverse operation to the one described in the above paragraph is that of removing an element of a numerical semigroup. If we want the resulting set to be a numerical semigroup, then the only thing we can remove is a minimal generator.

Let a, b, c, d be positive integers such that $a/b < c/d$, and let $I = [a/b, c/d]$. Then the set $S(I) = \mathbb{N} \cap \bigcup_{n \geq 0} nI$ is a numerical semigroup. This class of numerical semigroups coincides with that of sets of solutions to equations of the form $Ax \bmod B \leq Cx$ with A, B, C positive integers. A numerical semigroup in this class is said to be *proportionally modular*.

A sequence of positive rational numbers $a_1/b_1 < \dots < a_n/b_n$ with a_i, b_i positive integers is a *Bézout sequence* if $a_{i+1}b_i - a_i b_{i+1} = 1$ for all $i \in \{1, \dots, n-1\}$. If $a/b = a_1/b_1 < \dots < a_n/b_n = c/d$, then $S([a/b, c/d]) = \langle a_1, \dots, a_n \rangle$. Bézout sequences are not only interesting for this fact, they have shown to be a major tool in the study of proportionally modular numerical semigroups.

If S is a numerical semigroup and k is a positive integer, then the set $S/k = \{x \in \mathbb{N} \mid kx \in S\}$ is a

numerical semigroup, known as the *quotient* S by k .

Let m be a positive integer. A *subadditive* function with period m is a map $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(0) = 0$, $f(x+y) \leq f(x) + f(y)$ and $f(x+m) = f(x)$. If f is a subadditive function with period m , then the set $M_f = \{x \in \mathbb{N} \mid f(x) \leq x\}$ is a numerical semigroup. Moreover, every numerical semigroup is of this form. Thus a numerical semigroup can be given by a subadditive function with a given period. If S is a numerical semigroup and $s \in S, s \neq 0$, and $\text{Ap}(S, s) = \{w(0), w(1), \dots, w(s-1)\}$, then $f(x) = w(x \bmod s)$ is a subadditive function with period s such that $M_f = S$.

Let S be a numerical semigroup generated by $\{n_1, \dots, n_k\}$. Then we can define the following morphism (called sometimes the factorization morphism) by $\varphi : \mathbb{N}^k \rightarrow S$, $\varphi(a_1, \dots, a_k) = a_1 n_1 + \dots + a_k n_k$. If σ is the kernel congruence of φ (that is, $a \sigma b$ if $\varphi(a) = \varphi(b)$), then S is isomorphic to \mathbb{N}^k / σ . A *presentation* for S is a system of generators (as a congruence) of σ . If $\{n_1, \dots, n_p\}$ is a minimal system of generators, then a *minimal presentation* is a presentation such that none of its proper subsets is a presentation. Minimal presentations of numerical semigroups coincide with presentations with minimal cardinality, though in general these two concepts are not the same for an arbitrary commutative semigroup.

A set I of integers is an *ideal relative to a numerical semigroup* S provided that $I + S \subseteq I$ and that there exists $d \in S$ such that $d + I \subseteq S$. If $I \subseteq S$, we simply say that I is an *ideal* of S . If I and J are relative ideals of S , then so is $I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\}$, and it is tightly related to the operation ":" of ideals in a commutative ring.

In this package we have implemented the functions needed to deal with the elements exposed in this introduction.

Many of the algorithms, and the necessary background to understand them, can be found in the monograph [RGS09]. Some examples in this book have been illustrated with the help of this package. So the reader can also find there more examples on the usage of the functions implemented here.

This package was presented in [DGSM06].

Chapter 2

Numerical Semigroups

This chapter describes how to create numerical semigroups in **GAP** and perform some basic tests.

2.1 Generating Numerical Semigroups

Recalling some definitions from Chapter 1.

A numerical semigroup is a subset of the set \mathbb{N} of nonnegative integers that is closed under addition, contains 0 and whose complement in \mathbb{N} is finite.

We refer to the elements in a numerical semigroup that are less than or equal to the conductor as *small elements* of the semigroup.

A *gap* of a numerical semigroup S is a nonnegative integer not belonging to S . The *fundamental gaps* of S are those gaps that are maximal with respect to the partial order induced by division in \mathbb{N} .

Given a numerical semigroup S and a nonzero element s in it, one can consider for every integer i ranging from 0 to $s - 1$, the smallest element in S congruent with i modulo s , say $w(i)$ (this element exists since the complement of S in \mathbb{N} is finite). Clearly $w(0) = 0$. The set $\text{Ap}(S, s) = \{w(0), w(1), \dots, w(s - 1)\}$ is called the *Apéry set* of S with respect to s .

Let a, b, c, d be positive integers such that $a/b < c/d$, and let $I = [a/b, c/d]$. Then the set $S(I) = \mathbb{N} \cap \bigcup_{n \geq 0} nI$ is a numerical semigroup. This class of numerical semigroups coincides with that of sets of solutions to equations of the form $Ax \bmod B \leq Cx$ with A, B, C positive integers. A numerical semigroup in this class is said to be *proportionally modular*. If $C = 1$, then it is said to be *modular*.

There are several different ways to specify a numerical semigroup S , namely, by its generators; by its gaps, its fundamental or special gaps by its Apéry set, just to name some. In this section we describe functions that may be used to specify, in one of these ways, a numerical semigroup in **GAP**.

To create a numerical semigroup in **GAP** the function `NumericalSemigroup` is used.

2.1.1 NumericalSemigroup

▷ `NumericalSemigroup(Representation, List)` (function)

`Representation`

May be "generators", "minimalgenerators", "modular", "propmodular", "elements", "gaps", "fundamentalgaps", "subadditive" or "apery" according to whether the semigroup is to be given by means of a system of generators, a minimal system of generators, a condition of the

form $ax \bmod m \leq x$, a condition of the form $ax \bmod m \leq cx$, a set of all elements up to the conductor, the set of gaps, the set of fundamental gaps, a periodic subadditive function, or the Apéry set.

When no string is given as first argument it is assumed that the numerical semigroup will be given by means of a set of generators.

List

When the semigroup is given through a set of generators, this set may be given as a list or through its individual elements.

The set of all elements up to the conductor, the set of gaps, the set of fundamental gaps or the Apéry set are given through lists.

A periodic subadditive function with period m is given through the list of images of the elements, from 1 to m . The image of m has to be 0.

Numerical semigroups generated by an interval of positive integers and embedding dimension two numerical semigroups are known to be proportionally modular, and thus they are treated as such (unless the representation "minimalgenerators" is specified), since membership and other problems are solved faster for these semigroups.

Example

```
gap> s1 := NumericalSemigroup("generators",3,5,7);
<Numerical semigroup with 3 generators>
gap> s2 := NumericalSemigroup("generators",[3,5,7]);
<Numerical semigroup with 3 generators>
gap> s1=s2;
true
gap> s := NumericalSemigroup("minimalgenerators",3,7);
<Numerical semigroup with 2 generators>
gap> s := NumericalSemigroup("modular",3,5);
<Modular numerical semigroup satisfying 3x mod 5 <= x >
gap> s1:=NumericalSemigroup("generators",2,5);
<Modular numerical semigroup satisfying 5x mod 10 <= x >
gap> s = s1;
true
gap> s:=NumericalSemigroup(4,5,6);
<Proportionally modular numerical semigroup satisfying 6x mod 24 <= 2x >
```

Once it is known that a numerical semigroup contains the element 1, i.e. the semigroup is \mathbb{N} , the semigroup is treated as such.

Example

```
gap> NumericalSemigroup(1);
<The numerical semigroup N>
gap> NumericalSemigroupByInterval(1/3,1/2);
<The numerical semigroup N>
```

2.1.2 ModularNumericalSemigroup

▷ `ModularNumericalSemigroup(a, b)` (function)

Given two positive integers a and b , this function returns a modular numerical semigroup satisfying $ax \bmod b \leq x$.

Example

```
gap> ModularNumericalSemigroup(3,7);
<Modular numerical semigroup satisfying  $3x \bmod 7 \leq x$ >
```

2.1.3 ProportionallyModularNumericalSemigroup

▷ `ProportionallyModularNumericalSemigroup(a, b, c)` (function)

Given three positive integers a , b and c , this function returns a proportionally modular numerical semigroup satisfying $ax \bmod b \leq cx$.

Example

```
gap> ProportionallyModularNumericalSemigroup(3,7,12);
<Proportionally modular numerical semigroup satisfying  $3x \bmod 7 \leq 12x$ >
```

When $c = 1$, the semigroup is seen as a modular numerical semigroup.

Example

```
gap> NumericalSemigroup("propmodular",67,98,1);
<Modular numerical semigroup satisfying  $67x \bmod 98 \leq x$ >
```

2.1.4 NumericalSemigroupByGenerators

▷ `NumericalSemigroupByGenerators(List)` (function)
 ▷ `NumericalSemigroupByMinimalGenerators(List)` (function)
 ▷ `NumericalSemigroupByMinimalGeneratorsNC(List)` (function)
 ▷ `NumericalSemigroupByInterval(List)` (function)
 ▷ `NumericalSemigroupByOpenInterval(List)` (function)
 ▷ `NumericalSemigroupBySubAdditiveFunction(List)` (function)
 ▷ `NumericalSemigroupByAperyList(List)` (function)
 ▷ `NumericalSemigroupBySmallElements(List)` (function)
 ▷ `NumericalSemigroupByGaps(List)` (function)
 ▷ `NumericalSemigroupByFundamentalGaps(List)` (function)

The function `NumericalSemigroup` (2.1.1) is a front-end for these functions. The argument of each of these functions is a list representing an entity of the type to which the function's name refers.

Example

```
gap> s:=NumericalSemigroup(3,11);
<Modular numerical semigroup satisfying  $22x \bmod 33 \leq x$ >
gap> GapsOfNumericalSemigroup(s);
[ 1, 2, 4, 5, 7, 8, 10, 13, 16, 19 ]
gap> t:=NumericalSemigroupByGaps(last);
<Numerical semigroup>
gap> s=t;
true

gap> AperyListOfNumericalSemigroupWRTElement(s,20);;
gap> t:=NumericalSemigroupByAperyList(last);
<Numerical semigroup>
gap> s=t;
true
```

2.2 Some basic tests

This section describes some basic tests on numerical semigroups. The first described tests refer to the way the semigroup was created. Then are presented functions to test if a given list represents the small elements, gaps or the Apéry set (see 1) of a numerical semigroup; to test if an integer belongs to a numerical semigroup and if a numerical semigroup is a subsemigroup of another one.

2.2.1 IsNumericalSemigroup

- ▷ IsNumericalSemigroup(*NS*) (attribute)
- ▷ IsNumericalSemigroupByGenerators(*NS*) (attribute)
- ▷ IsNumericalSemigroupByMinimalGenerators(*NS*) (attribute)
- ▷ IsNumericalSemigroupByInterval(*NS*) (attribute)
- ▷ IsNumericalSemigroupByOpenInterval(*NS*) (attribute)
- ▷ IsNumericalSemigroupBySubAdditiveFunction(*NS*) (attribute)
- ▷ IsNumericalSemigroupByAperyList(*NS*) (attribute)
- ▷ IsNumericalSemigroupBySmallElements(*NS*) (attribute)
- ▷ IsNumericalSemigroupByGaps(*NS*) (attribute)
- ▷ IsNumericalSemigroupByFundamentalGaps(*NS*) (attribute)
- ▷ IsProportionallyModularNumericalSemigroup(*NS*) (attribute)
- ▷ IsModularNumericalSemigroup(*NS*) (attribute)

NS is a numerical semigroup and these attributes are available (their names should be self explanatory).

