SL2Reps

Constructs representations of SL2(Z).

0.1

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Chapter 1

Introduction

This package, SL2Reps, provides methods for constructing and testing matrix presentations of the representations of $\mathrm{SL}_2(\mathbb{Z})$ whose kernels are congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Irreducible representations of prime-power level are constructed individually by using the Weil representations of quadratic modules, and from these a list of all representations of a given degree or level can be produced. The format is designed for the study of modular tensor categories in particular.

1.1 Installation

To install SL2Reps, first download it from https://github.com/ontoclasm/sl2-reps, then place it in the pkg subdirectory of your GAP installation (or in the pkg subdirectory of any other GAP root directory, for example one added with the -l argument).

SL2Reps is then loaded with the GAP command

```
gap> LoadPackage( "SL2Reps" );
```

1.2 Usage

Specific irreducible representations may be constructed via the methods in Chapter 5, while lists of irreducible representations with a given degree or level may be constructed with those in Chapter 3.

This package uses an InfoClass, InfoSL2Reps. It may be set to 0 (silent), 1 (info), or 2 (verbose). To change it, use

```
gap> SetInfoLevel(InfoSL2Reps, k);
```

Chapter 2

Description

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\mathfrak{s}$ = `[[0,1],[-1,0]]` and $\mathfrak{t}$ = `[[1,1],[0,1]]` (which satisfy the relations $\mathfrak{s}^4 = (\mathfrak{s}\mathfrak{t})^3 = \mathrm{id}$). Thus, any complex representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n$ (where $n \in \mathbb{Z}^+$ is called the *degree* of ρ) is determined by the $n \times n$ matrices $S = \rho(\mathfrak{s})$ and $T = \rho(\mathfrak{t})$.

This package constructs irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ which factor through $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$ for some $\ell \in \mathbb{Z}^+$; the smallest such ℓ is called the *level* of the representation. One may equivalently say that the kernel of the representation is a congruence subgroup. It has been shown that any representation of $\mathrm{SL}_2(\mathbb{Z})$ arising from a modular tensor category has this property [DLN15].

We therefore present representations in the form of a record `rec(S, T, degree, level, name)`, where the name follows the conventions of [NW76].

Note that our definition of $\mathfrak{s}$ follows that of [Nob76]; other authors prefer the inverse, i.e. $\mathfrak{s}$ = `[[0,-1],[1,0]]` (under which convention the relations are $\mathfrak{s}^4 = \mathrm{id}$, $(\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2$). When working with that convention, one must invert the S matrices output by this package.

Throughout, we denote by $\mathbf{e}$ the map $k \mapsto e^{2\pi ik}$ (an isomorphism from $\mathbb{Q}/\mathbb{Z}$ to the group of finite roots of unity in $\mathbb{C}$).

2.1 Construction

Any representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ can be decomposed into a direct sum of irreducible representations (irreps). Further, if ρ has finite level, each irrep can be factorized into a tensor product of irreps whose levels are powers of distinct primes (using the Chinese remainder theorem). Therefore, to characterize all finite-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$ of finite level, it suffices to consider irreps of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ for primes p and positive integers λ.

2.1.1 Weil representations

Such representations may be constructed using Weil representations as described in [Nob76, Section 1]. We give a brief summary of the process here. First, if M is any additive abelian group, a *quadratic form* on M is a map $Q : M \to \mathbb{Q}/\mathbb{Z}$ such that $Q(-x) = Q(x)$ for all $x \in M$ and $B(x,y) = Q(x+y) - Q(x) - Q(y)$ defines a $\mathbb{Z}$-bilinear map $M \times M \to \mathbb{Q}/\mathbb{Z}$.

Now let p be a prime number and $\lambda \in \mathbb{Z}^+$. Choose a $\mathbb{Z}/p^\lambda\mathbb{Z}$-module M and a quadratic form Q on M such that the pair (M,Q) is of one of the three types described in Section 2.2. Each such M is a ring, and has at most 2 cyclic factors as an additive group. Those with 2 cyclic factors may be identified with a quotient of the quadratic integers, giving a norm on M. Then the *quadratic module*

(M,Q) gives rise to a representation of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ on the vector space $V=\mathbb{C}^M$ of complex-valued functions on M. This representation is denoted $W(M,Q)$.

