Linear Algebra Notes Compiled

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1. (a) The function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = arctan(x)$ is injective but not surjective. $arctan(x)$ is a montonoic function that is always increasing, because of thus given two arbitrary elements in source, they will map to two different values in the target. However note, that $arctan(x)$ is not surjective, as given a point in the target, there does not always exists a point in the source which maps to it. For example, let 5 be a value in the codomain. There does not exists a value in the domain that maps to this, as $\lim_{x\to\infty} arctan(x) = \frac{\pi}{2}$. This function is not invertible as by the first theorem of the joy of sets, for a function to be invertible it must be a bijection which means it is both surjective and injective, function f is only injective.

   (b) The function g is both injective and surjective. Note from above that $arctan(x)$ is a montonically increasing functions meaning that it is always increasing and thus there will never be two values at the same height. The function g is also surjective as given a value y in the codomain, we can find a value x where $f(x)$ then maps to y. While this did not hold for all real numbers, it does hold for the codomain $(\frac{-\pi}{2}, \frac{\pi}{2})$, as $\lim_{x\to-\infty} arctan(x) = \frac{-\pi}{2}$ and $\lim_{x\to\infty} arctan(x) = \frac{\pi}{2}$. This function is invertible as by the first theorem of the joy of sets, for a function to be invertible it must be a bijection which means it is both surjective and injective, function g is both.

   (c) The function h is neither surjective or injective. For it to be injective two values within the source could not map to the same target value. However $-2, -1$ both map to 0. Therefore this function cannot be injective. For h to be surjective, given an element in the target we should be able to find a value of x where $h(x)$ maps to the element in the target. However, assume we want to find an x value that maps to -2 in the target, the function h cannot result in a negative value, given a negative x, $h(x) = 0$, as there are no ways to get a negative value, there is no x value in the source that maps to -2 in the target, and thus this function is not surjective. This function is not invertible as by the first theorem of the joy of sets, for a function to be invertible it must be a bijection which means it is both surjective and injective, function f h is neither.

(d) The function q is injective but not surjective. The function is monotonically decreasing, with $\lim_{x\to 1} q(x) = \infty$ and $\lim_{x\to\infty} q(x) = 0$. As this function is monotonically decreasing any value of x maps to a unique value in the target space. Therefore, this is injective. This function is not surjective as the target space is all real numbers, but given a value in the source we cannot get a negative value. This can be validated by taking the limits to find that this function maps between $(0, \infty)$.By the first theorem of the joy of sets, for a function to be invertible it must be a bijection, however function q is only injective.

(e) The function F is both injective and a surjective. Given a value in $\mathbb{R}^{\nvDash}$ this will map to a unique value in $\mathbb{C}$. The function is also surjective as given a value $y \in \mathbb{C}$ there exists a value $(a, b) \in \mathbb{R}^{\nvDash}$ where $F(a, b) = y$. Note, that F can map to any value in C, as there are no limitations on what it can map to, it is surjective. By the first theorem in More joy of sets, as the function F is a bijection we know it then must be invertible.

(f) The function # is surjective but not injective. # maps from the powerset of a natural number to the cardinality of the set. Suppose we have $\#(\{1, 2, 3\})$ and $\#(\{3, 4, 5\})$, # would then map to 3 for both these sets as they both have a cardinality of 3. # can be shown to be surjective meaning that $\forall y \in \{0 \cup \mathbb{N} \cup \infty\}$ there exists some x in the source where $\#(x) = y$. Let the empty set map to 0, for all $x \in \mathbb{N}$ let the set contain all natural numbers up to and including x. Thus the cardinality of such x elements is x. Lastly, as the sum or product of $\mathbb{N}$ maps to ∞ we can conclude that for all y in the target there exists some value x in the source that maps to it. Therefore this is surjective. As this is surjective and not injective therefore by the first theorem of More joy of sets is not inveratable as it is not a bijection.

