SL2Reps

Constructs representations of SL2(Z).

0.1

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Chapter 1

Introduction

This package, SL2Reps, provides methods for constructing and testing matrix presentations of the representations of $\mathrm{SL}_2(\mathbb{Z})$.

Irreducible representations of prime-power level are constructed individually by means of Weil representations, and from these a list of all representations of a given degree or level may be produced. The format is designed to be useful in the study of modular tensor categories in particular.

1.1 Installation

To install SL2Reps, first download it from TODO, then place it in the pkg subdirectory of your GAP installation (or in the pkg subdirectory of any other GAP root directory, for example one added with the -l argument).

SL2Reps is then loaded with the GAP command

```
gap> LoadPackage( "SL2Reps" );
```

1.2 Usage

Specific irreducible representations may be constructed via the methods in Chapter 5, while lists of irreducible representations with a given degree or level may be constructed with those in Chapter 3.

This package uses an InfoClass, InfoSL2Reps. It may be set to 0 (silent), 1 (info), or 2 (verbose). To change it, use

```
gap> SetInfoLevel(InfoSL2Reps, k);
```

Chapter 2

Description

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\mathfrak{s}$ = `[[0,1],[-1,0]]` and $\mathfrak{t}$ = `[[1,1],[0,1]]` (which satisfy the relations $\mathfrak{s}^4 = (\mathfrak{s}\mathfrak{t})^3 = \mathrm{id}$). Thus, any complex representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n$ (where $n \in \mathbb{Z}^+$ is called the *degree* of ρ) is defined by the $n \times n$ matrices $S = \rho(\mathfrak{s})$ and $T = \rho(\mathfrak{t})$. In fact, any such representation factors through $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$ for some $\ell \in \mathbb{Z}^+$; the smallest such ℓ is called the *level* of ρ. We therefore present representations in the form of a record

```
rec(S, T, degree, level, name)
```

where the name follows the conventions of [NW76]. Note that our definition of $\mathfrak{s}$ follows that of [Nob76]; other authors prefer the inverse, i.e. `[[0,-1],[1,0]]`. When working with that convention, one must invert the S matrices output by this package.

Throughout, we denote by $\mathbf{e}$ the map $k \mapsto e^{2\pi i k}$ (an isomorphism from $\mathbb{Q}/\mathbb{Z}$ to the group of finite roots of unity in $\mathbb{C}$).

2.1 Construction

For any representation ρ of $\mathrm{SL}_2(\mathbb{Z})$, we may decompose ρ as a direct sum of irreducible representations of prime-power level using the Chinese remainder theorem. Therefore, to characterize all representations of $\mathrm{SL}_2(\mathbb{Z})$, it suffices to consider irreducible representations (irreps) of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$.

Such representations may be constructed using Weil representations as described in [Nob76, Section 1]. The Weil representations used by this package have the following form. Let p be a prime number and $\lambda \in \mathbb{Z}^+$. Choose a $\mathbb{Z}/p^\lambda\mathbb{Z}$-module M of rank 1 or 2, and a quadratic form Q on M, such that (M,Q) is of one of the three types described in Section 2.2. The *quadratic module* (M,Q) then gives rise to a representation of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ on the vector space $V = \mathbb{C}^M$ of complex-valued functions on M. This representation is denoted $W(M,Q)$.

With a finite number of exceptions, every representation of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ may be found as a subrepresentation of $W(M,Q)$ for an appropriate choice of (M,Q) [NW76, Hauptsatz 2] (the 18 exceptions may be found as the tensor product of two such subrepresentations; these may be generated with `SL2ExceptionalIrreps` (3.3.1)).

The subrepresentations in question are often of the form $W(M,Q,\chi)$, defined as follows. Let $\mathfrak{A}$ be an abelian subgroup of \[\operatorname{Aut}(M,Q) = \{ \varepsilon \in \operatorname{Aut}(M) \mid Q(\varepsilon x) = Q(x) \text{ for all } x \in M\}\] and $\chi \in \hat{\mathfrak{A}}$. Define \[V_\chi = \{f \in V \mid f(\varepsilon x) = \chi(\varepsilon) f(x) \text{ for all } x \in M \text{ and } \varepsilon \in \mathfrak{A}\}~,\] which is a $SL_2(\mathbb{Z}/p^\lambda\mathbb{Z})$-invariant subspace of V. We then denote by $W(M,Q,\chi)$ the subrepresentation of $W(M,Q)$ corresponding to V_χ.