Example

```
gap> s:=NumericalSemigroup(3,7);
<Modular numerical semigroup satisfying 7x mod 21 <= x >
gap> AperyListOfNumericalSemigroupWRTElement(s,30);;
gap> t:=NumericalSemigroupByAperyList(last);
<Numerical semigroup>
gap> IsNumericalSemigroupByGenerators(s);
true
gap> IsNumericalSemigroupByGenerators(t);
false
gap> IsNumericalSemigroupByAperyList(s);
false
gap> IsNumericalSemigroupByAperyList(t);
true
```

2.2.2 RepresentsSmallElementsOfNumericalSemigroup

- ▷ RepresentsSmallElementsOfNumericalSemigroup(*L*) (attribute)

Tests if the list *L* (which has to be a set) may represent the “small” # elements of a numerical semigroup.

Example

```
gap> L:=[ 0, 3, 6, 9, 11, 12, 14, 15, 17, 18, 20 ];
[ 0, 3, 6, 9, 11, 12, 14, 15, 17, 18, 20 ]
gap> RepresentsSmallElementsOfNumericalSemigroup(L);
```

```

true
gap> L:=[ 6, 9, 11, 12, 14, 15, 17, 18, 20 ];
[ 6, 9, 11, 12, 14, 15, 17, 18, 20 ]
gap> RepresentsSmallElementsOfNumericalSemigroup(L);
false

```

2.2.3 RepresentsGapsOfNumericalSemigroup

▷ RepresentsGapsOfNumericalSemigroup(L) (attribute)

Tests if the list L may represent the gaps (see 1) of a numerical semigroup.

Example

```

gap> s:=NumericalSemigroup(3,7);
<Modular numerical semigroup satisfying 7x mod 21 <= x >
gap> L:=GapsOfNumericalSemigroup(s);
[ 1, 2, 4, 5, 8, 11 ]
gap> RepresentsGapsOfNumericalSemigroup(L);
true
gap> L:=Set(List([1..21],i->RandomList([1..50])));
[ 2, 6, 7, 8, 10, 12, 14, 19, 24, 28, 31, 35, 42, 50 ]
gap> RepresentsGapsOfNumericalSemigroup(L);
false

```

2.2.4 IsAperyListOfNumericalSemigroup

▷ IsAperyListOfNumericalSemigroup(L) (function)

Tests whether a list L of integers may represent the Apéry list of a numerical semigroup. It returns true when the periodic function represented by L is subadditive (see RepresentsPeriodicSubAdditiveFunction (A.2.1)) and the remainder of the division of $L[i]$ by the length of L is i and returns false otherwise (the criterium used is the one explained in [Ros96b]).

Example

```

gap> IsAperyListOfNumericalSemigroup([0,21,7,28,14]);
true

```

2.2.5 IsSubsemigroupOfNumericalSemigroup

▷ IsSubsemigroupOfNumericalSemigroup(S , T) (function)

S and T are numerical semigroups. Tests whether T is contained in S .

Example

```

gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> T:=NumericalSemigroup(2,3);
<Modular numerical semigroup satisfying 3x mod 6 <= x >
gap> IsSubsemigroupOfNumericalSemigroup(T,S);
true
gap> IsSubsemigroupOfNumericalSemigroup(S,T);
false

```

2.2.6 BelongsToNumericalSemigroup

▷ `BelongsToNumericalSemigroup(n , S)` (operation)

n is an integer and S is a numerical semigroup. Tests whether n belongs to S . n in S is the short for `BelongsToNumericalSemigroup(n , S)`.

Example

```
gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> BelongsToNumericalSemigroup(15,S);
false
gap> 15 in S;
false
gap> SmallElementsOfNumericalSemigroup(S);
[ 0, 11, 12, 13, 22, 23, 24, 25, 26, 32, 33, 34, 35, 36, 37, 38, 39, 43 ]
gap> BelongsToNumericalSemigroup(13,S);
true
gap> 13 in S;
true
```

Chapter 3

Basic operations with numerical semigroups

3.1 The definitions

3.1.1 MultiplicityOfNumericalSemigroup

▷ `MultiplicityOfNumericalSemigroup(NS)` (attribute)

NS is a numerical semigroup. Returns the multiplicity of *NS*, which is the smallest positive integer belonging to *NS*.

Example

```
gap> S := NumericalSemigroup("modular", 7, 53);  
<Modular numerical semigroup satisfying 7x mod 53 <= x >  
gap> MultiplicityOfNumericalSemigroup(S);  
8
```

3.1.2 GeneratorsOfNumericalSemigroup

▷ `GeneratorsOfNumericalSemigroup(S)` (function)
▷ `GeneratorsOfNumericalSemigroupNC(S)` (function)
▷ `ReducedSetOfGeneratorsOfNumericalSemigroup(S[, bool, n])` (function)
▷ `MinimalGeneratingSystemOfNumericalSemigroup(S)` (attribute)

S is a numerical semigroup. `GeneratorsOfNumericalSemigroup` returns a set of generators of *S*, which may not be minimal. `GeneratorsOfNumericalSemigroupNC` returns the set of generators recorded in *S*!.generators, which may not be minimal. `ReducedSetOfGeneratorsOfNumericalSemigroupNC` Returns a set with possibly fewer generators than those recorded in *S*!.generators. It changes *S*!.generators to the set returned. The function has 1 to 3 arguments. One of them a numerical semigroup. Then an argument is a boolean (*true* means that all the elements not belonging to the Apéry set with respect to the multiplicity are removed; the default is "false") and another argument is a positive integer *n* (meaning that generators that can be written as the sum of *n* or less generators are removed; the default is "2"). The boolean or the integer may not be present. If a minimal generating set for *S* is known or no generating set is known, then

the minimal generating system is returned. `MinimalGeneratingSystemOfNumericalSemigroup` returns the minimal set of generators of `S`.

Example

```
gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> GeneratorsOfNumericalSemigroup(S);
[ 11, 12, 13, 32, 53 ]
gap> S := NumericalSemigroup(3, 5, 53);
<Numerical semigroup with 3 generators>
gap> GeneratorsOfNumericalSemigroup(S);
[ 3, 5, 53 ]
gap> MinimalGeneratingSystemOfNumericalSemigroup(S);
[ 3, 5 ]
gap> ReducedSetOfGeneratorsOfNumericalSemigroup(NumericalSemigroup(5,7,9,10,25));
[ 5, 7, 9, 25 ]
gap> ReducedSetOfGeneratorsOfNumericalSemigroup(true,NumericalSemigroup(5,7,9,10,25,28));
[ 5, 7, 9, 28 ]
gap> ReducedSetOfGeneratorsOfNumericalSemigroup(NumericalSemigroup(5,7,9,10,25,28),3);
[ 5, 7, 9 ]
```

3.1.3 EmbeddingDimensionOfNumericalSemigroup

▷ `EmbeddingDimensionOfNumericalSemigroup(NS)` (attribute)

`NS` is a numerical semigroup. It returns the cardinality of its minimal generating system.

3.1.4 SmallElementsOfNumericalSemigroup

▷ `SmallElementsOfNumericalSemigroup(NS)` (attribute)

`NS` is a numerical semigroup. It returns the list of small elements of `NS`. Of course, the time consumed to return a result may depend on the way the semigroup is given.

Example

```
gap> SmallElementsOfNumericalSemigroup(NumericalSemigroup(3,5,7));
[ 0, 3, 5 ]
```

3.1.5 FirstElementsOfNumericalSemigroup

▷ `FirstElementsOfNumericalSemigroup(n, NS)` (function)

`NS` is a numerical semigroup. It returns the list with the first `n` elements of `NS`.

Example

```
gap> FirstElementsOfNumericalSemigroup(2,NumericalSemigroup(3,5,7));
[ 0, 3 ]
gap> FirstElementsOfNumericalSemigroup(10,NumericalSemigroup(3,5,7));
[ 0, 3, 5, 6, 7, 8, 9, 10, 11, 12 ]
```

3.1.6 ApéryListOfNumericalSemigroupWRTElement

▷ `ApéryListOfNumericalSemigroupWRTElement(S , m)` (operation)

S is a numerical semigroup and m is a positive element of S . Computes the Apéry list of S with respect to m . It contains for every $i \in \{0, \dots, m-1\}$, in the $i+1$ th position, the smallest element in the semigroup congruent with i modulo m .

Example

```
gap> S := NumericalSemigroup("modular", 5, 53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> ApéryListOfNumericalSemigroupWRTElement(S, 12);
[ 0, 13, 26, 39, 52, 53, 54, 43, 32, 33, 22, 11 ]
```

3.1.7 ApéryListOfNumericalSemigroupWRTInteger

▷ `ApéryListOfNumericalSemigroupWRTInteger(S , m)` (operation)

S is a numerical semigroup and m is a positive integer. Computes the Apéry list of S with respect to m , that is, the set of elements x in S such that $x-m$ is not in S . If m is an element in S , then the output, as sets, is the same as `ApéryListOfNumericalSemigroupWRTInteger`, though without side effects, in the sense that this information is no longer used by the package.

Example

```
gap> s:=NumericalSemigroup(10,13,19,27);
<Numerical semigroup with 4 generators>
gap> ApéryListOfNumericalSemigroupWRTInteger(s,11);
[ 0, 10, 13, 19, 20, 23, 26, 27, 29, 32, 33, 36, 39, 42, 45, 46, 52, 55 ]
gap> Length(last);
18
gap> ApéryListOfNumericalSemigroupWRTInteger(s,10);
[ 0, 13, 19, 26, 27, 32, 38, 45, 51, 54 ]
gap> ApéryListOfNumericalSemigroupWRTElement(s,10);
[ 0, 51, 32, 13, 54, 45, 26, 27, 38, 19 ]
gap> Length(last);
10
```

3.1.8 ApéryListOfNumericalSemigroupAsGraph

▷ `ApéryListOfNumericalSemigroupAsGraph(ap)` (function)

ap is the Apéry list of a numerical semigroup. This function returns the adjacency list of the graph (ap, E) where the edge $u \rightarrow v$ is in E iff $v-u$ is in ap . The 0 is ignored.

Example

```
gap> s:=NumericalSemigroup(3,7);
gap> ApéryListOfNumericalSemigroupWRTElement(s,10);
[ 0, 21, 12, 3, 14, 15, 6, 7, 18, 9 ]
gap> ApéryListOfNumericalSemigroupAsGraph(last);
[ ., [ 3, 6, 9, 12, 15, 18, 21 ], ., [ 6, 9, 12, 15, 18, 21 ],
[ 7, 14, 21 ], ., [ 9, 12, 15, 18, 21 ], ., [ 12, 15, 18, 21 ],
[ 14, 21 ], [ 15, 18, 21 ], ., [ 18, 21 ], ., [ 21 ] ]
```


3.2 Frobenius Number

The largest nonnegative integer not belonging to a numerical semigroup S is the *Frobenius number* of S . If S is the set of nonnegative integers, then clearly its Frobenius number is -1 , otherwise its Frobenius number coincides with the maximum of the gaps (or fundamental gaps) of S . An integer z is a *pseudo-Frobenius number* of S if $z + S \setminus \{0\} \subseteq S$.

3.2.1 FrobeniusNumberOfNumericalSemigroup

▷ `FrobeniusNumberOfNumericalSemigroup(NS)` (attribute)

NS is a numerical semigroup. It returns the Frobenius number of NS . Of course, the time consumed to return a result may depend on the way the semigroup is given or on the knowledge already produced on the semigroup.

Example

```
gap> FrobeniusNumberOfNumericalSemigroup(NumericalSemigroup(3,5,7));
4
```

3.2.2 FrobeniusNumber

▷ `FrobeniusNumber(NS)` (attribute)

This is just a synonym of `FrobeniusNumberOfNumericalSemigroup` (3.2.1).

3.2.3 ConductorOfNumericalSemigroup

▷ `ConductorOfNumericalSemigroup(NS)` (attribute)

This is just a synonym of `FrobeniusNumberOfNumericalSemigroup(NS)+1`.