We may construct subrepresentations $W(M,Q,\chi)$ of $W(M,Q)$ as follows. Let

$$\mathrm{Aut}(M,Q)=\{\varepsilon\in\mathrm{Aut}(M)\mid Q(\varepsilon x)=Q(x)\text{ for all }x\in M\}$$

and denote by $\mathfrak{A}\leq\mathrm{Aut}(M,Q)$ the abelian subgroup defined in Section 2.2. Note that $\mathfrak{A}$ has at most two cyclic factors, whose generators we denote by α and β. Let $\chi\in\widehat{\mathfrak{A}}$ be a 1-dimensional representation (character) of $\mathfrak{A}$, and define

$$V_\chi=\{f\in V\mid f(\varepsilon x)=\chi(\varepsilon)f(x)\text{ for all }x\in M\text{ and }\varepsilon\in\mathfrak{A}\}\ ,$$

which is a $SL_2(\mathbb{Z}/p^\lambda\mathbb{Z})$-invariant subspace of V. We then denote by $W(M,Q,\chi)$ the subrepresentation of $W(M,Q)$ on V_χ.

2.1.2 Primitive characters

We will frequently refer to a character χ as being *primitive*. With the exception of a single family of modules of type R (the *extremal* case, for which see Section 2.2.3) primitivity amounts to the following: there exists some $\varepsilon\in\mathfrak{A}$ such that $\chi(\varepsilon)\neq 1$ and ε fixes the submodule pM pointwise. In the extremal case, $\mathfrak{A}=\langle\alpha\rangle\times\langle -1\rangle$ (see Section 2.2.3 for an explicit definition of α) and χ is primitive if $\chi(\alpha)=-1$.

Explicit descriptions of the primitive characters for each type are given in Section 2.2.

2.1.3 Irrep types

The prime-power irreps then fall into three cases.

- The overwhelming majority are of the form $W(M,Q,\chi)$ for χ primitive and $\chi^2\neq 1$; we call these *standard*. This includes the primitive characters from the extremal case.
- A finite number, and a single infinite family arising from the extremal case (Section 2.2.3), are instead constructed by using non-primitive characters or primitive characters χ with $\chi^2=1$. We call these *non-standard*.
- Finally, 18 *exceptional* irreps are constructed as tensor products of two irreps from the other two cases.

All the finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ of finite level can now be constructed by taking tensor products of these prime-power irreps. Note that, if two representations are defined by pairs `[S1,T1]` and `[S2,T2]`, then their tensor product may be calculated via the GAP command `KroneckerProduct`, namely as `[KroneckerProduct(S1,S2),KroneckerProduct(T1,T2)]`.

2.2 Weil representation types

2.2.1 Type D

Let p be prime. If $p=2$ or $p=3$, let $\lambda\geq 2$; otherwise, let $\lambda\geq 1$. Then the Weil representation arising from the quadratic module with $M=\mathbb{Z}/p^\lambda\mathbb{Z}\oplus\mathbb{Z}/p^\lambda\mathbb{Z}$ and $Q(x,y)=\frac{xy}{p^\lambda}$ is said to be of type D and denoted $D(p,\lambda)$. Information on (M,Q) may be obtained via `SL2ModuleD` (5.1.1), and subrepresentations of $D(p,\lambda)$ with level p^λ may be constructed via `SL2IrrepD` (5.1.2).

Here we define

$$\mathfrak{A} \cong (\mathbb{Z}/p^\lambda\mathbb{Z})^\times$$

acting on M by $a(x,y) = (a^{-1}x, ay)$; see [NW76, Section 2.1]. This group has the following structure. When $p=2$, $\mathfrak{A} = \langle\alpha\rangle \times \langle\beta\rangle$ with $\alpha = 5$ (of order $2^{\lambda-2}$; notably, for $\lambda = 2$, α is trivial) and $\beta = -1$. Otherwise, $\mathfrak{A}$ is cyclic; in this case, we write $\mathfrak{A} = \langle\alpha\rangle \times \langle\beta\rangle$ with $\alpha = p+1$ (of order $p^{\lambda-1}$; for $\lambda = 1$, α is trivial) and β an arbitrarily chosen element of order $p+1$.

For type D, a non-trivial character of $\mathfrak{A}$ is primitive if and only if it is injective on the cyclic subgroup of $\mathfrak{A}$ of maximal p-power order, which is generated by α. As a particular case, when α is trivial (i.e. $p=2$ and $\lambda=2$ or p odd and $\lambda=1$), all non-trivial characters are primitive.