In this context, we will frequently refer to a character χ as being *primitive*, which means that it can not be written in the form $\chi' \circ \pi$, where π is the canonical projection from $\mathfrak{A}$ to the corresponding automorphism group $\mathfrak{A}'$ of some quadratic module (pM, Q') with $Q'(px) = pQ(x)$ for all $x \in M$ and $\chi' \in \hat{\mathfrak{A}}'$.

All representations of $\mathrm{SL}_2(\mathbb{Z})$ may then be found as direct sums of these prime-power irreps.

2.2 Weil representation types

2.2.1 Type D

Let p be prime and $\lambda \geq 1$. Then the Weil representation arising from the quadratic module with $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^\lambda\mathbb{Z}$ and $Q(x,y) = \frac{xy}{p^\lambda}$ is said to be of type D and denoted $D(p,\lambda)$. Information on (M,Q) may be obtained via `SL2ModuleD` (5.1.1), and subrepresentations of $D(p,\lambda)$ with level p^λ may be constructed via `SL2IrrepD` (5.1.2).

2.2.2 Type N

Let p be prime and $\lambda \geq 1$. Then the Weil representation arising from the quadratic module with $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^\lambda\mathbb{Z}$ and $Q(x,y) = \frac{x^2+xy+\frac{1+u}{4}y^2}{p^\lambda}$ (where, for $p \neq 2$, u is chosen so that $u \equiv 3 \bmod 4$ with $\left(\frac{-u}{p}\right) = -1$, and for $p = 2$, $u = 3$) is said to be of type N and denoted $N(p,\lambda)$. Information on (M,Q) may be obtained via `SL2ModuleN` (5.2.1), and subrepresentations of $D(p,\lambda)$ with level p^λ may be constructed via `SL2IrrepN` (5.2.2).

2.2.3 Type R

The construction of type R varies depending on whether $p = 2$.

First, if p is an odd prime, let $\lambda \geq 2$, $\sigma \in \{1,\dots,\lambda\}$, and $r,t \in \{1,u\}$ with u a quadratic non-residue mod p. Then define $M = \mathbb{Z}/p^\lambda\mathbb{Z} \oplus \mathbb{Z}/p^{\lambda-\sigma}\mathbb{Z}$ and $Q(x,y) = \frac{r(x^2+p^\sigma ty^2)}{p^\lambda}$.

On the other hand, if $p = 2$, let $\lambda \geq 2$, $\sigma \in \{0,\dots,\lambda-2\}$ and $r,t \in \{1,3,5,7\}$. Then define $M = \mathbb{Z}/2^{\lambda-1}\mathbb{Z} \oplus \mathbb{Z}/2^{\lambda-\sigma-1}\mathbb{Z}$ and $Q(x,y) = \frac{r(x^2+2^\sigma ty^2)}{2^\lambda}$.

In either case, the resulting representation is said to be of type R and denoted $R(p,\lambda,\sigma,r,t)$. Information on (M,Q) may be obtained via `SL2ModuleR` (5.3.1), and subrepresentations of $R(p,\lambda,\sigma,r,t)$ with level p^λ may be constructed via `SL2IrrepR` (5.3.2). Note that if $\sigma = \lambda$ for $p \neq 2$, then the second factor of M is trivial (and hence t is irrelevant); this special case is handled by `SL2IrrepRUnary` (5.3.3).

Chapter 3

Lists of representations

3.1 Lists by degree

3.1.1 SL2PrimePowerIrrepsOfDegree

▷ `SL2PrimePowerIrrepsOfDegree(degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are exactly the given degree and have prime power level.

3.1.2 SL2PrimePowerIrrepsOfDegreeAtMost

▷ `SL2PrimePowerIrrepsOfDegreeAtMost(max_degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are at most the given degree and have prime power level.

3.1.3 SL2IrrepsOfDegree

▷ `SL2IrrepsOfDegree(degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are exactly the given degree.

3.1.4 SL2IrrepsOfDegreeAtMost

▷ `SL2IrrepsOfDegreeAtMost(degree)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ that are at most the given degree.

3.2 Lists by level

3.2.1 SL2PrimePowerIrrepsOfLevel

▷ `SL2PrimePowerIrrepsOfLevel(p, lambda)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of all irreps of $\mathrm{SL}_2(\mathbb{Z})$ with level exactly p^{λ}.

3.3 Lists of exceptional representations

3.3.1 SL2ExceptionalIrreps

▷ `SL2ExceptionalIrreps(arg)` (function)

Returns: a list of records of the form `rec(S, T, degree, level, name)`

Constructs a list of the 18 exceptional irreps of $\mathrm{SL}_2(\mathbb{Z})$.