3.2.4 PseudoFrobeniusOfNumericalSemigroup

▷ `PseudoFrobeniusOfNumericalSemigroup(S)` (attribute)

S is a numerical semigroup. It returns set of pseudo-Frobenius numbers of S .

Example

```
gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> PseudoFrobeniusOfNumericalSemigroup(S);
[ 21, 40, 41, 42 ]
```

3.2.5 TypeOfNumericalSemigroup

▷ `TypeOfNumericalSemigroup(NS)` (attribute)

Stands for `Length(PseudoFrobeniusOfNumericalSemigroup (NS))`.

3.3 Gaps

A *gap* of a numerical semigroup S is a nonnegative integer not belonging to S . The *fundamental gaps* of S are those gaps that are maximal with respect to the partial order induced by division in \mathbb{N} . The *special gaps* of a numerical semigroup S , are those fundamental gaps such that if they are added to the given numerical semigroup, then the resulting set is again a numerical semigroup.

3.3.1 GapsOfNumericalSemigroup

▷ `GapsOfNumericalSemigroup(NS)` (attribute)

NS is a numerical semigroup. It returns the set of gaps of NS .

Example

```
gap> GapsOfNumericalSemigroup(NumericalSemigroup(3,5,7));
[ 1, 2, 4 ]
```

3.3.2 GenusOfNumericalSemigroup

▷ `GenusOfNumericalSemigroup(NS)` (attribute)

NS is a numerical semigroup. It returns the number of gaps of NS .

3.3.3 FundamentalGapsOfNumericalSemigroup

▷ `FundamentalGapsOfNumericalSemigroup(S)` (attribute)

S is a numerical semigroup. It returns the set of fundamental gaps of S .

Example

```
gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> FundamentalGapsOfNumericalSemigroup(S);
[ 16, 17, 18, 19, 27, 28, 29, 30, 31, 40, 41, 42 ]
gap> GapsOfNumericalSemigroup(S);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20, 21, 27, 28, 29,
  30, 31, 40, 41, 42 ]
```

3.3.4 SpecialGapsOfNumericalSemigroup

▷ `SpecialGapsOfNumericalSemigroup(S)` (attribute)

S is a numerical semigroup. It returns the special gaps of S .

Example

```
gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> SpecialGapsOfNumericalSemigroup(S);
[ 40, 41, 42 ]
```

Chapter 4

Presentations of Numerical Semigroups

In this chapter we explain how to compute a minimal presentation of a numerical semigroup. There are three functions involved in this process.

4.1 Presentations of Numerical Semigroups

4.1.1 MinimalPresentationOfNumericalSemigroup

▷ `MinimalPresentationOfNumericalSemigroup(S)` (function)

S is a numerical semigroup. The output is a list of lists with two elements. Each list of two elements represents a relation between the minimal generators of the numerical semigroup. If $\{\{x_1, y_1\}, \dots, \{x_k, y_k\}\}$ is the output and $\{m_1, \dots, m_n\}$ is the minimal system of generators of the numerical semigroup, then $\{x_i, y_i\} = \{\{a_{i_1}, \dots, a_{i_n}\}, \{b_{i_1}, \dots, b_{i_n}\}\}$ and $a_{i_1}m_1 + \dots + a_{i_n}m_n = b_{i_1}m_1 + \dots + b_{i_n}m_n$.

Any other relation among the minimal generators of the semigroup can be deduced from the ones given in the output.

The algorithm implemented is described in [Ros96a] (see also [RGS99]).

Example

```
gap> s:=NumericalSemigroup(3,5,7);
<Numerical semigroup with 3 generators>
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 0, 2, 0 ], [ 1, 0, 1 ] ], [ [ 3, 1, 0 ], [ 0, 0, 2 ] ],
  [ [ 4, 0, 0 ], [ 0, 1, 1 ] ] ]
```

The first element in the list means that $1 \times 3 + 1 \times 7 = 2 \times 5$, and the others have similar meanings.

4.1.2 GraphAssociatedToElementInNumericalSemigroup

▷ `GraphAssociatedToElementInNumericalSemigroup(n, S)` (function)

S is a numerical semigroup and n is an element in S .

The output is a pair. If $\{m_1, \dots, m_n\}$ is the set of minimal generators of S , then the first component is the set of vertices of the graph associated to n in S , that is, the set $\{m_i \mid n - m_i \in S\}$, and the second component is the set of edges of this graph, that is, $\{\{m_i, m_j\} \mid n - (m_i + m_j) \in S\}$.

This function is used to compute a minimal presentation of the numerical semigroup S , as explained in [Ros96a].

Example

```
gap> s:=NumericalSemigroup(3,5,7);;
gap> GraphAssociatedToElementInNumericalSemigroup(10,s);
[ [ 3, 5, 7 ], [ [ 3, 7 ] ] ]
```

4.1.3 BettiElementsOfNumericalSemigroup

▷ BettiElementsOfNumericalSemigroup(S)

(function)

S is a numerical semigroup.

The output is the set of elements in S whose associated graph is nonconnected [GSO10].

Example

```
gap> s:=NumericalSemigroup(3,5,7);;
gap> BettiElementsOfNumericalSemigroup(s);
[ 10, 12, 14 ]
```

Chapter 5

Constructing numerical semigroups from others

5.1 Adding and removing elements of a numerical semigroup

In this section we show how to construct new numerical semigroups from a given numerical semigroup. Two dual operations are presented. The first one removes a minimal generator from a numerical semigroup. The second adds a special gap to a semigroup (see [RSGGJM03]).

5.1.1 RemoveMinimalGeneratorFromNumericalSemigroup

▷ RemoveMinimalGeneratorFromNumericalSemigroup(n , S) (function)

S is a numerical semigroup and n is one of its minimal generators.

The output is the numerical semigroup $S \setminus \{n\}$ (see [RSGGJM03]; $S \setminus \{n\}$ is a numerical semigroup if and only if n is a minimal generator of S).

Example

```
gap> s:=NumericalSemigroup(3,5,7);
<Numerical semigroup with 3 generators>
gap> RemoveMinimalGeneratorFromNumericalSemigroup(7,s);
<Numerical semigroup with 3 generators>
gap> MinimalGeneratingSystemOfNumericalSemigroup(last);
[ 3, 5 ]
```

5.1.2 AddSpecialGapOfNumericalSemigroup

▷ AddSpecialGapOfNumericalSemigroup(g , S) (function)

S is a numerical semigroup and g is a special gap of S

The output is the numerical semigroup $S \cup \{g\}$ (see [RSGGJM03], where it is explained why this set is a numerical semigroup).

Example

```
gap> s:=NumericalSemigroup(3,5,7);
gap> s2:=AddSpecialGapOfNumericalSemigroup(5,s);
```

```

<Numerical semigroup with 3 generators>
gap> s3:=AddSpecialGapOfNumericalSemigroup(5,s2);
<Numerical semigroup>
gap> SmallElementsOfNumericalSemigroup(s) =
> SmallElementsOfNumericalSemigroup(s3);
true
gap> s=s3;
true

```

5.1.3 IntersectionOfNumericalSemigroups

▷ IntersectionOfNumericalSemigroups(S , T) (function)

S and T are numerical semigroups. Computes the intersection of S and T (which is a numerical semigroup).

Example

```

gap> S := NumericalSemigroup("modular", 5,53);
<Modular numerical semigroup satisfying 5x mod 53 <= x >
gap> T := NumericalSemigroup(2,17);
<Modular numerical semigroup satisfying 17x mod 34 <= x >
gap> SmallElementsOfNumericalSemigroup(S);
[ 0, 11, 12, 13, 22, 23, 24, 25, 26, 32, 33, 34, 35, 36, 37, 38, 39, 43 ]
gap> SmallElementsOfNumericalSemigroup(T);
[ 0, 2, 4, 6, 8, 10, 12, 14, 16 ]
gap> IntersectionOfNumericalSemigroups(S,T);
<Numerical semigroup>
gap> SmallElementsOfNumericalSemigroup(last);
[ 0, 12, 22, 23, 24, 25, 26, 32, 33, 34, 35, 36, 37, 38, 39, 43 ]

```

5.1.4 QuotientOfNumericalSemigroup

▷ QuotientOfNumericalSemigroup(S , n) (function)

S is a numerical semigroup and n is an integer. Computes the quotient of S by n , that is, the set $\{x \in \mathbb{N} \mid nx \in S\}$, which is again a numerical semigroup. S / n may be used as a short for QuotientOfNumericalSemigroup(S , n).

Example

```

gap> s:=NumericalSemigroup(3,29);
<Modular numerical semigroup satisfying 58x mod 87 <= x >
gap> SmallElementsOfNumericalSemigroup(s);
[ 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 29, 30, 32, 33, 35, 36, 38,
  39, 41, 42, 44, 45, 47, 48, 50, 51, 53, 54, 56 ]
gap> t:=QuotientOfNumericalSemigroup(s,7);
<Numerical semigroup>
gap> SmallElementsOfNumericalSemigroup(t);
[ 0, 3, 5, 6, 8 ]
gap> u := s / 7;
<Numerical semigroup>

```

```
gap> SmallElementsOfNumericalSemigroup(u);
[ 0, 3, 5, 6, 8 ]
```

5.2 Constructing the set of all numerical semigroups containing a given numerical semigroup

In order to construct the set of numerical semigroups containing a fixed numerical semigroup S , one first constructs its unitary extensions, that is to say, the sets $S \cup \{g\}$ that are numerical semigroups with g a positive integer. This is achieved by constructing the special gaps of the semigroup, and then adding each of them to the numerical semigroup. Then we repeat the process for each of this new numerical semigroups until we reach \mathbb{N} .

These procedures are described in [RSGGJM03].

5.2.1 OverSemigroupsNumericalSemigroup

▷ OverSemigroupsNumericalSemigroup(s) (function)

s is a numerical semigroup. The output is the set of numerical semigroups containing it.

Example

```
gap> OverSemigroupsNumericalSemigroup(NumericalSemigroup(3,5,7));
[ <The numerical semigroup N>, <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup with 3 generators> ]
gap> List(last,s->MinimalGeneratingSystemOfNumericalSemigroup(s));
[ [ 1 ], [ 2, 3 ], [ 3 .. 5 ], [ 3, 5, 7 ] ]
```

5.2.2 NumericalSemigroupsWithFrobeniusNumber

▷ NumericalSemigroupsWithFrobeniusNumber(f) (function)

f is a non zero integer greater than or equal to -1. The output is the set of numerical semigroups with Frobenius number f . The algorithm implemented is given in [RSGGJM04].

Example

```
gap> Length(NumericalSemigroupsWithFrobeniusNumber(20));
900
```

Given a numerical semigroup of genus g , removing minimal generators, one obtains numerical semigroups of genus $g+1$. In order to avoid repetitions, we only remove minimal generators greater than the Frobenius number of the numerical semigroup (this is accomplished with the local function `sons`).

These procedures are described in [RSGGB03] and [BA08].

5.2.3 NumericalSemigroupsWithGenus

▷ NumericalSemigroupsWithGenus(g) (function)

g is a nonnegative integer. The output is the set of numerical semigroups with genus g .