2.2.2 Type N

Let p be prime and $\lambda \geq 1$. If $p \neq 2$, choose an integer u so that $u \equiv 3 \bmod 4$ with $-u$ a quadratic non-residue mod p; if $p=2$, define $u=3$. Then the Weil representation arising from the quadratic module with $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^\lambda\mathbb{Z}$ and $Q(x,y) = \frac{x^2+xy+\frac{1+u}{4}y^2}{p^\lambda}$ is said to be of type N and denoted $N(p,\lambda)$. Information on (M,Q) may be obtained via `SL2ModuleN` (5.2.1), and subrepresentations of $N(p,\lambda)$ with level p^λ may be constructed via `SL2IrrepN` (5.2.2).

Here we define

$$\mathfrak{A} = \{\varepsilon \in M^\times \mid \mathrm{Nm}(\varepsilon) = 1\}$$

acting on M by multiplication; see [NW76, Section 2.2]. This group has the following structure. When $\lambda \geq 2$, it is generated by α and β: for $p=2$, $|\alpha| = 2^{\lambda-2}$ and $|\beta| = 6$, while for $p \neq 2$, $|\alpha| = p^{\lambda-1}$ and $|\beta| = p+1$. On the other hand, when $\lambda = 1$, $\mathfrak{A}$ is cyclic; in this case, we choose a generator β with $|\beta| = p+1$ and (for simplicity) say $\alpha = 1$.

For type N with $p=2$, $\lambda=2$, a non-trivial character of $\mathfrak{A}$ is primitive if and only if $\chi(-1) = -1$. For all other type N cases, a character of $\mathfrak{A}$ is primitive if and only if it is injective on the subgroup $\langle\alpha\rangle \leq \mathfrak{A}$. Note that, when $\lambda = 1$, $\alpha = 1$, so all non-trivial characters are primitive.

2.2.3 Type R

The structure of (M,Q) of type R depends upon three additional parameters: σ, r, and t. The relevant values thereof depend on whether $p=2$, as follows.

First, if p is an odd prime, let $\lambda \geq 2$, $\sigma \in \{1,\dots,\lambda\}$, and $r,t \in \{1,u\}$ with u a quadratic non-residue mod p. Then define $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^{\lambda-\sigma}\mathbb{Z}$ and $Q(x,y) = \frac{r(x^2+p^\sigma t y^2)}{p^\lambda}$.

On the other hand, if $p=2$, let $\lambda \geq 2$, $\sigma \in \{0,\dots,\lambda-2\}$ and $r,t \in \{1,3,5,7\}$. Then define $M = \mathbb{Z}/2^{\lambda-1}\mathbb{Z} \oplus \mathbb{Z}/2^{\lambda-\sigma-1}\mathbb{Z}$ and $Q(x,y) = \frac{r(x^2+2^\sigma t y^2)}{2^\lambda}$.

In either case, the resulting representation is said to be of type R and denoted $R(p,\lambda,\sigma,r,t)$. Information on (M,Q) may be obtained via `SL2ModuleR` (5.3.1), and subrepresentations of $R(p,\lambda,\sigma,r,t)$ with level p^λ may be constructed via `SL2IrrepR` (5.3.2).

There are two special cases to consider. First, if $\sigma = \lambda$ for $p \neq 2$, then the second factor of M is trivial (and hence t is irrelevant); this special case is handled by `SL2IrrepRUnary` (5.3.3) (which is called by `SL2IrrepR` (5.3.2) when appropriate).

Second, if $\sigma = \lambda - 2$ for $p=2$, $\lambda \geq 2$, then the second factor of M is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and collapses in $2M$; we call this case the *extremal* family. Here, $\mathrm{Aut}(M,Q)$ is itself abelian, so we let $\mathfrak{A} = \mathrm{Aut}(M,Q)$. This group has the following structure:

- For $\lambda = 2$, $t = 1$, $\mathfrak{A} = \langle \tau \rangle$ where $\tau : (x,y) \mapsto (y,x)$; a character χ is primitive if $\chi(\tau) = -1$.
- For $\lambda = 2$, $t = 3$, $\mathfrak{A}$ is trivial; there are no primitive characters.
- For $\lambda = 3$ or 4, $\mathfrak{A} = \{\pm 1\}$; there are no primitive characters.
- Finally, for $\lambda \geq 5$, $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with $\alpha \neq -1$ of order 2, and a character $\chi \in \widehat{\mathfrak{A}}$ is primitive if $\chi(\alpha) = -1$. Note that, for this special case, the usual test for primitivity (described in Section 2.1) fails, as there are no elements of $\mathfrak{A}$ fixing $2M$ pointwise.

