

# SL2Reps

**Constructs representations of  $SL_2(\mathbb{Z})$ .**

0.1

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# Chapter 1

## Introduction

This package, `SL2Reps`, provides methods for constructing and testing matrix presentations of the representations of  $\mathrm{SL}_2(\mathbb{Z})$ .

Irreducible representations of prime-power level are constructed individually by means of Weyl representations, and from these a list of all representations of a given degree or level may be produced.

### 1.1 Installation

To install `SL2Reps`, first download it from `TODO`, then place it in the `pkg` subdirectory of your GAP installation (or in the `pkg` subdirectory of any other GAP root directory, for example one added with the `-l` argument).

`SL2Reps` is then loaded with the GAP command

```
gap> LoadPackage( "SL2Reps" );
```

### 1.2 Usage

`TODO`: Link to other sections.

This package uses an `InfoClass`, `InfoSL2Reps`. It may be set to 0 (silent), 1 (info), or 2 (verbose). To change it, use

```
gap> SetInfoLevel(InfoSL2Reps, k);
```

## Chapter 2

# Description

The group  $\mathrm{SL}_2(\mathbb{Z})$  is generated by  $\mathfrak{s} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $\mathfrak{t} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , subject to the relations  $\mathfrak{s}^4 = \mathrm{id}$ ,  $(\mathfrak{st})^3 = \mathfrak{s}^2$ . Thus, any complex representation  $\rho : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^n$  (where  $n \in \mathbb{Z}^+$  is called the *degree* of  $\rho$ ) is defined by the  $n \times n$  matrices  $S = \rho(\mathfrak{s})$  and  $T = \rho(\mathfrak{t})$ . In fact, any such representation factors through  $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$  for some  $\ell \in \mathbb{Z}^+$ ; the smallest such  $\ell$  is called the *level* of  $\rho$ . We therefore present representations in the form of a record

```
rec(S, T, degree, level, name)
```

where the name follows the conventions of [NW76]. Note that our definition of  $\mathfrak{s}$  follows that of [Nob76]; other authors prefer the inverse, i.e.  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . When working with that convention, one must invert the  $S$  matrices output by this package.

### 2.1 Construction

For any representation  $\rho$  of  $\mathrm{SL}_2(\mathbb{Z})$ , we may decompose  $\rho$  as a direct sum of irreducible representations of prime-power level using the Chinese remainder theorem. Therefore, to characterize all representations of  $\mathrm{SL}_2(\mathbb{Z})$ , it suffices to consider irreducible representations of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ .

Such representations may be constructed using Weyl representations as described in [Nob76, Section 1]. The Weyl representations used by this package have the following form. Let  $p$  be a prime number and  $\lambda \in \mathbb{Z}^+$ . Choose a  $\mathbb{Z}/p^\lambda\mathbb{Z}$ -module  $M$  of rank 1 or 2, and a quadratic form  $Q$  on  $M$ , such that  $(M, Q)$  is of one of the three types described in 2.2. The *quadratic module*  $(M, Q)$  then gives rise to a representation of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ , denoted  $W(M, Q)$ .

With a finite number of exceptions, every representation of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$  may be found as a subrepresentation of  $W(M, Q)$  for an appropriate choice of  $(M, Q)$  [NW76, Hauptsatz 2]. The 18 exceptions may be found as the tensor product of two such subrepresentations; these may be generated with `SL2Reps_Exceptions` (3.3.1).

Representations of  $\mathrm{SL}_2(\mathbb{Z})$  may then be found as direct sums of these prime-power representations.

### 2.2 Weyl representation types

#### 2.2.1 Type D

Let  $p$  be prime and  $\lambda \geq 1$ . Then the Weyl representation arising from the quadratic module with  $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^\lambda\mathbb{Z}$  and  $Q(x, y) = \frac{xy}{p^\lambda}$  is said to be of type  $D$  and denoted  $D(p, \lambda)$ . Information on

$(M, Q)$  may be obtained via `SL2Reps_ModuleD` (5.1.1), and subrepresentations of  $D(p, \lambda)$  with level  $p^\lambda$  may be constructed via `SL2Reps_RepD` (5.1.2).