Chapter 4

Methods for testing

4.1 Testing

4.1.1 SL2WithConjClasses

▷ `SL2WithConjClasses(p, ld)` (function)

Returns: the group $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ with conjugacy classes set to the format we use.

4.1.2 SL2ChiST

▷ `SL2ChiST(S, T, p, ld)` (function)

Returns: a list representing a character of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$

Converts the modular data (S,T), which must have level dividing p^λ, into a character of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$, presented in a form matching the conjugacy classes used in `SL2WithConjClasses`.

4.1.3 SL2IrrepPositionTest

▷ `SL2IrrepPositionTest(p, lambda)` (function)

Returns: a boolean

Constructs and tests all irreps of level dividing p^λ by checking their positions in `Irr(G)`.

Chapter 5

Irreducible representations of prime-power level

Methods for generating individual irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^{\lambda}\mathbb{Z})$ for a given level p^{λ}.

In each case (except the unary type R, for which see `SL2IrrepRUnary` (5.3.3)), the underlying module M is of rank 2, so its elements have the form (x,y) and are thus represented by lists $[x,y]$.

5.1 Representations of type D

See 2.2.1.

5.1.1 SL2ModuleD

▷ `SL2ModuleD(p, ld)` (function)

Returns: a record `rec(Agrp, Bp, Char, IsPrim)` describing (M,Q)

Constructs information about the underlying quadratic module (M,Q) of type D.

`Agrp` is a list describing the elements of $\mathfrak{A} = (\mathbb{Z}/p^{\lambda}\mathbb{Z})^{\times}$ (see [NW76, Section 2.1]). The group $\mathfrak{A}$ has the following form. When $p=2$ and $\lambda \geq 3$, it is generated by $\alpha = -1$ and $\zeta = 5$. Otherwise, it is cyclic; in this case, we choose a generator α and (for simplicity) say $\zeta = 1$. Each element a of $\mathfrak{A}$ is represented in `Agrp` by a list `[v, a, a_inv]`, where `v` is a list defined by $a = \alpha^{\mathrm{v}[1]}\zeta^{\mathrm{v}[2]}$.

`Bp` is a list of representatives for the $\mathfrak{A}$-orbits on $M^{\times}$, which correspond to a basis the $\mathrm{SL}_2(\mathbb{Z}/p^{\lambda}\mathbb{Z})$-invariant subspace associated to any primitive character $\chi \in \hat{\mathfrak{A}}$ with $\chi^2 \not\equiv 1$. For other characters, we must use different bases which are particular to each case.

`Char(i,j)` converts two integers i, j to a function representing a character of $\mathfrak{A}$. Each character in $\hat{\mathfrak{A}}$ is of the form $\chi_{i,j}$, given by \[\chi_{i,j}(\alpha^{v}\zeta^{w}) \mapsto \mathbf{e}\left(\frac{vi}{|\alpha|}\right) \mathbf{e}\left(\frac{wj}{|\zeta|}\right)~.\] Note that j is irrelevant in the cases where $\mathfrak{A}$ is cyclic.

`IsPrim(chi)` tests whether the output of `Char(i,j)` represents a primitive character. A character is primitive if it is injective on $\langle \mathrm{o} \rangle \leq \mathfrak{A}$, where o is defined as follows. If $p=2$ and $\lambda \geq 3$, $\mathrm{o} = 5$. If $\lambda = 1$, $\mathrm{o} = \alpha$. In all other cases, $\mathrm{o} = 1+p$.

5.1.2 SL2IrrepD

▷ `SL2IrrepD(p, ld, chi_index)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of type D with level p^λ corresponding to the character χ indexed by `chi_index = [i,j]` (see the discussion of `Char(i,j)` in `SL2ModuleD` (5.1.1)).

Depending on the parameters, $W(M,Q)$ will contain either 1 or 2 such irreps.

5.2 Representations of type N

See 2.2.2.

5.2.1 SL2ModuleN

▷ `SL2ModuleN(p, ld)` (function)

Returns: a record `rec(Agrp, Bp, Char, IsPrim, Nm, Prod)` describing (M,Q)

Constructs information about the underlying quadratic module (M,Q) of type N.