Example

```

gap> NumericalSemigroupsWithGenus(5);
[ <Proportionally modular numerical semigroup satisfying  $11x \bmod 66 \leq 5x$ >,
  <Numerical semigroup with 5 generators>,
  <Numerical semigroup with 5 generators>,
  <Numerical semigroup with 5 generators>,
  <Numerical semigroup with 5 generators>,
  <Numerical semigroup with 4 generators>,
  <Numerical semigroup with 4 generators>,
  <Numerical semigroup with 4 generators>,
  <Numerical semigroup with 4 generators>,
  <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 3 generators>,
  <Modular numerical semigroup satisfying  $11x \bmod 22 \leq x$ > ]
gap> List(last, MinimalGeneratingSystemOfNumericalSemigroup);
[ [ 6 .. 11 ], [ 5, 7, 8, 9, 11 ], [ 5, 6, 8, 9 ], [ 5, 6, 7, 9 ],
  [ 5, 6, 7, 8 ], [ 4, 6, 7 ], [ 4, 7, 9, 10 ], [ 4, 6, 9, 11 ],
  [ 4, 5, 11 ], [ 3, 8, 10 ], [ 3, 7, 11 ], [ 2, 11 ] ]

```


Chapter 6

Irreducible numerical semigroups

6.1 Irreducible numerical semigroups

An irreducible numerical semigroup is a semigroup that cannot be expressed as the intersection of numerical semigroups properly containing it.

It is not difficult to prove that a semigroup is irreducible if and only if it is maximal (with respect to set inclusion) in the set of all numerical semigroup having its same Frobenius number (see [RB03]). Hence, according to [FGR87] (respectively [BDF97]), symmetric (respectively pseudo-symmetric) numerical semigroups are those irreducible numerical semigroups with odd (respectively even) Frobenius number.

In [RSGGJM03] it is shown that a nontrivial numerical semigroup is irreducible if and only if it has only one special gap. We use this characterization.

In this section we show how to construct the set of all numerical semigroups with a given Frobenius number. In old versions of the package, we first constructed an irreducible numerical semigroup with the given Frobenius number (as explained in [RGS04]), and then we constructed the rest from it. That is why we separated both functions. The present version uses a faster procedure presented in [BR13].

Every numerical semigroup can be expressed as an intersection of irreducible numerical semigroups. If S can be expressed as $S = S_1 \cap \dots \cap S_n$, with S_i irreducible numerical semigroups, and no factor can be removed, then we say that this decomposition is minimal. Minimal decompositions can be computed by using Algorithm 26 in [RSGGJM03].

6.1.1 IsIrreducibleNumericalSemigroup

▷ IsIrreducibleNumericalSemigroup(s) (function)

s is a numerical semigroup. The output is true if s is irreducible, false otherwise.

Example

```
gap> IsIrreducibleNumericalSemigroup(NumericalSemigroup(4,6,9));
true
gap> IsIrreducibleNumericalSemigroup(NumericalSemigroup(4,6,7,9));
false
```

6.1.2 IsSymmetricNumericalSemigroup

▷ IsSymmetricNumericalSemigroup(s) (function)

s is a numerical semigroup. The output is true if s is symmetric, false otherwise.

Example

```
gap> IsSymmetricNumericalSemigroup(NumericalSemigroup(10,23));
true
gap> IsSymmetricNumericalSemigroup(NumericalSemigroup(10,11,23));
false
```

6.1.3 IsPseudoSymmetricNumericalSemigroup

▷ IsPseudoSymmetricNumericalSemigroup(s) (function)

s is a numerical semigroup. The output is true if s is pseudo-symmetric, false otherwise.

Example

```
gap> IsPseudoSymmetricNumericalSemigroup(NumericalSemigroup(6,7,8,9,11));
true
gap> IsPseudoSymmetricNumericalSemigroup(NumericalSemigroup(4,6,9));
false
```

6.1.4 AnIrreducibleNumericalSemigroupWithFrobeniusNumber

▷ AnIrreducibleNumericalSemigroupWithFrobeniusNumber(f) (function)

f is an integer greater than or equal to -1. The output is an irreducible numerical semigroup with frobenius number f . From the way the procedure is implemented, the resulting semigroup has at most four generators (see [RGS04]).

Example

```
gap> FrobeniusNumber(AnIrreducibleNumericalSemigroupWithFrobeniusNumber(28));
28
```

6.1.5 IrreducibleNumericalSemigroupsWithFrobeniusNumber

▷ IrreducibleNumericalSemigroupsWithFrobeniusNumber(f) (function)

f is an integer greater than or equal to -1. The output is the set of all irreducible numerical semigroups with frobenius number f .

Example

```
gap> Length(IrreducibleNumericalSemigroupsWithFrobeniusNumber(39));
227
```

6.1.6 DecomposeIntoIrreducibles

▷ DecomposeIntoIrreducibles(s) (function)

s is a numerical semigroup. The output is a set of irreducible numerical semigroups containing it. These elements appear in a minimal decomposition of s as intersection into irreducibles.

Example

```
gap> DecomposeIntoIrreducibles(NumericalSemigroup(5,6,8));
[ <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 4 generators> ]
```

6.2 Complete intersection numerical semigroups

The cardinality of a minimal presentation of a numerical semigroup is always greater than or equal to its embedding dimension minus one. Complete intersection numerical semigroups are numerical semigroups reaching this bound, and they are irreducible. It can be shown that every complete intersection (other than \mathbb{N}) is a complete intersection if and only if it is the gluing of two complete intersections. When in this gluing, one of the copies is isomorphic to \mathbb{N} , then we obtain a free semigroup in the sense of [BC77]. Two special kinds of free semigroups are telescopic semigroups ([KP95]) and those associated to an irreducible planar curve ([Zar86]). We use the algorithms presented in [AGS13] to find the set of all complete intersections (also free, telescopic and associated to irreducible planar curves) numerical semigroups with given Frobenius number.

6.2.1 AsGluingOfNumericalSemigroups

▷ `AsGluingOfNumericalSemigroups(s)` (function)

s is a numerical semigroup. Returns all partitions $\{A_1, A_2\}$ of the minimal generating set of s such that s is a gluing of A_1 and A_2 by $\gcd(A_1)\gcd(A_2)$

Example

```
gap> s := NumericalSemigroup( 10, 15, 16 );
<Numerical semigroup with 3 generators>
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 10, 15 ], [ 16 ] ], [ [ 10, 16 ], [ 15 ] ] ]
gap> s := NumericalSemigroup( 18, 24, 34, 46, 51, 61, 74, 8 );
<Numerical semigroup with 8 generators>
gap> AsGluingOfNumericalSemigroups(s);
[ ]
```

6.2.2 IsACompleteIntersectionNumericalSemigroup

▷ `IsACompleteIntersectionNumericalSemigroup(s)` (function)

s is a numerical semigroup. The output is true if the numerical semigroup is a complete intersection, that is, the cardinality of a (any) minimal presentation equals its embedding dimension minus one.

Example

```
gap> s := NumericalSemigroup( 10, 15, 16 );
<Numerical semigroup with 3 generators>
gap> IsACompleteIntersectionNumericalSemigroup(s);
true
gap> s := NumericalSemigroup( 18, 24, 34, 46, 51, 61, 74, 8 );
<Numerical semigroup with 8 generators>
gap> IsACompleteIntersectionNumericalSemigroup(s);
false
```

6.2.3 CompleteIntersectionNumericalSemigroupsWithFrobeniusNumber

▷ CompleteIntersectionNumericalSemigroupsWithFrobeniusNumber(f) (function)

f is an integer greater than or equal to -1. The output is the set of all complete intersection numerical semigroups with frobenius number f .

Example

```
gap> Length(CompleteIntersectionNumericalSemigroupsWithFrobeniusNumber(57));
34
```

6.2.4 IsFreeNumericalSemigroup

▷ IsFreeNumericalSemigroup(s) (function)

s is a numerical semigroup. The output is true if the numerical semigroup is free in the sense of [BC77]: it is either \mathbb{N} or the gluing of a copy of \mathbb{N} with a free numerical semigroup.

Example

```
gap> IsFreeNumericalSemigroup(NumericalSemigroup(10,15,16));
true
gap> IsFreeNumericalSemigroup(NumericalSemigroup(3,5,7));
false
```

6.2.5 FreeNumericalSemigroupsWithFrobeniusNumber

▷ FreeNumericalSemigroupsWithFrobeniusNumber(f) (function)

f is an integer greater than or equal to -1. The output is the set of all free numerical semigroups with frobenius number f .

Example

```
gap> Length(FreeNumericalSemigroupsWithFrobeniusNumber(57));
33
```

6.2.6 IsTelescopicNumericalSemigroup

▷ IsTelescopicNumericalSemigroup(s) (function)

s is a numerical semigroup. The output is true if the numerical semigroup is telescopic in the sense of [KP95]: it is either \mathbb{N} or the gluing of $\langle n_e \rangle$ and $s' = \langle n_1/d, \dots, n_{e-1}/d \rangle$, and s' is again a telescopic numerical semigroup, where $n_1 \cdots n_e$ are the minimal generators of s .

Example

```
gap> IsTelescopicNumericalSemigroup(NumericalSemigroup(4,11,14));
false
gap> IsFreeNumericalSemigroup(NumericalSemigroup(4,11,14));
true
```

6.2.7 TelescopicNumericalSemigroupsWithFrobeniusNumber

▷ `TelescopicNumericalSemigroupsWithFrobeniusNumber(f)` (function)

f is an integer greater than or equal to -1. The output is the set of all telescopic numerical semigroups with frobenius number f .

Example

```
gap> Length(TelescopicNumericalSemigroupsWithFrobeniusNumber(57));
20
```

6.2.8 IsNumericalSemigroupAssociatedIrreduciblePlanarCurveSingularity

▷ `IsNumericalSemigroupAssociatedIrreduciblePlanarCurveSingularity(s)` (function)

s is a numerical semigroup. The output is true if the numerical semigroup is associated to an irreducible planar curve singularity ([Zar86]). These semigroups are telescopic.

Example

```
gap> IsNumericalSemigroupAssociatedIrreduciblePlanarCurveSingularity(NumericalSemigroup(4,11,14))
false
gap> IsNumericalSemigroupAssociatedIrreduciblePlanarCurveSingularity(NumericalSemigroup(4,11,19))
true
```

6.2.9 NumericalSemigroupsAssociatedIrreduciblePlanarCurveSingularityWithFrobeniusNumber

▷ `NumericalSemigroupsAssociatedIrreduciblePlanarCurveSingularityWithFrobeniusNumber(f)` (function)

f is an integer greater than or equal to -1. The output is the set of all numerical semigroups associated to irreducible planar curves singularities with frobenius number f .

Example

```
gap> Length(NumericalSemigroupsAssociatedIrreduciblePlanarCurveSingularityWithFrobeniusNumber(57))
7
```

6.3 Almost-symmetric numerical semigroups

A numerical semigroup is almost-symmetric ([BR97]) if its genus is the arithmetic mean of its Frobenius number and type. We use a procedure presented in [RGS13] to determine the set of all almost-symmetric numerical semigroups with given Frobenius number. In order to do this, we first calculate the set of all almost-symmetric numerical semigroups that can be constructed from an irreducible numerical semigroup.

6.3.1 AlmostSymmetricNumericalSemigroupsFromIrreducible

▷ `AlmostSymmetricNumericalSemigroupsFromIrreducible(s)` (function)

s is an irreducible numerical semigroup. The output is the set of almost-symmetric numerical semigroups that can be constructed from s by removing some of its generators as explained in [RGS13].

Example

```
gap> AlmostSymmetricNumericalSemigroupsFromIrreducible(NumericalSemigroup(5,8,9,11));
[ <Numerical semigroup>, <Numerical semigroup>, <Numerical semigroup> ]
gap> List(last, MinimalGeneratingSystemOfNumericalSemigroup);
[ [ 5, 8, 9, 11 ], [ 5, 8, 11, 14, 17 ], [ 5, 9, 11, 13, 17 ] ]
```

6.3.2 IsNumericalSemigroupAssociatedIrreduciblePlanarCurveSingularity

▷ IsNumericalSemigroupAssociatedIrreduciblePlanarCurveSingularity(s) (function)

s is a numerical semigroup. The output is true if the numerical semigroup is almost symmetric.