2. (a) Suppose that f is a linear transformation therefore by the definition of a linear transformation f must preserve both vector addition and scalar multiplication. Let $\vec{0} \in \mathbb{R}^m$. Therefore as a linear function preserves vector addition $f(\vec{0}+\vec{0}) = f(\vec{0})+f(\vec{0})$ As $\vec{0}+\vec{0} = \vec{0}$, it follows that $f(\vec{0}) = f(\vec{0}) + f(\vec{0})$. Subtracting $f(\vec{0})$ from both sides we find that $f\vec{0} = \vec{0}_{\mathbb{R}^n}$ and since $f(\vec{0}) \in \mathbb{R}^n$ we know that $f(\vec{0}) - f(\vec{0}) = \vec{0}_{\mathbb{R}^n}$.

(b) Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. Let $c, d \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$ and $f = T(\vec{x})$. As f is a linear transformation that preserves scalar multiplication then $c(T\vec{x}) = T(c\vec{x})$. It follows that $cT(\vec{x}) = T(\vec{x})_1 + T(\vec{x})_2 + T(\vec{x})_3 + ... + T(\vec{x})_c$. As $T(\vec{x})$ is just a linear transformation it follows that c, linear transformations preserve vector addition and thus $cT(\vec{x}) = T(\vec{x})_1 + T(\vec{x})_2 + T(\vec{x})_3 + ... + T(\vec{x})_c = T(\vec{x}_1 + \vec{x}_2 + ... + \vec{x}_c)$. Summing these identical vectors we find that $T(c\vec{x})$ Therefore as $cT(\vec{x})$ preserves both vector addition and scalar multiplication we can conclude that $cT(\vec{x})$ is a linear transformation itself.

(c) Suppose the function f and g are both linear transformation. Let $c \in \mathbb{R}$ Therefore $cf + cg = cf(\vec{x}) + cg(\vec{x})$. As f and g both linear transformations we know that they preserve vector addition such that $cf(\vec{x}) + cg(\vec{x}) = f + g(\vec{x})$ Further as they are both linear transformation we know they preserve scalar multiplication and thus $cf(\vec{x}) = f(c\vec{x})$ and $cg(\vec{x}) = g(c\vec{x})$. Again as f and g are linear transformations and therefore by the definition of linear transformation, preserve vector addition we know that we can write this as $f + g(c\vec{x})$. As $f + g$ preserves both vector addition and scalar multiplication we know that it must be a linear transformation.

(d) Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are linear transformations. Let $\vec{x}, \vec{x_2} \in \mathbb{R}^m, c \in \mathbb{R}$. Suppose then we have $g(f(\vec{x}+f(\vec{x_2})$. As f is a linear function and preserves vector addition we can rewrite this as $g(f(\vec{x} + \vec{x_2}))$. Likewise as g is a linear transformation we know it also preserves vector addition and thus $g(f(\vec{x} + \vec{x_2})) = g(f(\vec{x})) + g(f(\vec{x_2})$. Therefore we can conclude that $g \circ f$ preserves vector addition. Suppose we have $cg(f(\vec{x})$. As g is just a linear transformation we know that it preserves scalar multiplication and thus equal to $g(cf(\vec{x})$. As f is a linear transformation we know that it must preserve scalar multiplication and thus $g(f(c\vec{x}))$. Therfore $g \circ f$ preserves scalar multiplication. As $g \circ f$ preserves both vector addition and scalar multiplication that it must be a linear transformation itself.

