

# SL2Reps

**Constructs representations of  $SL_2(\mathbb{Z})$ .**

0.1

24 September 2021

**Siu-Hung Ng**

**Yilong Wang**

**Samuel Wilson**

**Siu-Hung Ng**

Email: [rng@math.lsu.edu](mailto:rng@math.lsu.edu)

**Yilong Wang**

Email: [wyl@bimsa.cn](mailto:wyl@bimsa.cn)

**Samuel Wilson**

Email: [swil311@lsu.edu](mailto:swil311@lsu.edu)

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Installation . . . . .	3
1.2	Usage . . . . .	3
<b>2</b>	<b>Description</b>	<b>4</b>
2.1	Construction . . . . .	4
2.2	Weyl representation types . . . . .	4
<b>3</b>	<b>Lists of representations</b>	<b>6</b>
3.1	Lists by degree . . . . .	6
3.2	Lists by level . . . . .	6
3.3	Lists of exceptional representations . . . . .	7
<b>4</b>	<b>Methods for testing</b>	<b>8</b>
4.1	Testing . . . . .	8
<b>5</b>	<b>Irreducible representations of prime-power level</b>	<b>9</b>
5.1	Representations of type D . . . . .	9
5.2	Representations of type N . . . . .	9
5.3	Representations of type R . . . . .	10
	<b>References</b>	<b>12</b>
	<b>Index</b>	<b>13</b>

# Chapter 1

## Introduction

This package, `SL2Reps`, provides methods for constructing and testing matrix presentations of the representations of  $SL_2(\mathbb{Z})$ .

Irreducible representations of prime-power level are constructed individually by means of Weyl representations, and from these a list of all representations of a given degree or level may be produced. The format is designed to be useful in the study of modular tensor categories in particular.

### 1.1 Installation

To install `SL2Reps`, first download it from `TODD`, then place it in the `pkg` subdirectory of your GAP installation (or in the `pkg` subdirectory of any other GAP root directory, for example one added with the `-l` argument).

`SL2Reps` is then loaded with the GAP command

```
gap> LoadPackage( "SL2Reps" );
```

### 1.2 Usage

Specific irreducible representations may be constructed via the methods in Chapter 5, while lists of irreducible representations with a given degree or level may be constructed with those in Chapter 3.

This package uses an `InfoClass`, `InfoSL2Reps`. It may be set to 0 (silent), 1 (info), or 2 (verbose). To change it, use

```
gap> SetInfoLevel(InfoSL2Reps, k);
```

## Chapter 2

# Description

The group  $\mathrm{SL}_2(\mathbb{Z})$  is generated by  $\mathfrak{s} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and  $\mathfrak{t} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  (which satisfy the relations  $\mathfrak{s}^4 = (\mathfrak{st})^3 = \mathrm{id}$ ). Thus, any complex representation  $\rho : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^n$  (where  $n \in \mathbb{Z}^+$  is called the *degree* of  $\rho$ ) is defined by the  $n \times n$  matrices  $S = \rho(\mathfrak{s})$  and  $T = \rho(\mathfrak{t})$ . In fact, any such representation factors through  $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$  for some  $\ell \in \mathbb{Z}^+$ ; the smallest such  $\ell$  is called the *level* of  $\rho$ . We therefore present representations in the form of a record

```
rec(S, T, degree, level, name)
```

where the name follows the conventions of [NW76]. Note that our definition of  $\mathfrak{s}$  follows that of [Nob76]; other authors prefer the inverse, i.e.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . When working with that convention, one must invert the  $S$  matrices output by this package.

### 2.1 Construction

For any representation  $\rho$  of  $\mathrm{SL}_2(\mathbb{Z})$ , we may decompose  $\rho$  as a direct sum of irreducible representations of prime-power level using the Chinese remainder theorem. Therefore, to characterize all representations of  $\mathrm{SL}_2(\mathbb{Z})$ , it suffices to consider irreducible representations of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ .

Such representations may be constructed using Weyl representations as described in [Nob76, Section 1]. The Weyl representations used by this package have the following form. Let  $p$  be a prime number and  $\lambda \in \mathbb{Z}^+$ . Choose a  $\mathbb{Z}/p^\lambda\mathbb{Z}$ -module  $M$  of rank 1 or 2, and a quadratic form  $Q$  on  $M$ , such that  $(M, Q)$  is of one of the three types described in 2.2. The *quadratic module*  $(M, Q)$  then gives rise to a representation of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ , denoted  $W(M, Q)$ .