Example

```
gap> IsAlmostSymmetricNumericalSemigroup(NumericalSemigroup(5,8,11,14,17));
true
```

6.3.3 AlmostSymmetricNumericalSemigroupsWithFrobeniusNumber

▷ AlmostSymmetricNumericalSemigroupsWithFrobeniusNumber(f) (function)

f is an integer greater than or equal to -1. The output is the set of all almost symmetric numerical semigroups with Frobenius number f .

Example

```
gap> Length(AlmostSymmetricNumericalSemigroupsWithFrobeniusNumber(12));
15
gap> Length(IrreducibleNumericalSemigroupsWithFrobeniusNumber(12));
2
```

Chapter 7

Ideals of numerical semigroups

7.1 Ideals of numerical semigroups

Let S be a numerical semigroup. A set I of integers is an *ideal relative* to a numerical semigroup S provided that $I + S \subseteq I$ and that there exists $d \in S$ such that $d + I \subseteq S$.

If $\{i_1, \dots, i_k\}$ is a subset of \mathbb{Z} , then the set $I = \{i_1, \dots, i_k\} + S = \bigcup_{n=1}^k i_n + S$ is an ideal relative to S , and $\{i_1, \dots, i_k\}$ is a system of generators of I . A system of generators M is minimal if no proper subset of M generates the same ideal. Usually, ideals are specified by means of its generators and the ambient numerical semigroup to which they are ideals (for more information see for instance [BDF97]).

7.1.1 IdealOfNumericalSemigroup

▷ `IdealOfNumericalSemigroup(l, S)` (function)

S is a numerical semigroup and *l* a list of integers.

The output is the ideal of *S* generated by *l*

There are several shortcuts for this function, as shown in the example.

Example

```
gap> IdealOfNumericalSemigroup([3,5],NumericalSemigroup(9,11));
<Ideal of numerical semigroup>
gap> [3,5]+NumericalSemigroup(9,11);
<Ideal of numerical semigroup>
gap> last=last2;
true
gap> 3+NumericalSemigroup(5,9);
<Ideal of numerical semigroup>
```

7.1.2 IsIdealOfNumericalSemigroup

▷ `IsIdealOfNumericalSemigroup(Obj)` (function)

Tests if the object *Obj* is an ideal of a numerical semigroup.

Example

```
gap> I:=[1..7]+NumericalSemigroup(7,19);;
gap> IsIdealOfNumericalSemigroup(I);
true
```

```
gap> IsIdealOfNumericalSemigroup(2);
false
```

7.1.3 MinimalGeneratingSystemOfIdealOfNumericalSemigroup

▷ `MinimalGeneratingSystemOfIdealOfNumericalSemigroup(I)` (function)

I is an ideal of a numerical semigroup.
The output is the minimal system of generators of I .

Example

```
gap> I:=[3,5,9]+NumericalSemigroup(2,11);;
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(I);
[ 3 ]
```

7.1.4 GeneratorsOfIdealOfNumericalSemigroup

▷ `GeneratorsOfIdealOfNumericalSemigroup(I)` (function)
▷ `GeneratorsOfIdealOfNumericalSemigroupNC(I)` (function)

I is an ideal of a numerical semigroup.

The output of `GeneratorsOfIdealOfNumericalSemigroup` is a system of generators of the ideal. If the minimal system of generators is known, then it is used as output. `GeneratorsOfIdealOfNumericalSemigroupNC` always returns the set of generators stored in $I!.generators$.

Example

```
gap> I:=[3,5,9]+NumericalSemigroup(2,11);;
gap> GeneratorsOfIdealOfNumericalSemigroup(I);
[ 3, 5, 9 ]
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(I);
[ 3 ]
gap> GeneratorsOfIdealOfNumericalSemigroup(I);
[ 3 ]
gap> GeneratorsOfIdealOfNumericalSemigroupNC(I);
[ 3, 5, 9 ]
```

7.1.5 AmbientNumericalSemigroupOfIdeal

▷ `AmbientNumericalSemigroupOfIdeal(I)` (function)

I is an ideal of a numerical semigroup, say S .
The output is S .

Example

```
gap> I:=[3,5,9]+NumericalSemigroup(2,11);;
gap> AmbientNumericalSemigroupOfIdeal(I);
<Modular numerical semigroup satisfying 11x mod 22 <= x >
```


7.1.6 SmallElementsOfIdealOfNumericalSemigroup

▷ SmallElementsOfIdealOfNumericalSemigroup(*I*) (function)

I is an ideal of a numerical semigroup.

The output is a list with the elements in *I* that are less than or equal to the greatest integer not belonging to the ideal plus one.

Example

```
gap> I:=[3,5,9]+NumericalSemigroup(2,11);;
gap> SmallElementsOfIdealOfNumericalSemigroup(I);
[ 3, 5, 7, 9, 11, 13 ]
gap> J:=[2,11]+NumericalSemigroup(2,11);;
gap> SmallElementsOfIdealOfNumericalSemigroup(J);
[ 2, 4, 6, 8, 10 ]
```

7.1.7 BelongsToIdealOfNumericalSemigroup

▷ BelongsToIdealOfNumericalSemigroup(*n*, *I*) (function)

I is an ideal of a numerical semigroup, *n* is an integer.

The output is true if *n* belongs to *I*.

n in I can be used for short.

Example

```
gap> J:=[2,11]+NumericalSemigroup(2,11);;
gap> BelongsToIdealOfNumericalSemigroup(9,J);
false
gap> 9 in J;
false
gap> BelongsToIdealOfNumericalSemigroup(10,J);
true
gap> 10 in J;
true
```

7.1.8 SumIdealsOfNumericalSemigroup

▷ SumIdealsOfNumericalSemigroup(*I*, *J*) (function)

I, *J* are ideals of a numerical semigroup.

The output is the sum of both ideals $\{i+j \mid i \in I, j \in J\}$.

I + J is a synonym of this function.

Example

```
gap> I:=[3,5,9]+NumericalSemigroup(2,11);;
gap> J:=[2,11]+NumericalSemigroup(2,11);;
gap> I+J;
<Ideal of numerical semigroup>
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(last);
[ 5, 14 ]
gap> SumIdealsOfNumericalSemigroup(I,J);
<Ideal of numerical semigroup>
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(last);
[ 5, 14 ]
```

7.1.9 MultipleOfIdealOfNumericalSemigroup

▷ `MultipleOfIdealOfNumericalSemigroup(n, I)` (function)

I is an ideal of a numerical semigroup, *n* is a non negative integer.

The output is the ideal $I + \dots + I$ (*n* times).

$n * I$ can be used for short.

Example

```
gap> I:=[0,1]+NumericalSemigroup(3,5,7);;
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(2*I);
[ 0, 1, 2 ]
```

7.1.10 SubtractIdealsOfNumericalSemigroup

▷ `SubtractIdealsOfNumericalSemigroup(I, J)` (function)

I, *J* are ideals of a numerical semigroup.

The output is the ideal $\{z \in \mathbb{Z} \mid z + J \subseteq I\}$.

$I - J$ is a synonym of this function.

$S - J$ is a synonym of $(0 + S) - J$, if *S* is the ambient semigroup of *I* and *J*. The following example appears in [HS04].

Example

```
gap> S:=NumericalSemigroup(14, 15, 20, 21, 25);;
gap> I:=[0,1]+S;;
gap> II:=S-I;;
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(I);
[ 0, 1 ]
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(II);
[ 14, 20 ]
gap> MinimalGeneratingSystemOfIdealOfNumericalSemigroup(I+II);
[ 14, 15, 20, 21 ]
```

7.1.11 DifferenceOfIdealsOfNumericalSemigroup

▷ `DifferenceOfIdealsOfNumericalSemigroup(I, J)` (function)

I, *J* are ideals of a numerical semigroup. *J* must be contained in *I*.

The output is the set $I \setminus J$.

Example

```
gap> S:=NumericalSemigroup(14, 15, 20, 21, 25);;
gap> I:=[0,1]+S;
<Ideal of numerical semigroup>
gap> 2*I-2*I;
<Ideal of numerical semigroup>
gap> I-I;
<Ideal of numerical semigroup>
gap> DifferenceOfIdealsOfNumericalSemigroup(last2,last);
[ 26, 27, 37, 38 ]
```

7.1.12 TranslationOfIdealOfNumericalSemigroup

▷ TranslationOfIdealOfNumericalSemigroup(k, I) (function)

Given an ideal I of a numerical semigroup S and an integer k returns an ideal of the numerical semigroup S generated by $\{i_1 + k, \dots, i_n + k\}$ where $\{i_1, \dots, i_n\}$ is the system of generators of I .

As a synonym to TranslationOfIdealOfNumericalSemigroup(k, I) the expression $k + I$ may be used.

Example

```
gap> s:=NumericalSemigroup(13,23);;
gap> l:=List([1..6], _ -> Random([8..34]));
[ 22, 29, 34, 25, 10, 12 ]
gap> I:=IdealOfNumericalSemigroup(l, s);;
gap> It:=TranslationOfIdealOfNumericalSemigroup(7,I);
<Ideal of numerical semigroup>
gap> It2:=7+I;
<Ideal of numerical semigroup>
gap> It2=It;
true
```

7.1.13 HilbertFunctionOfIdealOfNumericalSemigroup

▷ HilbertFunctionOfIdealOfNumericalSemigroup(n, I) (function)

I is an ideal of a numerical semigroup, n is a non negative integer. I must be contained in its ambient semigroup.

The output is the cardinality of the set $nI \setminus (n+1)I$.

Example

```
gap> I:=[6,9,11]+NumericalSemigroup(6,9,11);;
gap> List([1..7], n->HilbertFunctionOfIdealOfNumericalSemigroup(n,I));
[ 3, 5, 6, 6, 6, 6, 6 ]
```

7.1.14 BlowUpIdealOfNumericalSemigroup

▷ BlowUpIdealOfNumericalSemigroup(I) (function)

I is an ideal of a numerical semigroup.

The output is the ideal $\bigcup_{n \geq 0} nI - nI$.

Example

```
gap> I:=[0,2]+NumericalSemigroup(6,9,11);;
gap> BlowUpIdealOfNumericalSemigroup(I);;
gap> SmallElementsOfIdealOfNumericalSemigroup(last);
[ 0, 2, 4, 6, 8 ]
```

7.1.15 ReductionNumberIdealNumericalSemigroup

▷ ReductionNumberIdealNumericalSemigroup(I) (function)

I is an ideal of a numerical semigroup.

The output is the least integer such that $nI + i = (n+1)I$, where $i = \min(I)$.

Example

```
gap> I:=[0,2]+NumericalSemigroup(6,9,11);;
gap> ReductionNumberIdealNumericalSemigroup(I);
2
```

7.1.16 MaximalIdealOfNumericalSemigroup

▷ MaximalIdealOfNumericalSemigroup(S)

(function)

Returns the maximal ideal of the numerical semigroup S .

Example

```
gap> MaximalIdealOfNumericalSemigroup(NumericalSemigroup(3,7));
<Ideal of numerical semigroup>
```

7.1.17 BlowUpOfNumericalSemigroup

▷ BlowUpOfNumericalSemigroup(S)

(function)

If M is the maximal ideal of the numerical semigroup, then the output is the numerical semigroup $\bigcup_{n \geq 0} nM - nM$.