### 2.2.2 Type N

Let  $p$  be prime and  $\lambda \geq 1$ . Then the Weyl representation arising from the quadratic module with  $M = \mathbb{Z}/p^\lambda \mathbb{Z} \oplus \mathbb{Z}/p^\lambda \mathbb{Z}$  and  $Q(x, y) = \frac{x^2 + xy + \frac{1+u}{4}y^2}{p^\lambda}$  (where, for  $p \neq 2$ ,  $u$  is chosen so that  $u \equiv 3 \pmod{4}$  with  $\left(\frac{-u}{p}\right) = -1$ , and for  $p = 2$ ,  $u = 3$ ) is said to be of type  $N$  and denoted  $N(p, \lambda)$ . Information on  $(M, Q)$  may be obtained via `SL2Reps_ModuleN` (5.2.1), and subrepresentations of  $D(p, \lambda)$  with level  $p^\lambda$  may be constructed via `SL2Reps_RepN` (5.2.2).

### 2.2.3 Type R

The construction of type  $R$  varies depending on whether  $p = 2$ .

First, if  $p$  is an odd prime, let  $\lambda \geq 2$ ,  $\sigma \in \{1, \dots, \lambda\}$ , and  $r, t \in \{1, u\}$  with  $u$  a quadratic non-residue mod  $p$ . Then define  $M = \mathbb{Z}/p^\lambda \mathbb{Z} \oplus \mathbb{Z}/p^{\lambda-\sigma} \mathbb{Z}$  and  $Q(x, y) = \frac{r(x^2 + p^\sigma t y^2)}{p^\lambda}$ .

On the other hand, if  $p = 2$ , let  $\lambda \geq 2$ ,  $\sigma \in \{0, \dots, \lambda - 2\}$  and  $r, t \in \{1, 3, 5, 7\}$ . Then define  $M = \mathbb{Z}/2^{\lambda-1} \mathbb{Z} \oplus \mathbb{Z}/2^{\lambda-\sigma-1} \mathbb{Z}$  and  $Q(x, y) = \frac{r(x^2 + 2^\sigma t y^2)}{2^\lambda}$ .

In either case, the resulting representation is said to be of type  $R$  and denoted  $R(p, \lambda, \sigma, r, t)$ . Information on  $(M, Q)$  may be obtained via `SL2Reps_ModuleR` (5.3.1), and subrepresentations of  $R(p, \lambda, \sigma, r, t)$  with level  $p^\lambda$  may be constructed via `SL2Reps_RepR` (5.3.2). Note that if  $\sigma = \lambda$  for  $p \neq 2$ , then the second factor of  $M$  is trivial (and hence  $t$  is irrelevant); this special case is handled by `SL2Reps_RepRUnary` (5.3.3).

## Chapter 3

# Lists of representations

### 3.1 Lists by degree

#### 3.1.1 SL2Reps\_PrimePowerIrrepsOfDegree

▷ SL2Reps\_PrimePowerIrrepsOfDegree(*degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are exactly the given degree and have prime power level.

#### 3.1.2 SL2Reps\_PrimePowerIrrepsOfDegreeAtMost

▷ SL2Reps\_PrimePowerIrrepsOfDegreeAtMost(*max\_degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are at most the given degree and have prime power level.

#### 3.1.3 SL2Reps\_IrrepsOfDegree

▷ SL2Reps\_IrrepsOfDegree(*degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are exactly the given degree.

#### 3.1.4 SL2Reps\_IrrepsOfDegreeAtMost

▷ SL2Reps\_IrrepsOfDegreeAtMost(*degree*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  that are at most the given degree.

### 3.2 Lists by level

#### 3.2.1 SL2Reps\_PrimePowerIrrepsOfLevel

▷ SL2Reps\_PrimePowerIrrepsOfLevel(*p*, *lambda*) (function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of  $SL_2(\mathbb{Z})$  with level exactly  $p^\lambda$ .

### 3.3 Lists of exceptional representations

#### 3.3.1 SL2Reps\_Exceptions

▷ `SL2Reps_Exceptions(arg)`

(function)

**Returns:** a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of the 18 exceptional irreps of  $\mathrm{SL}_2(\mathbb{Z})$ .