With a finite number of exceptions, every representation of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$  may be found as a subrepresentation of  $W(M, Q)$  for an appropriate choice of  $(M, Q)$  [NW76, Hauptsatz 2]. The 18 exceptions may be found as the tensor product of two such subrepresentations; these may be generated with `SL2Reps_Exceptions` (3.3.1).

Representations of  $\mathrm{SL}_2(\mathbb{Z})$  may then be found as direct sums of these prime-power representations.

### 2.2 Weyl representation types

#### 2.2.1 Type D

Let  $p$  be prime and  $\lambda \geq 1$ . Then the Weyl representation arising from the quadratic module with  $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^\lambda\mathbb{Z}$  and  $Q(x, y) = \frac{xy}{p^\lambda}$  is said to be of type  $D$  and denoted  $D(p, \lambda)$ . Information on

$(M, Q)$  may be obtained via `SL2Reps_ModuleD` (5.1.1), and subrepresentations of  $D(p, \lambda)$  with level  $p^\lambda$  may be constructed via `SL2Reps_RepD` (5.1.2).

### 2.2.2 Type N

Let  $p$  be prime and  $\lambda \geq 1$ . Then the Weyl representation arising from the quadratic module with  $M = \mathbb{Z}/p^\lambda \mathbb{Z} \oplus \mathbb{Z}/p^\lambda \mathbb{Z}$  and  $Q(x, y) = \frac{x^2 + xy + \frac{1+u}{4}y^2}{p^\lambda}$  (where, for  $p \neq 2$ ,  $u$  is chosen so that  $u \equiv 3 \pmod{4}$  with  $\left(\frac{-u}{p}\right) = -1$ , and for  $p = 2$ ,  $u = 3$ ) is said to be of type  $N$  and denoted  $N(p, \lambda)$ . Information on  $(M, Q)$  may be obtained via `SL2Reps_ModuleN` (5.2.1), and subrepresentations of  $D(p, \lambda)$  with level  $p^\lambda$  may be constructed via `SL2Reps_RepN` (5.2.2).

### 2.2.3 Type R

The construction of type  $R$  varies depending on whether  $p = 2$ .

First, if  $p$  is an odd prime, let  $\lambda \geq 2$ ,  $\sigma \in \{1, \dots, \lambda\}$ , and  $r, t \in \{1, u\}$  with  $u$  a quadratic non-residue mod  $p$ . Then define  $M = \mathbb{Z}/p^\lambda \mathbb{Z} \oplus \mathbb{Z}/p^{\lambda-\sigma} \mathbb{Z}$  and  $Q(x, y) = \frac{r(x^2 + p^\sigma t y^2)}{p^\lambda}$ .

On the other hand, if  $p = 2$ , let  $\lambda \geq 2$ ,  $\sigma \in \{0, \dots, \lambda - 2\}$  and  $r, t \in \{1, 3, 5, 7\}$ . Then define  $M = \mathbb{Z}/2^{\lambda-1} \mathbb{Z} \oplus \mathbb{Z}/2^{\lambda-\sigma-1} \mathbb{Z}$  and  $Q(x, y) = \frac{r(x^2 + 2^\sigma t y^2)}{2^\lambda}$ .

In either case, the resulting representation is said to be of type  $R$  and denoted  $R(p, \lambda, \sigma, r, t)$ . Information on  $(M, Q)$  may be obtained via `SL2Reps_ModuleR` (5.3.1), and subrepresentations of  $R(p, \lambda, \sigma, r, t)$  with level  $p^\lambda$  may be constructed via `SL2Reps_RepR` (5.3.2). Note that if  $\sigma = \lambda$  for  $p \neq 2$ , then the second factor of  $M$  is trivial (and hence  $t$  is irrelevant); this special case is handled by `SL2Reps_RepRUnary` (5.3.3).

## Chapter 3

# Lists of representations

### 3.1 Lists by degree

#### 3.1.1 SL2Reps\_PrimePowerIrrepsOfDegree

▷ SL2Reps\_PrimePowerIrrepsOfDegree(*degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are exactly the given degree and have prime power level.

#### 3.1.2 SL2Reps\_PrimePowerIrrepsOfDegreeAtMost

▷ SL2Reps\_PrimePowerIrrepsOfDegreeAtMost(*max\_degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are at most the given degree and have prime power level.

#### 3.1.3 SL2Reps\_IrrepsOfDegree

▷ SL2Reps\_IrrepsOfDegree(*degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are exactly the given degree.

#### 3.1.4 SL2Reps\_IrrepsOfDegreeAtMost

▷ SL2Reps\_IrrepsOfDegreeAtMost(*degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are at most the given degree.