Outside of the above two special cases, we define

$$\mathfrak{A} = \{\varepsilon \in M^\times \mid \mathrm{Nm}(\varepsilon) = 1\}$$

acting on M by multiplication; see [NW76, Section 2.3 - 2.4]. The structure of $\mathfrak{A}$ is as follows, and a character is primitive if it is injective on $\langle \omega \rangle \leq \mathfrak{A}$, with $\omega = \alpha$ except when specified:

- Suppose $p \neq 2$, $\lambda \geq 2$, $\sigma \in \{1, \dots, \lambda - 1\}$. As a special case, if $p = 3$, $\lambda \geq 3$, and $\sigma = t = 1$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle \beta \rangle$ with α of order $3^{\lambda-2}$ and β of order 6. Otherwise, $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with α of order $p^{\lambda-\sigma}$.
- Suppose $p = 2$, $\lambda \geq 3$, $\sigma \in \{0, \dots, \lambda - 3\}$. Then the structure of $(M, Q))$ depends on r, t, but only up to an equivalence class described in [Nob76, Satz 4]; we assume without loss of generality that r, t are minimal positive representatives thereunder. Then:
  - If $\sigma = 0$, $r \in \{1, 3\}$, $t = 1$, and $\lambda = 3$, then $\mathfrak{A} = \langle \beta \rangle$ with $\beta = (0, 1)$ of order 4. Here $\omega = -1 = \beta^2$.
  - If $\sigma = 0$, $r \in \{1, 3\}$, $t = 1$, and $\lambda \geq 4$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle \beta \rangle$ with $\alpha = (1 \mod 4, 4)$ of order $2^{\lambda-3}$ and $\beta = (0, 1)$ of order 4.
  - If $\sigma = 0$, $r \in \{1, 3\}$, $t = 5$, and $\lambda = 3$, then $\mathfrak{A} = \langle \beta \rangle$ with $\beta = (2, 1)$ of order 4. Here $\omega = -1$.
  - If $\sigma = 0$, $r \in \{1, 3\}$, $t = 5$, and $\lambda \geq 4$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with $\alpha = (2, 3 \mod 4)$ of order $2^{\lambda-2}$. Here $\omega = -\alpha^2$ when $\lambda = 4$ and $\omega = \alpha$ otherwise.
  - If $\sigma = 0$, $r = 1$, $t \in \{3, 7\}$, and $\lambda = 3$, then $\mathfrak{A} = \langle -1 \rangle$. Here $\omega = -1$.
  - If $\sigma = 0$, $r = 1$, $t \in \{3, 7\}$, and $\lambda \geq 4$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with $\alpha = (1 \mod 4, 4)$ of order $2^{\lambda-3}$.
  - If $\sigma = 1$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with $\alpha = (1 \mod 4, 2)$ of order $2^{\lambda-3}$.
  - If $\sigma = 2$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with $\alpha = (1 \mod 4, 2)$ of order $2^{\lambda-4}$.
  - If $\sigma \geq 3$, then $\mathfrak{A} = \langle \alpha \rangle \times \langle -1 \rangle$ with $\alpha = (1 \mod 4, 1)$ of order $2^{\lambda-\sigma-1}$.

Chapter 3

Lists of representations

3.1 Lists by degree

3.1.1 SL2PrimePowerIrrepsOfDegree

▷ `SL2PrimePowerIrrepsOfDegree(degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are exactly the given degree and have prime power level.

3.1.2 SL2PrimePowerIrrepsOfDegreeAtMost

▷ `SL2PrimePowerIrrepsOfDegreeAtMost(max_degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are at most the given degree and have prime power level.

3.1.3 SL2IrrepsOfDegree

▷ `SL2IrrepsOfDegree(degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are exactly the given degree.

3.1.4 SL2IrrepsOfDegreeAtMost

▷ `SL2IrrepsOfDegreeAtMost(degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are at most the given degree.

3.2 Lists by level

3.2.1 SL2PrimePowerIrrepsOfLevel

▷ `SL2PrimePowerIrrepsOfLevel(p, lambda)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ with level exactly p^{λ}.