(e) *Proof.* Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, as f is a bijective function, we know that it must have an inverse such that $f(f^{-1}(\vec{x})) = \vec{x}$. Given a vector $\vec{x}, \vec{y} \in \mathbb{R}^n$, $f^{-1}(\vec{x}) = \vec{v}$ where $\vec{v}$ is a unique vector in $\mathbb{R}^n$, and $f^{-1}(\vec{y}) = \vec{w}$ where $\vec{w}$ is a unique vector in $\mathbb{R}^n$. Therefore $f^{-1}(\vec{x}) + f^{-1}(\vec{y}) = \vec{v} + \vec{w}$. Applying f to this equation we find $f(f^{-1}(\vec{x}) + f^{-1}(\vec{y})) = f(\vec{v} + \vec{w})$. And as f is linear and thus preserves vector addition we can rewrite this as $f(f^{-1}(\vec{x})) + f(f^{-1}(\vec{y})) = f(\vec{v} + \vec{w})$. Thus by the definition of inverse we can rewrite this as $\vec{x} + \vec{y} = f(\vec{v} + \vec{w})$. Applying f^{-1} to both sides we find that $f^{-1}(\vec{x}) + f^{-1}(\vec{y}) = f^{-1}(f(\vec{v} + \vec{w})) \to f^{-1}(\vec{x}) + f^{-1}(\vec{y}) = f^{-1}(x + y)$ as f is linear transformation such that $f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}) = \vec{x} + \vec{y}$ by the definition of an inverse. Therefore f^{-1} preserves vector addition.

Let C be an arbitrary element in $\mathbb{R}$.Note from above that $f^{-1}(\vec{x}) = \vec{v}$. Applying f to both sides of this equation we find we have $f(f^{-1}(\vec{x}) = f(\vec{v})$. As f is a linear transformation it preserves scalar multiplication such that $cf(f^{-1}(\vec{x}) = f(c\vec{v})$. By the definition of an inverse we can then rewrite this $c(\vec{x} = f(c\vec{v})$. Applying f^{-1} to both sides we find that $cf^{-1}(\vec{x}) = f^{-1}(f(c\vec{v}))$. By the definition of scalar multiplication we know that $f(c\vec{v}) = cf(\vec{v})$, therefore$cf^{-1}(\vec{x}) = f^{-1}(cf(\vec{v}))$. By the definition of inverse we know that $f(\vec{v}) = \vec{x}$. Thus we can rewrite this as $cf^{-1}(\vec{x}) = f^{-1}(c\vec{x})$, and thus f^{-1} preserves scalar multiplication. As f^{-1} preserves both vector addition and scalar multiplication it is therefore a linear transformation. $\square$

3. (a) *Proof.* Suppose that f is left invertible and f is not injective. Therefore there exists a function $g : Y \to X$ such that $g \circ f = id_x$, $fg = id_x$ Let there exist $x \in X$ such that $f(x) = y$, where $y \in Y$. Suppose then that $x \neq x'$ but $y = y'$ Therefore by the definition of what it means to be left invertible then $g(f(x)) = x$ and $g(f(x') = x'$, however this is not the case as this would mean g would map y and y' to two different values when $y = y'$. Therefore this contradicts the definition of a function which can not map one element to two values, and thus f must be injective.

   Suppose that f is injective. Let $g : Y \to X$ where $(y) = x$ where $y \in Y$ and $x \in X$. As f is injective we know that $x \neq x$ and $y \neq y'$, therefore it follows that such function g must also be injective. Therefore $g \circ f(x) = x$ as $f(x) = y$, $g(y) = x$ thus that $g(f(x) = x$ therefore $g \circ f(x) = x$ and Thus f must be left invertible.

   Therfore as if f is left invertible implies that f is injective, and if f is injective implies that f is left invertible. We can conclude that f is left invertible if and only if f is injective. □

   (b) *Proof.* Suppose that f is right invertible and f is not surjective.Therefore there exists $y \in Y$ such that $\forall x \in X f(x) \neq y$. Given the definition of right inverability we know that there exists a function $h : Y \to X$ such that $f \circ h = id_y$. However, while $h(y)$ will map to a unique value $x \in X$ we know infact that as f is not surjective that for all $x \in X$, $f(x) \neq y$ thus there cannot be a function f that maps to y. This then contradicts the fact that f is right invertable as this implies that there is no function such that $h \circ f \neq id_y$. Thus our intial supposition must be wrong. Therefore f must be surjective.