### 3.2 Lists by level

#### 3.2.1 SL2Reps\_PrimePowerIrrepsOfLevel

▷ SL2Reps\_PrimePowerIrrepsOfLevel(*p*, *lambda*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  with level exactly  $p^\lambda$ .

### 3.3 Lists of exceptional representations

#### 3.3.1 SL2Reps\_Exceptions

▷ `SL2Reps_Exceptions(arg)`

(function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of the 18 exceptional irreps of  $\mathrm{SL}_2(\mathbb{Z})$ .

## Chapter 4

# Methods for testing

### 4.1 Testing

#### 4.1.1 SL2Reps\_SL2Conj

▷ `SL2Reps_SL2Conj( $p$ ,  $ld$ )` (function)

**Returns:** the group  $SL_2(\mathbb{Z}/p^\lambda\mathbb{Z})$  with conjugacy classes set to the format we use.

#### 4.1.2 SL2Reps\_ChiST

▷ `SL2Reps_ChiST( $S$ ,  $T$ ,  $p$ ,  $ld$ )` (function)

**Returns:** a list representing a character of  $SL_2(\mathbb{Z}/p^\lambda\mathbb{Z})$

Converts the modular data  $(S, T)$ , which must have level dividing  $p^\lambda$ , into a character of  $SL_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ , presented in a form matching the conjugacy classes used in `SL2Reps_SL2Conj`.

#### 4.1.3 SL2Reps\_IrrepPositionTest

▷ `SL2Reps_IrrepPositionTest( $p$ ,  $lambda$ )` (function)

**Returns:** a boolean

Constructs and tests all irreps of level dividing  $p^\lambda$  by checking their positions in  $\text{Irr}(G)$ .



## Chapter 5

# Irreducible representations of prime-power level

### 5.1 Representations of type D

See [2.2.1](#).

#### 5.1.1 SL2Reps\_ModuleD

▷ `SL2Reps_ModuleD(p, ld)` (function)  
**Returns:** a record `rec(Agrp, Bp, Char, IsPrim)` describing  $(M, Q)$   
Constructs information about the underlying quadratic module  $(M, Q)$  of type  $D$ .  
`Agrp` describes the elements of  $\mathfrak{A} = (\mathbb{Z}/p^\lambda\mathbb{Z})^\times$  (see [NW76, Section 2.1]).  
`Bp` describes a set of representatives for the  $\mathfrak{A}$ -orbits on  $M^\times$ , which correspond to a basis the  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ -invariant subspace associated to any primitive character  $\chi \in \hat{\mathfrak{A}}$  with  $\chi^2 \neq 1$ . For other characters, we must use different bases which are particular to each case.  
`Char(i, j)` converts the `chi_index` used in `SL2Reps_RepD` (5.1.2) to a function.  
`IsPrim(chi)` tests whether a given character (e.g. from `Char(i, j)`) is primitive.

#### 5.1.2 SL2Reps\_RepD

▷ `SL2Reps_RepD(p, ld, chi_index)` (function)  
**Returns:** a list of lists of the form  $[S, T]$   
Constructs the modular data for the irreducible representation(s) of type  $D$  with level  $p^\lambda$  corresponding to the character  $\chi$  indexed by `chi_index`.

### 5.2 Representations of type N

See [2.2.2](#).

#### 5.2.1 SL2Reps\_ModuleN

▷ `SL2Reps_ModuleN(p, ld)` (function)  
**Returns:** a record `rec(Agrp, Bp, Char, Nm, Prod)` describing  $(M, Q)$

Constructs information about the underlying quadratic module  $(M, Q)$  of type  $N$ .

$\text{Agrp}$  describes the elements of  $\mathfrak{A} = \{\varepsilon \in M^\times \mid \text{Nm}(\varepsilon) = 1\}$  (see [NW76, Section 2.2]).

$\text{Bp}$  describes a set of representatives for the  $\mathfrak{A}$ -orbits on  $M^\times$ , which correspond to a basis the  $\text{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ -invariant subspace associated to any primitive character  $\chi \in \hat{\mathfrak{A}}$  with  $\chi^2 \neq 1$ . For other characters, we must use different bases which are particular to each case.

$\text{Char}(i, j)$  converts the  $\text{chi\_index}$  used in  $\text{SL2Reps\_RepN}$  (5.2.2) to a function.

$\text{Nm}(a)$  and  $\text{Prod}(a, b)$  are the norm and product functions on  $M$ , respectively.