## Chapter 4

# Methods for testing

### 4.1 Testing

#### 4.1.1 SL2Reps\_SL2Conj

▷ `SL2Reps_SL2Conj(p, ld)` (function)

**Returns:** the group  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$  with conjugacy classes set to the format we use.

#### 4.1.2 SL2Reps\_ChiST

▷ `SL2Reps_ChiST(S, T, p, ld)` (function)

**Returns:** a list representing a character of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$

Converts the modular data  $(S, T)$ , which must have level dividing  $p^\lambda$ , into a character of  $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ , presented in a form matching the conjugacy classes used in `SL2Reps_SL2Conj`.

#### 4.1.3 SL2Reps\_IrrepPositionTest

▷ `SL2Reps_IrrepPositionTest(p, lambda)` (function)

**Returns:** a boolean

Constructs and tests all irreps of level dividing  $p^\lambda$  by checking their positions in  $\mathrm{Irr}(G)$ .



## Chapter 5

# Irreducible representations of prime-power level

### 5.1 Representations of type D

See [2.2.1](#).

#### 5.1.1 SL2Reps\_ModuleD

- ▷ `SL2Reps_ModuleD( $p$ ,  $ld$ )` (function)  
**Returns:** a record describing  $(M, Q)$   
Constructs information about the underlying quadratic module  $(M, Q)$  of type  $D$ .

#### 5.1.2 SL2Reps\_RepD

- ▷ `SL2Reps_RepD( $p$ ,  $ld$ ,  $chi\_index$ )` (function)  
**Returns:** a list of lists of the form  $[S, T]$   
Constructs the irreducible representation(s) of type  $D$  with level  $p^\lambda$  corresponding to the character  $\chi$  indexed by  $chi\_index$ .

### 5.2 Representations of type N

See [2.2.2](#).

#### 5.2.1 SL2Reps\_ModuleN

- ▷ `SL2Reps_ModuleN( $p$ ,  $ld$ )` (function)  
**Returns:** a record describing  $(M, Q)$   
Constructs information about the underlying quadratic module  $(M, Q)$  of type  $N$ .

#### 5.2.2 SL2Reps\_RepN

- ▷ `SL2Reps_RepN( $p$ ,  $ld$ ,  $chi\_index$ )` (function)  
**Returns:** a list of lists of the form  $[S, T]$

Constructs the irreducible representation(s) of type  $N$  with level  $p^\lambda$  corresponding to the character  $\chi$  indexed by `chi_index`.

## 5.3 Representations of type R

See 2.2.3.

### 5.3.1 SL2Reps\_ModuleR

- ▷ `SL2Reps_ModuleR(p, ld, sigma, r, t)` (function)  
**Returns:** a record describing  $(M, Q)$   
Constructs information about the underlying quadratic module  $(M, Q)$  of type  $R$ .

### 5.3.2 SL2Reps\_RepR

- ▷ `SL2Reps_RepR(p, ld, sigma, r, t, chi_index)` (function)  
**Returns:** a list of lists of the form  $[S, T]$   
Constructs the irreducible representation(s) of type  $R$  with level  $p^\lambda$  corresponding to the character  $\chi$  indexed by `chi_index`.

### 5.3.3 SL2Reps\_RepRUnary

- ▷ `SL2Reps_RepRUnary(p, ld, r)` (function)  
**Returns:** a list of lists of the form  $[S, T]$   
Constructs the irreducible representation(s) of unary type  $R$  (that is, with  $\sigma = \lambda$ ) with level  $p^\lambda$ .

# References

- [Nob76] Alexandre Nobs. Die irreduziblen Darstellungen der Gruppen  $SL_2(Z_p)$ , insbesondere  $SL_2(Z_2)$ . I. *Comment. Math. Helv.*, 51(4):465–489, 1976. [4](#)
- [NW76] Alexandre Nobs and Jürgen Wolfart. Die irreduziblen Darstellungen der Gruppen  $SL_2(Z_p)$ , insbesondere  $SL_2(Z_p)$ . II. *Comment. Math. Helv.*, 51(4):491–526, 1976. [4](#)

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