3.3 Lists of exceptional representations

3.3.1 SL2ExceptionalIrreps

▷ `SL2ExceptionalIrreps(arg)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of the 18 exceptional irreps of $\mathrm{SL}_2(\mathbb{Z})$.

Chapter 4

Methods for testing

4.1 Testing

4.1.1 SL2WithConjClasses

▷ `SL2WithConjClasses(p, ld)` (function)

Returns: the group $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ with conjugacy classes set to the format we use.

4.1.2 SL2ChiST

▷ `SL2ChiST(S, T, p, ld)` (function)

Returns: a list representing a character of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$

Converts the modular data (S,T), which must have level dividing p^λ, into a character of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$, presented in a form matching the conjugacy classes used in `SL2WithConjClasses`.

4.1.3 SL2IrrepPositionTest

▷ `SL2IrrepPositionTest(p, lambda)` (function)

Returns: a boolean

Constructs and tests all irreps of level dividing p^λ by checking their positions in `Irr(G)`.

Chapter 5

Irreducible representations of prime-power level

Methods for generating individual irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ for a given level p^λ.

In each case (except the unary type R, for which see `SL2IrrepRUnary` (5.3.3)), the underlying module M is of rank 2, so its elements have the form (x,y) and are thus represented by lists `[x,y]`.

Characters of the abelian group $\mathfrak{A} = \langle\alpha\rangle \times \langle\beta\rangle$, have the form $\chi_{i,j}$, given by \[\chi_{i,j}(\alpha^{v}\beta^{w}) \mapsto \mathbf{e}\left(\frac{vi}{|\alpha|}\right) \mathbf{e}\left(\frac{wj}{|\beta|}\right)~,\] where i and j are integers. We therefore represent each character by a list `[i,j]`. Note that in some cases α or β is trivial, and the corresponding index i or j is therefore irrelevant.

5.1 Representations of type D

See Section 2.2.1.

5.1.1 SL2ModuleD

▷ `SL2ModuleD(p, ld)` (function)

Returns: a record `rec(Agrp, Bp, Char, IsPrim)` describing (M,Q)

Constructs information about the underlying quadratic module (M,Q) of type D, for p a prime and $\lambda \geq 1$.

`Agrp` is a list describing the elements of $\mathfrak{A}$. Each element $a \in \mathfrak{A}$ is represented in `Agrp` by a list `[v, a, a_inv]`, where `v` is a list defined by $a = \alpha^{\mathrm{v}[1]}\beta^{\mathrm{v}[2]}$. Note that β is trivial, and hence `v[2]` is irrelevant, when $\mathfrak{A}$ is cyclic.

`Bp` is a list of representatives for the $\mathfrak{A}$-orbits on $M^\times$, which correspond to a basis for the $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$-invariant subspace associated to any primitive character $\chi \in \widehat{\mathfrak{A}}$ with $\chi^2 \not\equiv 1$. For other characters, we must use different bases which are particular to each case.

`Char(i,j)` converts two integers i, j to a function representing the character $\chi_{i,j} \in \widehat{\mathfrak{A}}$.

`IsPrim(chi)` tests whether the output of `Char(i,j)` represents a primitive character.

5.1.2 SL2IrrepD

▷ `SL2IrrepD(p, ld, chi_index)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of type D with level p^λ, for p a prime and $\lambda \geq 1$, corresponding to the character χ indexed by `chi_index = [i,j]` (see the discussion of `Char(i,j)` in `SL2ModuleD` (5.1.1)).

Depending on the parameters, $W(M,Q)$ will contain either 1 or 2 such irreps.

5.2 Representations of type N

See Section 2.2.2.

5.2.1 SL2ModuleN

▷ `SL2ModuleN(p, ld)` (function)

Returns: a record `rec(Agrp, Bp, Char, IsPrim, Nm, Prod)` describing (M,Q)

Constructs information about the underlying quadratic module (M,Q) of type N, for p a prime and $\lambda \geq 1$.

`Agrp` is a list describing the elements of $\mathfrak{A}$. Each element $a \in \mathfrak{A}$ is represented in `Agrp` by a list `[v, a]`, where `v` is a list defined by $a = \alpha^{\mathtt{v[1]}}\beta^{\mathtt{v[2]}}$. Note that α is trivial, and hence `v[1]` is irrelevant, when $\mathfrak{A}$ is cyclic.