### 5.2.2 SL2Reps\_RepN

▷  $\text{SL2Reps\_RepN}(p, ld, \text{chi\_index})$  (function)

**Returns:** a list of lists of the form  $[S, T]$

Constructs the modular data for the irreducible representation(s) of type  $N$  with level  $p^\lambda$  corresponding to the character  $\chi$  indexed by  $\text{chi\_index}$ .

## 5.3 Representations of type R

See 2.2.3.

### 5.3.1 SL2Reps\_ModuleR

▷  $\text{SL2Reps\_ModuleR}(p, ld, \text{sigma}, r, t)$  (function)

**Returns:** a record  $\text{rec}(\text{Agrp}, \text{Char}, \text{IsPrim}, \text{Nm}, \text{Ord}, \text{Prod}, c, \text{tM})$  describing  $(M, Q)$

Constructs information about the underlying quadratic module  $(M, Q)$  of type  $R$ .

$\text{Agrp}$  describes the elements of  $\mathfrak{A} = \{\varepsilon \in M^\times \mid \text{Nm}(\varepsilon) = 1\}$  (see [NW76, Section 2.3 - 2.5]).

Representatives for the  $\mathfrak{A}$ -orbits on  $M^\times$  can depend on the choice of character, even for primitive characters  $\chi$  with  $\chi^2 \neq 1$ . Thus, we cannot provide them here, and they are instead calculated by  $\text{SL2Reps\_RepR}$  (5.3.2).

$\text{Char}(i, j)$  converts the  $\text{chi\_index}$  used in  $\text{SL2Reps\_RepR}$  (5.3.2) to a function.

$\text{IsPrim}(\text{chi})$  tests whether a given character (e.g. from  $\text{Char}$ ) is primitive.

$\text{Nm}(a)$ ,  $\text{Ord}(a)$ , and  $\text{Prod}(a, b)$  are the norm, order, and product functions on  $M$ , respectively.

$c$  is a scalar used in calculating the  $S$ -matrix; namely  $c = \frac{1}{|M|} \sum_{x \in M} \mathbf{e}(Q(x))$ .

$\text{tM}$  is the group  $M - pM$ .

### 5.3.2 SL2Reps\_RepR

▷  $\text{SL2Reps\_RepR}(p, ld, \text{sigma}, r, t, \text{chi\_index})$  (function)

**Returns:** a list of lists of the form  $[S, T]$

Constructs the modular data for the irreducible representation(s) of type  $R$  with level  $p^\lambda$  corresponding to the character  $\chi$  indexed by  $\text{chi\_index}$ .

When  $\sigma = \lambda$ , this falls through to  $\text{SL2Reps\_RepRUnary}$  (5.3.3).

### 5.3.3 SL2Reps\_RepRUnary

▷  $\text{SL2Reps\_RepRUnary}(p, ld, r)$  (function)

**Returns:** a list of lists of the form  $[S, T]$

Constructs the modular data for the irreducible representation(s) of unary type  $R$  (that is, with  $\sigma = \lambda$ ) with level  $p^\lambda$ .

# References

- [Nob76] Alexandre Nobs. Die irreduziblen Darstellungen der Gruppen  $SL_2(Z_p)$ , insbesondere  $SL_2(Z_2)$ . I. *Comment. Math. Helv.*, 51(4):465–489, 1976. [4](#)
- [NW76] Alexandre Nobs and Jürgen Wolfart. Die irreduziblen Darstellungen der Gruppen  $SL_2(Z_p)$ , insbesondere  $SL_2(Z_p)$ . II. *Comment. Math. Helv.*, 51(4):491–526, 1976. [4](#), [9](#), [10](#)

# Index

SL2Reps\_ChiST, [8](#)  
SL2Reps\_Exceptions, [7](#)  
SL2Reps\_IrrepPositionTest, [8](#)  
SL2Reps\_IrrepsOfDegree, [6](#)  
SL2Reps\_IrrepsOfDegreeAtMost, [6](#)  
SL2Reps\_ModuleD, [9](#)  
SL2Reps\_ModuleN, [9](#)  
SL2Reps\_ModuleR, [10](#)  
SL2Reps\_PrimePowerIrrepsOfDegree, [6](#)  
SL2Reps\_PrimePowerIrrepsOfDegreeAt-  
Most, [6](#)  
SL2Reps\_PrimePowerIrrepsOfLevel, [6](#)  
SL2Reps\_RepD, [9](#)  
SL2Reps\_RepN, [10](#)  
SL2Reps\_RepR, [10](#)  
SL2Reps\_RepRUnary, [10](#)  
SL2Reps\_SL2Conj, [8](#)