   Suppose that f is surjective. Therefore $\forall y \in Y$ such that $\exists x \in X$ such that $f(x) = y$. Therefore as f is surjective given $b \in Y$ we then know that there exists some element $a \in X$ such that $f(a) = b$ Suppose there exists a function $g : Y \to X$ such that $g(b = a$. Once again as f is surjective we know that $f(a) = b$ and it follows that $g(f(a)) = a$. Therefore we know that given f is surjective that f is right invertable.

   Therfore as if f is right invertible implies that f is surjective, and if f is surjective implies that f is right invertible. We can conclude that f is right invertible if and only if f is surjective. □

   (c) *Proof.* Suppose that f is invertible, then we know that there is a single function $F : Y \to X$ such that $F \circ f = id_x$ and $f \circ F = id_y$. Therefore there exists a function $h : Y \to X$ such that $f \circ h = id_y$ and f is right invertible. Since f is invertible we know that there is only such a single function thus we know that f is also left-invertible with h. Therefore as f is both right and left invertible we know that f is then surjective and injective and thus it must be bijective.

Suppose that f is a bijective. Therefore it is both injective and surjective. If f is surjective and injective we therefore know that f is right invertible, and left invertible. As f is left invertible we know by definition that there exists a function g such that $g \circ f = id_x$, and as f is right invertible we know by definition that there exists a function h such that $f \circ h = id_y$. . Suppose we have an element $y \in Y$ and $y \in X$ where $y_X = y_Y$. Therefore as f is right invertible we know that $g \circ f = y$ and as f is left invertible then $f \circ h = y$. Therefore we know that $g = h$. Thus it is a single function. Therefore there exists a single function, called $F : Y \to X$ such that $F \circ f = id_X$ and $fF = id_Y$. Therefore by definition f is invertible.

As f being bijective implies that f is invertible, and as f is invertible implies that f is bijective. Therefore we know that f is invertible if and only if f is bijective.

$\square$

4. (a) *Proof.* When $l = \{\begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix} | t \in \mathbb{R}\}$ Then $T(l) = T(\begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix})$. Therefore as liner transformations preserve vector addition and scalar multiplication we can rewrite this as $T(l) = T(\begin{bmatrix}1\\2\\3\end{bmatrix}) + tT(\begin{bmatrix}1\\1\\1\end{bmatrix})$. Therefore while $t = 0$, $T(l) = T(\begin{bmatrix}1\\2\\3\end{bmatrix})$ which is just $\begin{bmatrix}-1\\5\\2\end{bmatrix})$ by plugging this vector into $A\vec{x}$ where it is $\vec{x}$, to get the point shown above. When $t \neq 0$ we will get a line in the form of $T(\begin{bmatrix}1\\2\\3\end{bmatrix}) + tT(\begin{bmatrix}1\\1\\1\end{bmatrix})$. $\square$

(b) *Proof.* I claim that any line $l \in \mathbb{R}^3$ can be written as $l = \{\begin{bmatrix}b1\\b2\\b3\end{bmatrix} + t\begin{bmatrix}m1\\m2\\m3\end{bmatrix} | t \in \mathbb{R}\}$. Suppose that $x \in l$ then there exists t_1 such that $x = \vec{b} + t_1\vec{m}/$ Therefore $T(\vec{x}) = T(\vec{b} + t_1\vec{m})$. By the definition of linear transformations we can then write this as $T(\vec{b}) + T(t_1\vec{m})$. As linear transformations preserve scalar multiplication we can rewrite this as $T(\vec{b}) + t_1T(\vec{m})$. Therefore it is now clear to see that $T(\vec{b})$ and $t_1T(\vec{m})$ are both just fixed vectors. Therefore we know that $T(l) = \{T(\vec{b}) + tT(\vec{m}) | t \in \mathbb{R}\}$. Geometrically when $T(\vec{m}) \neq 0$ then $T(l)$ is a line. When $T(\vec{m}) = 0$, then $T(l)$ is just the point $T(\vec{b})$. $\square$