`Bp` is a list of representatives for the $\mathfrak{A}$-orbits on $M^\times$, which correspond to a basis for the $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$-invariant subspace associated to any primitive character $\chi \in \widehat{\mathfrak{A}}$ with $\chi^2 \not\equiv 1$. For other characters, we must use different bases which are particular to each case.

`Char(i,j)` converts two integers i, j to a function representing the character $\chi_{i,j} \in \widehat{\mathfrak{A}}$.

`IsPrim(chi)` tests whether the output of `Char(i,j)` represents a primitive character.

`Nm(a)` and `Prod(a,b)` are the norm and product functions on M, respectively.

5.2.2 SL2IrrepN

▷ `SL2IrrepN(p, ld, chi_index)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of type N with level p^λ, for p a prime and $\lambda \geq 1$, corresponding to the character χ indexed by `chi_index = [i,j]` (see the discussion of `Char(i,j)` in `SL2ModuleN` (5.2.1)).

Depending on the parameters, $W(M,Q)$ will contain either 1 or 2 such irreps.

5.3 Representations of type R

See Section 2.2.3.

5.3.1 SL2ModuleR

▷ `SL2ModuleR(p, ld, sigma, r, t)` (function)

Returns: a record `rec(Agrp, Bp, Char, IsPrim, Nm, Ord, Prod, c, tM)` describing (M,Q)

Constructs information about the underlying quadratic module (M,Q) of type R, for p a prime. The additional parameters λ, σ, r, and t should be integers chosen as follows.

If p is an odd prime, let $\lambda \geq 2$, $\sigma \in \{1,\ldots,\lambda-1\}$, and $r,t \in \{1,u\}$ with u a quadratic non-residue mod p. Note that $\sigma = \lambda$ is a valid choice for type R, however, this gives the unary case, and so is not handled by this function, as it is decomposed in a different way; for this case, use `SL2IrrepRUnary` (5.3.3) instead.

If $p = 2$, let $\lambda \geq 2$, $\sigma \in \{0,\ldots,\lambda-2\}$ and $r,t \in \{1,3,5,7\}$.

`Agrp` is a list describing the elements of $\mathfrak{A}$. Each element a of $\mathfrak{A}$ is represented in `Agrp` by a list `[v, a]`, where `v` is a list defined by $a = \alpha^{\mathtt{v}[1]}\beta^{\mathtt{v}[2]}$.

`Bp` is a list of representatives for the $\mathfrak{A}$-orbits on $M^\times$, which correspond to a basis for the $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$-invariant subspace associated to any primitive character $\chi \in \widehat{\mathfrak{A}}$ with $\chi^2 \not\equiv 1$. For other characters, we must use different bases which are particular to each case.

`Char(i,j)` converts two integers i, j to a function representing the character $\chi_{i,j} \in \widehat{\mathfrak{A}}$.

`IsPrim(chi)` tests whether the output of `Char(i,j)` represents a primitive character.

`Nm(a)`, `Ord(a)`, and `Prod(a,b)` are the norm, order, and product functions on M, respectively.

`c` is a scalar used in calculating the S-matrix; namely $c = \frac{1}{|M|}\sum_{x\in M}\mathbf{e}(Q(x))$. Note that this is equal to $S_Q(-1)/\sqrt{|M|}$, where $S_Q(-1) = \frac{1}{\sqrt{|M|}}\sum_{x\in M}\mathbf{e}(Q(x))$ is also known as the central charge.

`tM` is a list describing the elements of the group $M - pM$.

5.3.2 SL2IrrepR

▷ `SL2IrrepR(p, ld, sigma, r, t, chi_index)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of type R with parameters p, λ, σ, r, and t, corresponding to the character χ indexed by `chi_index = [i,j]` (see the discussions of σ, r, t, and `Char(i,j)` in `SL2ModuleN` (5.2.1)).

Depending on the parameters, $W(M,Q)$ will contain either 1 or 2 such irreps.

If $\sigma = \lambda$ for $p \neq 2$, then the second factor of M is trivial (and hence t is irrelevant), so this falls through to `SL2IrrepRUnary` (5.3.3).

5.3.3 SL2IrrepRUnary

▷ `SL2IrrepRUnary(p, ld, r)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of unary type R (that is, the special case where $\sigma = \lambda$) with p an odd prime, λ a positive integer, and $r \in \{1,u\}$ with u a quadratic non-residue mod p.

In this case, $W(M,Q)$ always contains exactly 2 such irreps.

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