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> BlowUpOfNumericalSemigroup(s);
<Numerical semigroup with 10 generators>
gap> SmallElementsOfNumericalSemigroup(last);
[ 0, 5, 10, 12, 15, 17, 20, 22, 24, 25, 27, 29, 30, 32, 34, 35, 36, 37, 39,
  40, 41, 42, 44 ]
gap> m:=MaximalIdealOfNumericalSemigroup(s);
<Ideal of numerical semigroup>
gap> BlowUpIdealOfNumericalSemigroup(m);
<Ideal of numerical semigroup>
gap> SmallElementsOfIdealOfNumericalSemigroup(last);
[ 0, 5, 10, 12, 15, 17, 20, 22, 24, 25, 27, 29, 30, 32, 34, 35, 36, 37, 39,
  40, 41, 42, 44 ]
```

7.1.18 MicroInvariantsOfNumericalSemigroup

▷ MicroInvariantsOfNumericalSemigroup(S)

(function)

Returns the microinvariants of the numerical semigroup S defined in [Eli01]. For their computation we have used the formula given in [BF06]. The Ap\`ery set of S and its blow up are involved in this computation.

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> bu:=BlowUpOfNumericalSemigroup(s);;
gap> ap:=AperyListOfNumericalSemigroupWRTElement(s,30);;
gap> apbu:=AperyListOfNumericalSemigroupWRTElement(bu,30);;
gap> (ap-apbu)/30;
[ 0, 4, 4, 3, 2, 1, 3, 4, 4, 3, 2, 3, 1, 4, 4, 3, 3, 1, 4, 4, 4, 3, 2, 4, 2,
```

```

5, 4, 3, 3, 2 ]
gap> MicroInvariantsOfNumericalSemigroup(s)=last;
true

```

7.1.19 IsGradedAssociatedRingNumericalSemigroupCM

▷ IsGradedAssociatedRingNumericalSemigroupCM(S) (function)

Returns true if the graded ring associated to $K[[S]]$ is Cohen-Macaulay, and false otherwise. This test is the implementation of the algorithm given in [BF06].

Example

```

gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsGradedAssociatedRingNumericalSemigroupCM(s);
false
gap> MicroInvariantsOfNumericalSemigroup(s);
[ 0, 4, 4, 3, 2, 1, 3, 4, 4, 3, 2, 3, 1, 4, 4, 3, 3, 1, 4, 4, 4, 3, 2, 4, 2,
  5, 4, 3, 3, 2 ]
gap> List(AperyListOfNumericalSemigroupWRTElement(s,30),
> w->MaximumDegreeOfElementWRTNumericalSemigroup(w,s));
[ 0, 1, 4, 1, 2, 1, 3, 1, 4, 3, 2, 3, 1, 1, 4, 3, 3, 1, 4, 1, 4, 3, 2, 4, 2,
  5, 4, 3, 1, 2 ]
gap> last=last2;
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsGradedAssociatedRingNumericalSemigroupCM(s);
true
gap> MicroInvariantsOfNumericalSemigroup(s);
[ 0, 2, 1, 1 ]
gap> List(AperyListOfNumericalSemigroupWRTElement(s,4),
> w->MaximumDegreeOfElementWRTNumericalSemigroup(w,s));
[ 0, 2, 1, 1 ]

```

7.1.20 CanonicalIdealOfNumericalSemigroup

▷ CanonicalIdealOfNumericalSemigroup(S) (function)

Computes a canonical ideal of S ([BF06]): $\{x \in \mathbb{Z} \mid g - x \notin S\}$.

Example

```

gap> s:=NumericalSemigroup(4,6,11);;
gap> m:=MaximalIdealOfNumericalSemigroup(s);;
gap> c:=CanonicalIdealOfNumericalSemigroup(s);
<Ideal of numerical semigroup>
gap> (m-c)-c=m;
true
gap> id:=3+s;
<Ideal of numerical semigroup>
gap> (id-c)-c=id;
true

```

7.1.21 IntersectionIdealsOfNumericalSemigroup

▷ IntersectionIdealsOfNumericalSemigroup(I, J) (function)

Given two ideals I and J of a numerical semigroup S returns the ideal of the numerical semigroup S which is the intersection of the ideals I and J .

Example

```
gap> i:=IdealOfNumericalSemigroup([75,89],s);;
gap> j:=IdealOfNumericalSemigroup([115,289],s);;
gap> IntersectionIdealsOfNumericalSemigroup(i,j);
<Ideal of numerical semigroup>
```

7.1.22 IsMonomialNumericalSemigroup

▷ IsMonomialNumericalSemigroup(S) (function)

S is a numerical semigroup.

Tests whether S a monomial numerical semigroup.

Let R a Noetherian ring such that $K \subseteq R \subseteq K[[t]]$, K is a field of characteristic zero, the algebraic closure of R is $K[[t]]$, and the conductor $(R : K[[t]])$ is not zero. If $v : K((t)) \rightarrow \mathbb{Z}$ is the natural valuation for $K((t))$, then $v(R)$ is a numerical semigroup.

Let S be a numerical semigroup minimally generated by $\{n_1, \dots, n_e\}$. The semigroup ring associated to S is $K[[S]] = K[[t^{n_1}, \dots, t^{n_e}]]$. A ring is called a semigroup ring if it is of the form $K[[S]]$, for some numerical semigroup S . We say that S is a monomial numerical semigroup if for any R as above with $v(R) = S$, R is a semigroup ring. See [Mic02] for details.

Example

```
gap> IsMonomialNumericalSemigroup(NumericalSemigroup(4,6,7));
true
gap> IsMonomialNumericalSemigroup(NumericalSemigroup(4,6,11));
false
```

7.1.23 ApéryListOfIdealOfNumericalSemigroupWRTElement

▷ ApéryListOfIdealOfNumericalSemigroupWRTElement(I, n) (function)

Computes the sets of elements x of I such that $x - n$ not in the ideal I , where n is supposed to be in the ambient semigroup of I . The element in the i th position of the output list (starting in 0) is congruent with i modulo n .

Example

```
gap> s:=NumericalSemigroup(10,11,13);;
gap> i:=[12,14]+s;;
gap> ApéryListOfIdealOfNumericalSemigroupWRTElement(i,10);
[ 40, 51, 12, 23, 14, 25, 36, 27, 38, 49 ]
```

7.1.24 ApéryListOfIdealOfNumericalSemigroupWRTElement

▷ ApéryListOfIdealOfNumericalSemigroupWRTElement(s) (function)

Computes the Apéry table associated to the numerical semigroup s as explained in [CBJZA13], that is, a list containing the Apéry list of s with respect to its multiplicity and the Apéry lists of kM (with M the maximal ideal of s) with respect to the multiplicity of s , for $k \in \{1, \dots, r\}$, where r is the reduction number of M (see `ReductionNumberIdealNumericalSemigroup`).

Example

```
gap> s:=NumericalSemigroup(10,11,13);;
gap> ApéryTableOfNumericalSemigroup(s);
[ [ 0, 11, 22, 13, 24, 35, 26, 37, 48, 39 ],
  [ 10, 11, 22, 13, 24, 35, 26, 37, 48, 39 ],
  [ 20, 21, 22, 23, 24, 35, 26, 37, 48, 39 ],
  [ 30, 31, 32, 33, 34, 35, 36, 37, 48, 39 ],
  [ 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 ] ]
```

Chapter 8

Numerical semigroups with maximal embedding dimension

8.1 Numerical semigroups with maximal embedding dimension

If S is a numerical semigroup and m is its multiplicity (the least positive integer belonging to it), then the embedding dimension e of S (the cardinality of the minimal system of generators of S) is less than or equal to m . We say that S has maximal embedding dimension (MED for short) when $e = m$. The intersection of two numerical semigroups with the same multiplicity and maximal embedding dimension is again of maximal embedding dimension. Thus we define the MED closure of a non-empty subset of positive integers $M = \{m < m_1 < \cdots < m_n < \cdots\}$ with $\gcd(M) = 1$ as the intersection of all MED numerical semigroups with multiplicity m .

Given a MED numerical semigroup S , we say that $M = \{m_1 < \cdots < m_k\}$ is a MED system of generators if the MED closure of M is S . Moreover, M is a minimal MED generating system for S provided that every proper subset of M is not a MED system of generators of S . Minimal MED generating systems are unique, and in general are smaller than the classical minimal generating systems (see [RGSGB03]).

8.1.1 IsMEDNumericalSemigroup

▷ `IsMEDNumericalSemigroup(S)` (function)

S is a numerical semigroup.

Returns true if S is a MED numerical semigroup and false otherwise.

Example

```
gap> IsMEDNumericalSemigroup(NumericalSemigroup(3,5,7));
true
gap> IsMEDNumericalSemigroup(NumericalSemigroup(3,5));
false
```


8.1.2 MEDNumericalSemigroupClosure

▷ MEDNumericalSemigroupClosure(S) (function)

S is a numerical semigroup.
Returns the MED closure of S .

Example

```
gap> MEDNumericalSemigroupClosure(NumericalSemigroup(3,5));
<Numerical semigroup>
gap> MinimalGeneratingSystemOfNumericalSemigroup(last);
[ 3, 5, 7 ]
```

8.1.3 MinimalMEDGeneratingSystemOfMEDNumericalSemigroup

▷ MinimalMEDGeneratingSystemOfMEDNumericalSemigroup(S) (function)

S is a MED numerical semigroup.
Returns the minimal MED generating system of S .

Example

```
gap> MinimalMEDGeneratingSystemOfMEDNumericalSemigroup(
> NumericalSemigroup(3,5,7));
[ 3, 5 ]
```

8.2 Numerical semigroups with the Arf property and Arf closures

Numerical semigroups with the Arf property are a special kind of numerical semigroups with maximal embedding dimension. A numerical semigroup S is Arf if for every x, y, z in S with $x \geq y \geq z$, one has that $x + y - z \in S$.

The intersection of two Arf numerical semigroups is again Arf, and thus we can consider the Arf closure of a set of nonnegative integers with greatest common divisor equal to one. Analogously as with MED numerical semigroups, we define Arf systems of generators and minimal Arf generating system for an Arf numerical semigroup. These are also unique (see [RGSGB04]).

8.2.1 IsArfNumericalSemigroup

▷ IsArfNumericalSemigroup(S) (function)

S is a numerical semigroup.
Returns true if S is an Arf numerical semigroup and false otherwise.

Example

```
gap> IsArfNumericalSemigroup(NumericalSemigroup(3,5,7));
true
gap> IsArfNumericalSemigroup(NumericalSemigroup(3,7,11));
false
gap> IsMEDNumericalSemigroup(NumericalSemigroup(3,7,11));
```

```
true
```

8.2.2 ArfNumericalSemigroupClosure

▷ `ArfNumericalSemigroupClosure(S)` (function)

S is a numerical semigroup.

Returns the Arf closure of S .

Example

```
gap> ArfNumericalSemigroupClosure(NumericalSemigroup(3,7,11));
<Numerical semigroup>
gap> MinimalGeneratingSystemOfNumericalSemigroup(last);
[ 3, 7, 8 ]
```

8.2.3 MinimalArfGeneratingSystemOfArfNumericalSemigroup

▷ `MinimalArfGeneratingSystemOfArfNumericalSemigroup(S)` (function)

S is an Arf numerical semigroup.

Returns the minimal MED generating system of S .

Example

```
gap> MinimalArfGeneratingSystemOfArfNumericalSemigroup(
> NumericalSemigroup(3,7,8));
[ 3, 7 ]
```

8.2.4 ArfNumericalSemigroupsWithFrobeniusNumber

▷ `ArfNumericalSemigroupsWithFrobeniusNumber(f)` (function)

f is an integer greater than or equal to -1. The output is the set of all Arf numerical semigroups with Frobenius number f .

Example

```
gap> ArfNumericalSemigroupsWithFrobeniusNumber(10);
[ <Numerical semigroup>, <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>, <Numerical semigroup> ]
gap> List(last, MinimalGeneratingSystemOfNumericalSemigroup);
[ [ 7, 9, 11, 12, 13, 15, 17 ], [ 3, 11, 13 ], [ 6, 9, 11, 13, 14, 16 ],
  [ 9, 11, 12, 13, 14, 15, 16, 17, 19 ], [ 4, 11, 13, 14 ],
  [ 8, 11, 12, 13, 14, 15, 17, 18 ], [ 7, 11, 12, 13, 15, 16, 17 ],
  [ 6, 11, 13, 14, 15, 16 ], [ 11 .. 21 ] ]
```

8.3 Saturated numerical semigroups

Saturated numerical semigroups with the Arf property are a special kind of numerical semigroups with maximal embedding dimension. A numerical semigroup S is saturated if the following condition holds: s, s_1, \dots, s_r in S are such that $s_i \leq s$ for all i in $\{1, \dots, r\}$ and z_1, \dots, z_r in \mathbb{Z} are such that $z_1 s_1 + \dots + z_r s_r \geq 0$, then $s + z_1 s_1 + \dots + z_r s_r$ in S .