`Agrp` is a list describing the elements of $\mathfrak{A} = \{\varepsilon \in M^\times \mid \mathrm{Nm}(\varepsilon) = 1\}$ (see [NW76, Section 2.2]). The group $\mathfrak{A}$ has the following form. When $\lambda \geq 2$, it is generated by α and ζ: for $p = 2$, $|\alpha| = 2^{\lambda-2}$ and $|\zeta| = 6$, while for $p \neq 2$, $|\alpha| = p^{\lambda-1}$ and $|\zeta| = p+1$. On the other hand, when $\lambda = 1$, $\mathfrak{A}$ is cyclic; in this case, we choose a generator ζ with $|\zeta| = p+1$ and (for simplicity) say $\alpha = 1$. Each element a of $\mathfrak{A}$ is represented in `Agrp` by a list `[v, a]`, where `v` is a list defined by $a = \alpha^{\mathtt{v}[1]}\zeta^{\mathtt{v}[2]}$.

`Bp` is a list of representatives for the $\mathfrak{A}$-orbits on $M^\times$, which correspond to a basis the $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$-invariant subspace associated to any primitive character $\chi \in \hat{\mathfrak{A}}$ with $\chi^2 \not\equiv 1$. For other characters, we must use different bases which are particular to each case.

`Char(i,j)` converts two integers i, j to a function representing a character of $\mathfrak{A}$. Each character in $\hat{\mathfrak{A}}$ is of the form $\chi_{i,j}$, given by \[\chi_{i,j}(\alpha^{v}\zeta^{w}) \mapsto \mathbf{e}\left(\frac{vi}{|\alpha|}\right) \mathbf{e}\left(\frac{wj}{|\zeta|}\right)~.\] Note that i is irrelevant in the cases where $\mathfrak{A}$ is cyclic.

`IsPrim(chi)` tests whether the output of `Char(i,j)` represents a primitive character. As a special case, for $p = 2$, $\lambda = 2$, a character is primitive if $\chi(-1) = -1$. For all other cases, a character is primitive if it is injective on $\langle \mathrm{o} \rangle \leq \mathfrak{A}$, where o is defined as follows. If $\lambda = 1$, $\mathrm{o} = \zeta$; otherwise $\mathrm{o} = \alpha$.

`Nm(a)` and `Prod(a,b)` are the norm and product functions on M, respectively.

5.2.2 SL2IrrepN

▷ `SL2IrrepN(p, ld, chi_index)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of type N with level p^λ corresponding to the character χ indexed by `chi_index = [i,j]` (see the discussion of `Char(i,j)` in `SL2ModuleN` (5.2.1)).

Depending on the parameters, $W(M,Q)$ will contain either 1 or 2 such irreps.

5.3 Representations of type R

See 2.2.3.

5.3.1 SL2ModuleR

▷ `SL2ModuleR(p, ld, sigma, r, t)` (function)

Returns: a record `rec(Agrp, Char, IsPrim, Nm, Ord, Prod, c, tM)` describing (M,Q)

Constructs information about the underlying quadratic module (M,Q) of type R.

`Agrp` describes the elements of $\mathfrak{A} = \{\varepsilon \in M^\times \mid \mathrm{Nm}(\varepsilon) = 1\}$ (see [NW76, Section 2.3 - 2.5]).

Representatives for the $\mathfrak{A}$-orbits on $M^\times$ can depend on the choice of character, even for primitive characters χ with $\chi^2 \not\equiv 1$. Thus, we cannot provide them here, and they are instead calculated by `SL2IrrepR` (5.3.2).

`Char(i,j)` converts the `chi_index` used in `SL2IrrepR` (5.3.2) to a function.

`IsPrim(chi)` tests whether a given character (e.g. from `Char`) is primitive.

`Nm(a)`, `Ord(a)`, and `Prod(a,b)` are the norm, order, and product functions on M, respectively.

`c` is a scalar used in calculating the S-matrix; namely $c = \frac{1}{|M|}\sum_{x\in M} \mathbf{e}(Q(x))$.

`tM` is the group $M - pM$.

5.3.2 SL2IrrepR

▷ `SL2IrrepR(p, ld, sigma, r, t, chi_index)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of type R with level p^λ corresponding to the character χ indexed by `chi_index`.

When $\sigma = \lambda$, this falls through to `SL2IrrepRUnary` (5.3.3).

5.3.3 SL2IrrepRUnary

▷ `SL2IrrepRUnary(p, ld, r)` (function)

Returns: a list of lists of the form $[S,T]$

Constructs the modular data for the irreducible representation(s) of unary type R (that is, with $\sigma = \lambda$) with level p^λ.

References

[Nob76] Alexandre Nobs. Die irreduziblen Darstellungen der Gruppen $SL_2(Z_p)$, insbesondere $SL_2(Z_2)$. I. *Comment. Math. Helv.*, 51(4):465–489, 1976. 4

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