The intersection of two saturated numerical semigroups is again saturated, and thus we can consider the saturated closure of a set of nonnegative integers with greatest common divisor equal to one (see [RGS09]).

8.3.1 IsSaturatedNumericalSemigroup

▷ IsSaturatedNumericalSemigroup(S) (function)

S is a numerical semigroup.

Returns true if S is a saturated numerical semigroup and false otherwise.

Example

```
gap> IsSaturatedNumericalSemigroup(NumericalSemigroup(4,6,9,11));
true
gap> IsSaturatedNumericalSemigroup(NumericalSemigroup(8, 9, 12, 13, 15, 19 ));
false
```

8.3.2 SaturatedNumericalSemigroupClosure

▷ SaturatedNumericalSemigroupClosure(S) (function)

S is a numerical semigroup.

Returns the saturated closure of S .

Example

```
gap> SaturatedNumericalSemigroupClosure(NumericalSemigroup(8, 9, 12, 13, 15));
<Numerical semigroup>
gap> MinimalGeneratingSystemOfNumericalSemigroup(last);
[ 8 .. 15 ]
```

8.3.3 SaturatedNumericalSemigroupsWithFrobeniusNumber

▷ SaturatedNumericalSemigroupsWithFrobeniusNumber(f) (function)

f is an integer greater than or equal to -1. The output is the set of all Saturated numerical semigroups with Frobenius number f .

Example

```
gap> SaturatedNumericalSemigroupsWithFrobeniusNumber(10);
[ <Numerical semigroup>, <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup> ]
gap> List(last, MinimalGeneratingSystemOfNumericalSemigroup);
[ [ 3, 11, 13 ], [ 4, 11, 13, 14 ], [ 6, 9, 11, 13, 14, 16 ],
  [ 6, 11, 13, 14, 15, 16 ], [ 7, 11, 12, 13, 15, 16, 17 ],
  [ 8, 11, 12, 13, 14, 15, 17, 18 ], [ 9, 11, 12, 13, 14, 15, 16, 17, 19 ],
  [ 11 .. 21 ] ]
```

Chapter 9

Nonunique invariants for factorizations in numerical semigroups

9.1 Factorizations in Numerical Semigroups

Let S be a numerical semigroup minimally generated by $\{m_1, \dots, m_n\}$. A factorization of an element $s \in S$ is an n -tuple $a = (a_1, \dots, a_n)$ of nonnegative integers such that $s = a_1 m_1 + \dots + a_n m_n$. The length of a is $|a| = a_1 + \dots + a_n$. Given two factorizations a and b of s , the distance between a and b is $d(a, b) = \max\{|a - \gcd(a, b)|, |b - \gcd(a, b)|\}$, where $\gcd((a_1, \dots, a_n), (b_1, \dots, b_n)) = (\min(a_1, b_1), \dots, \min(a_n, b_n))$.

If $l_1 > \dots > l_k$ are the lengths of all the factorizations of $s \in S$, the Delta set associated to s is $\Delta(s) = \{l_1 - l_2, \dots, l_k - l_{k-1}\}$.

The catenary degree of an element in S is the least positive integer c such that for any two of its factorizations a and b , there exists a chain of factorizations starting in a and ending in b and so that the distance between two consecutive links is at most c . The catenary degree of S is the supremum of the catenary degrees of the elements in S .

The tame degree of S is the least positive integer t for any factorization a of an element s in S , and any i such that $s - m_i \in S$, there exists another factorization b of s so that the distance to a is at most t and $b_i \neq 0$.

The ω -primality of an element s in S is the least positive integer k such that if $(\sum_{i \in I} s_i) - s \in S$, $s_i \in S$, then there exists $\Omega \subseteq I$ with cardinality k such that $(\sum_{i \in \Omega} s_i) - s \in S$. The ω -primality is the maximum of the ω -primality of its minimal generators.

The basic properties of these constants can be found in [GHK06]. The algorithm used to compute the catenary and tame degree is an adaptation of the algorithms appearing in [CGSL⁺06] for numerical semigroup (see [CGSD07]). The computation of the elasticity of a numerical semigroup reduces to m/n with m the multiplicity of the semigroup and n its largest minimal generator (see [CHM06] or [GHK06]).

9.1.1 FactorizationsIntegerWRTList

▷ FactorizationsIntegerWRTList(n , ls) (function)

ls is a list of integers and n an integer. The output is the set of factorizations of n in terms of the elements in the list ls . This function uses RestrictedPartitions.

Example

```
gap> FactorizationsIntegerWRTList(100,[11,13,15,19]);
[[ 2, 6, 0, 0 ], [ 3, 4, 1, 0 ], [ 4, 2, 2, 0 ], [ 5, 0, 3, 0 ],
 [ 5, 2, 0, 1 ], [ 6, 0, 1, 1 ], [ 0, 1, 2, 3 ], [ 1, 1, 0, 4 ]]
```

9.1.2 FactorizationsElementWRTNumericalSemigroup

▷ FactorizationsElementWRTNumericalSemigroup(n , S) (function)

S is a numerical semigroup and n a nonnegative integer. The output is the set of factorizations of n in terms of the minimal generating set of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> FactorizationsElementWRTNumericalSemigroup(1100,s);
[[ 0, 8, 1, 0, 0, 0 ], [ 0, 0, 0, 2, 2, 0 ], [ 5, 1, 1, 0, 0, 1 ],
 [ 0, 2, 3, 0, 0, 1 ]]
```

9.1.3 LengthsOfFactorizationsIntegerWRTList

▷ LengthsOfFactorizationsIntegerWRTList(n , ls) (function)

ls is a list of integers and n an integer. The output is the set of lengths of the factorizations of n in terms of the elements in ls .

Example

```
gap> LengthsOfFactorizationsIntegerWRTList(100,[11,13,15,19]);
[ 6, 8 ]
```

9.1.4 RClassesOfSetOfFactorizations

▷ RClassesOfSetOfFactorizations(ls) (function)

ls is a set of factorizations (a list of lists of nonnegative integers with the same length). The output is the set of \mathcal{R} -classes of this set of factorizations as defined in Chapter 7 of [RGS09].

Example

```
gap> s:=NumericalSemigroup(10,11,19,23);;
gap> BettiElementsOfNumericalSemigroup(s);
[ 30, 33, 42, 57, 69 ]
gap> FactorizationsElementWRTNumericalSemigroup(69,s);
[[ 5, 0, 1, 0 ], [ 2, 1, 2, 0 ], [ 0, 0, 0, 3 ] ]
gap> RClassesOfSetOfFactorizations(last);
[[ [ 2, 1, 2, 0 ], [ 5, 0, 1, 0 ] ], [ [ 0, 0, 0, 3 ] ] ]
```

9.1.5 LengthsOfFactorizationsElementWRTNumericalSemigroup

▷ LengthsOfFactorizationsElementWRTNumericalSemigroup(n , S) (function)

S is a numerical semigroup and n a nonnegative integer. The output is the set of lengths of the factorizations of n in terms of the minimal generating set of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> LengthsOfFactorizationsElementWRTNumericalSemigroup(1100,s);
[ 4, 6, 8, 9 ]
```

9.1.6 ElasticityOfFactorizationsElementWRTNumericalSemigroup

▷ ElasticityOfFactorizationsElementWRTNumericalSemigroup(n , S) (function)

S is a numerical semigroup and n a positive integer. The output is the maximum length divided by the minimum length of the factorizations of n in terms of the minimal generating set of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> ElasticityOfFactorizationsElementWRTNumericalSemigroup(1100,s);
9/4
```

9.1.7 ElasticityOfNumericalSemigroup

▷ ElasticityOfNumericalSemigroup(S) (function)

S is a numerical semigroup. The output is the elasticity of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> ElasticityOfNumericalSemigroup(s);
286/101
```

9.1.8 DeltaSetOfSetOfIntegers

▷ DeltaSetOfSetOfIntegers(ls) (function)

ls is list of integers. The output is the Delta set of the elements in ls , that is, the set of differences of consecutive elements in the list.

Example

```
gap> LengthsOfFactorizationsIntegerWRTList(100,[11,13,15,19]);
[ 6, 8 ]
gap> DeltaSetOfSetOfIntegers(last);
[ 2 ]
```

9.1.9 DeltaSetOfFactorizationsElementWRTNumericalSemigroup

▷ DeltaSetOfFactorizationsElementWRTNumericalSemigroup(n , S) (function)

S is a numerical semigroup and n a nonnegative integer. The output is the Delta set of the factorizations of n in terms of the minimal generating set of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> DeltaSetOfFactorizationsElementWRTNumericalSemigroup(1100,s);
[ 1, 2 ]
```

9.1.10 MaximumDegreeOfElementWRTNumericalSemigroup

▷ MaximumDegreeOfElementWRTNumericalSemigroup(n , S) (function)

S is a numerical semigroup and n a nonnegative integer. The output is the maximum length of the factorizations of n in terms of the minimal generating set of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> MaximumDegreeOfElementWRTNumericalSemigroup(1100,s);
9
```

9.1.11 CatenaryDegreeOfSetOfFactorizations

▷ CatenaryDegreeOfSetOfFactorizations(ls) (function)

ls is a set of factorizations (a list of lists of nonnegative integers with the same length). The output is the catenary degree of this set of factorizations.

Example

```
gap> FactorizationsIntegerWRTList(100,[11,13,15,19]);
[ [ 2, 6, 0, 0 ], [ 3, 4, 1, 0 ], [ 4, 2, 2, 0 ], [ 5, 0, 3, 0 ],
  [ 5, 2, 0, 1 ], [ 6, 0, 1, 1 ], [ 0, 1, 2, 3 ], [ 1, 1, 0, 4 ] ]
gap> CatenaryDegreeOfSetOfFactorizations(last);
5
```

9.1.12 CatenaryDegreeOfNumericalSemigroup

▷ CatenaryDegreeOfNumericalSemigroup(S) (function)

S is a numerical semigroup. The output is the catenary degree of S .

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> CatenaryDegreeOfNumericalSemigroup(s);
8
```

9.1.13 CatenaryDegreeOfElementInNumericalSemigroup

▷ CatenaryDegreeOfElementInNumericalSemigroup(n , S) (function)

n is a nonnegative integer and S is a numerical semigroup. The output is the catenary degree of n relative to S .

Example

```
gap> CatenaryDegreeOfElementInNumericalSemigroup(157,NumericalSemigroup(13,18));
0
gap> CatenaryDegreeOfElementInNumericalSemigroup(1157,NumericalSemigroup(13,18));
18
```

9.1.14 TameDegreeOfSetOfFactorizations

▷ `TameDegreeOfSetOfFactorizations(ls)` (function)

ls is a set of factorizations (a list of lists of nonnegative integers with the same length). The output is the tame degree of this set of factorizations.

Example

```
gap> FactorizationsIntegerWRTList(100,[11,13,15,19]);
[[ 2, 6, 0, 0 ], [ 3, 4, 1, 0 ], [ 4, 2, 2, 0 ], [ 5, 0, 3, 0 ], [ 5, 2, 0, 1 ], [ 6, 0, 1, 1 ],
[ 1, 1, 0, 4 ] ]
gap> TameDegreeOfSetOfFactorizations(last);
4
```

9.1.15 TameDegreeOfNumericalSemigroup

▷ `TameDegreeOfNumericalSemigroup(S)` (function)

S is a numerical semigroup. The output is the tame degree of *S*.

Example

```
gap> s:=NumericalSemigroup(101,113,196,272,278,286);
<Numerical semigroup with 6 generators>
gap> TameDegreeOfNumericalSemigroup(s);
14
```

9.1.16 TameDegreeOfElementInNumericalSemigroup

▷ `TameDegreeOfElementInNumericalSemigroup(n, S)` (function)

n is an element of the numerical semigroup *S*. The output is the tame degree of *n* in *S*.

Example

```
gap> s:=NumericalSemigroup(10,11,13);
<Numerical semigroup with 3 generators>
gap> TameDegreeOfElementInNumericalSemigroup(100,s);
5
```

9.1.17 OmegaPrimalityOfElementInNumericalSemigroup

▷ `OmegaPrimalityOfElementInNumericalSemigroup(n, S)` (function)

n is an element of the numerical semigroup *S*. The output is the ω -primality of *n* in *S* as explained in [BGS10].

Example

```
gap> s:=NumericalSemigroup(10,11,13);  
<Numerical semigroup with 3 generators>  
gap> OmegaPrimalityOfElementInNumericalSemigroup(100,s);  
13
```

9.1.18 OmegaPrimalityOfNumericalSemigroup

▷ OmegaPrimalityOfNumericalSemigroup(n , S)

(function)

S is a numerical semigroup. The output is the maximum of the ω -primalities of the minimal generators of S .

Example

```
gap> s:=NumericalSemigroup(10,11,13);  
<Numerical semigroup with 3 generators>  
gap> OmegaPrimalityOfNumericalSemigroup(s);  
5
```

Appendix A

Generalities

Here we describe some functions which are not specific for numerical semigroups but are used to do computations with them. As they may have interest by themselves, we describe them here.

A.1 Bézout sequences

A sequence of positive rational numbers $a_1/b_1 < \dots < a_n/b_n$ with a_i, b_i positive integers is a *Bézout sequence* if $a_{i+1}b_i - a_ib_{i+1} = 1$ for all $i \in \{1, \dots, n-1\}$.

The following function uses an algorithm presented in [BR08].

A.1.1 BezoutSequence

▷ `BezoutSequence(arg)` (function)

`arg` consists of two rational numbers or a list of two rational numbers. The output is a Bézout sequence with ends the two rational numbers given. (Warning: rational numbers are silently transformed into irreducible fractions.)

Example

```
gap> BezoutSequence(4/5, 53/27);  
[ 4/5, 1, 3/2, 5/3, 7/4, 9/5, 11/6, 13/7, 15/8, 17/9, 19/10, 21/11, 23/12,  
 25/13, 27/14, 29/15, 31/16, 33/17, 35/18, 37/19, 39/20, 41/21, 43/22,  
 45/23, 47/24, 49/25, 51/26, 53/27 ]
```

A.1.2 IsBezoutSequence

▷ `IsBezoutSequence(L)` (function)

`L` is a list of rational numbers. `IsBezoutSequence` returns true or false according to whether `L` is a Bézout sequence or not.

Example

```
gap> IsBezoutSequence([ 4/5, 1, 3/2, 5/3, 7/4, 9/5, 11/6]);  
true  
gap> IsBezoutSequence([ 4/5, 1, 3/2, 5/3, 7/4, 9/5, 11/3]);  
false  
Take the 6 and the 7 elements of the sequence
```

A.1.3 CeilingOfRational

▷ CeilingOfRational(r) (function)

Returns the smallest integer greater than or equal to the rational r .

Example

```
gap> CeilingOfRational(3/5);
1
```

A.2 Periodic subadditive functions

A periodic function f of period m from the set \mathbb{N} of natural numbers into itself may be specified through a list of m natural numbers. The function f is said to be *subadditive* if $f(i+j) \leq f(i) + f(j)$ and $f(0) = 0$.

A.2.1 RepresentsPeriodicSubAdditiveFunction

▷ RepresentsPeriodicSubAdditiveFunction(L) (function)

L is a list of integers. RepresentsPeriodicSubAdditiveFunction returns true or false according to whether L represents a periodic subAdditive function f periodic of period m or not. To avoid defining $f(0)$ (which we assume to be 0) we define $f(m) = 0$ and so the last element of the list must be 0. This technical need is due to the fact that positions in a list must be positive (not a 0).

Example

```
gap> RepresentsPeriodicSubAdditiveFunction([1,2,3,4,0]);
true
```

A.2.2 IsListOfIntegersNS

▷ IsListOfIntegersNS(L) (function)

Detects wheter L is a nonempty list of integers.

Example

```
gap> IsListOfIntegersNS([1,-1,0]);
true
gap> IsListOfIntegersNS(2);
false
gap> IsListOfIntegersNS([[2],3]);
false
gap> IsListOfIntegersNS([]);
false
```

Appendix B

Random functions

Here we describe some functions which allow to create several "random" objects.

B.1 Random functions

B.1.1 RandomNumericalSemigroup

▷ `RandomNumericalSemigroup(n, a [, b])` (function)

Returns a “random” numerical semigroup with no more than *n* generators in $[1..a]$ (or in $[a..b]$, if *b* is present).

Example

```
gap> RandomNumericalSemigroup(3,9);
<Numerical semigroup with 3 generators>
gap> RandomNumericalSemigroup(3,9,55);
<Numerical semigroup with 3 generators>
```

B.1.2 RandomListForNS

▷ `RandomListForNS(n, a, b)` (function)

Returns a set of length not greater than *n* of random integers in $[a..b]$ whose GCD is 1. It is used to create "random" numerical semigroups.

Example

```
gap> RandomListForNS(13,1,79);
[ 22, 26, 29, 31, 34, 46, 53, 61, 62, 73, 76 ]
```

B.1.3 RandomModularNumericalSemigroup

▷ `RandomModularNumericalSemigroup(k [, m])` (function)

Returns a “random” modular numerical semigroup $S(a,b)$ with $a \leq k$ (see 1) and multiplicity at least *m*, were *m* is the second argument, which may not be present..

Example

```
gap> RandomModularNumericalSemigroup(9);
<Modular numerical semigroup satisfying 5x mod 6 <= x >
```

```
gap> RandomModularNumericalSemigroup(10,25);
<Modular numerical semigroup satisfying 4x mod 157 <= x >
```

B.1.4 RandomProportionallyModularNumericalSemigroup

▷ RandomProportionallyModularNumericalSemigroup(k , m) (function)

Returns a “random” proportionally modular numerical semigroup $S(a,b,c)$ with $a \leq k$ (see 1) and multiplicity at least m , where m is the second argument, which may not be present.

Example

```
gap> RandomProportionallyModularNumericalSemigroup(9);
<Proportionally modular numerical semigroup satisfying 2x mod 3 <= 2x >
gap> RandomProportionallyModularNumericalSemigroup(10,25);
<Proportionally modular numerical semigroup satisfying 6x mod 681 <= 2x >
```

B.1.5 RandomListRepresentingSubAdditiveFunction

▷ RandomListRepresentingSubAdditiveFunction(m , a) (function)

Produces a “random” list representing a subadditive function (see 1) which is periodic with period m (or less). When possible, the images are in $[a..20*a]$. (Otherwise, the list of possible images is enlarged.)

Example

```
gap> RandomListRepresentingSubAdditiveFunction(7,9);
[ 173, 114, 67, 0 ]
gap> RepresentsPeriodicSubAdditiveFunction(last);
true
```

Appendix C

Contributions

C.1 Functions implemented by A. Sammartano

C.1.1 IsGradedAssociatedRingNumericalSemigroupBuchsbaum

▷ IsGradedAssociatedRingNumericalSemigroupBuchsbaum(S) (function)

S is a numerical semigroup.

Returns true if the graded ring associated to $K[[S]]$ is Buchsbaum, and false otherwise. This test is the implementation of the algorithm given in [DMV09].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsGradedAssociatedRingNumericalSemigroupBuchsbaum(s);
true
```

C.1.2 IsMpureNumericalSemigroup

▷ IsMpureNumericalSemigroup(S) (function)

S is a numerical semigroup.

Test for the M-Purity of the numerical semigroup S . This test is based on [Bry10].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsMpureNumericalSemigroup(s);
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsMpureNumericalSemigroup(s);
true
```

C.1.3 IsPureNumericalSemigroup

▷ IsPureNumericalSemigroup(S) (function)

S is a numerical semigroup.

Test for the purity of the numerical semigroup S . This test is based on [Bry10].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsPureNumericalSemigroup(s);
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsPureNumericalSemigroup(s);
true
```

C.1.4 IsGradedAssociatedRingNumericalSemigroupGorenstein

▷ IsGradedAssociatedRingNumericalSemigroupGorenstein(S) (function)

S is a numerical semigroup.

Returns true if the graded ring associated to $K[[S]]$ is Gorenstein, and false otherwise. This test is the implementation of the algorithm given in [DMS11].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsGradedAssociatedRingNumericalSemigroupGorenstein(s);
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsGradedAssociatedRingNumericalSemigroupGorenstein(s);
true
```

C.1.5 IsGradedAssociatedRingNumericalSemigroupCI

▷ IsGradedAssociatedRingNumericalSemigroupCI(S) (function)

S is a numerical semigroup.

Returns true if the Complete Intersection property of the associated graded ring of a numerical semigroup ring associated to $K[[S]]$, and false otherwise. This test is the implementation of the algorithm given in [DMS13b].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsGradedAssociatedRingNumericalSemigroupCI(s);
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsGradedAssociatedRingNumericalSemigroupCI(s);
true
```

C.1.6 IsAperySetGammaRectangular

▷ IsAperySetGammaRectangular(S) (function)

S is a numerical semigroup.

Test for the Gamma-Rectangularity of the Apéry Set of a numerical semigroup. This test is the implementation of the algorithm given in [DMS13a].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsAperySetGammaRectangular(s);
```

```
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsAperySetGammaRectangular(s);
true
```

C.1.7 IsAperySetBetaRectangular

▷ IsAperySetBetaRectangular(S) (function)

S is a numerical semigroup.

Test for the Beta-Rectangularity of the Apéry Set of a numerical semigroup. This test is the implementation of the algorithm given in [DMS13a].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsAperySetBetaRectangular(s);
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsAperySetBetaRectangular(s);
true
```

C.1.8 IsAperySetAlphaRectangular

▷ IsAperySetAlphaRectangular(S) (function)

S is a numerical semigroup.

Test for the Alpha-Rectangularity of the Apéry Set of a numerical semigroup. This test is the implementation of the algorithm given in [DMS13a].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> IsAperySetAlphaRectangular(s);
false
gap> s:=NumericalSemigroup(4,6,11);;
gap> IsAperySetAlphaRectangular(s);
true
```

C.1.9 TypeSequenceOfNumericalSemigroup

▷ TypeSequenceOfNumericalSemigroup(S) (function)

S is a numerical semigroup.

Computes the type sequence of a numerical semigroup. This test is the implementation of the algorithm given in [BDF97].

Example

```
gap> s:=NumericalSemigroup(30, 35, 42, 47, 148, 153, 157, 169, 181, 193);;
gap> TypeSequenceOfNumericalSemigroup(s);
[ 13, 3, 4, 4, 7, 3, 3, 3, 2, 2, 2, 3, 3, 2, 4, 3, 2, 1, 3, 2, 1, 1, 2, 2, 1,
  1, 1, 2, 2, 1, 3, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1,
  1, 1, 1 ]
gap> s:=NumericalSemigroup(4,6,11);;
```



```
gap> TypeSequenceOfNumericalSemigroup(s);  
[ 1, 1, 1, 1, 1, 1, 1 ]
```

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