(c) *Proof.* I claim that any plane $p \in \mathbb{R}^3$ can be written as $p = \{\begin{bmatrix}b1\\b2\\b3\end{bmatrix} + t\begin{bmatrix}m1\\m2\\m3\end{bmatrix} + s\begin{bmatrix}n1\\n2\\n3\end{bmatrix} | t, s \in \mathbb{R}\}$. Suppose that $x \in P$, then there exists t_1, s_1 such that $x = \vec{b} + t_1\vec{m} + s_1\vec{n}$. Therefore $T(\vec{x}) = T(\vec{b} + t_1\vec{m} + s_1\vec{n}$. By the definition of linear transformation we can rewrite this as $T(\vec{b}) + T(t_1\vec{m}) + T(s_1\vec{n})$ as linear transformations preserve vector addition. As by definition of linear transformations, they preserve scalar multiplication, we can rewrite this further as $T(\vec{b}) + t_1T(\vec{m}) + s_1T(\vec{n})$. Now it clear to see that these are just fixed vectors. Therefore $T(p) = \{T(\vec{b}) + tT(\vec{m}) + sT(\vec{n}) | t, s \in \mathbb{R}\}$. Geometrically when $T(\vec{m}) \neq 0$ and $T(\vec{n}) \neq 0$ then $T(p)$ is a plane. When $T\vec{m} = 0$ then $T(p)$ is a line; $T(\vec{b}) + sT(\vec{n})$. Similarly, when $T(\vec{n}) = 0$, then $T(p)$ is just the line $T(\vec{b}) + tT(\vec{m})$. Lastly, when $T(\vec{m}) = 0$ and $T(\vec{n}) = 0$ then $T(p)$ is just a point; $T(\vec{b})$ $\square$

(d) *Proof.* Let $C = \{all(x_1, x_2) : (x_1 - c_1)^2 + (x_2 - c_2)^2 = r^2$ where $c_1, c_2 \in \mathbb{R}$. Therefore the geometric meaning of $T(C)$ cannot be conclude be unlike a plane or a line. The function representing a circle in $\mathbb{R}^2$ does not only rely on scalar multiplication and vector addition. For example what we $T(x^2)(T(x))^2$. Suppose we write a different formula for a circle making using of trigonometric functions. In such a case $T(cos(x)) \neq cos(T(x))$ and therefore does not preserve scalar multiplication. Therefore there are not clear geometric understanding of the shape of $T(C)$. $\square$

5. (a) *Proof.* Given A $= [a_{ij}] \in \mathbb{R}^{m\times n}, B = [b_{ij}] \in \mathbb{R}^{n\times p}$ we know by definition of matrix multiplication that $AB = \begin{bmatrix} | & & | \\ A\vec{b_1} & \cdots & A\vec{b_p} \\ | & & | \end{bmatrix}$ where $B = \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_p} \\ | & & | \end{bmatrix}$. Looking at the jth column of AB we see $A\vec{b_j}$. By the definition above we know that this is equal to $\sum_{k=1}^{n} b_k\vec{a_k}$.. This would compute an entire jth column vector for AB thus consdering only the ith row. we find it equal to $b_{1j}a_{i1} + b_{2j}a_{i2} + ... + b_{nj}a_{in}$ through basic expansion of the summation. We can then rewrite this as $\sum_{k=1}^{n} b_{kj}a_{ik}$, and as these are elements of the real numbers multiplication is commutative therefore we can write this as $\sum_{k=1}^{n} a_{ik}b_{kj}$. □

(b) *Proof.* Let $A = \begin{bmatrix} - & \vec{a_1} & - \\ & \vdots & \\ - & \vec{a_m} & - \end{bmatrix}$ and $B = \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_p} \\ | & & | \end{bmatrix}$ where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ where $n = n$. Therefore by the definition of matrix products given above we know that $AB = \begin{bmatrix} | & & | \\ A\vec{b_1} & \cdots & A\vec{b_p} \\ | & & | \end{bmatrix}$ where $AB \in \mathbb{R}^{m\times p}$. Using the properties of vector multiplication we can expand this out to see that

$AB = \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_{11}b_{1p} + ... + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + ..a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$. By the definition of transpose we know that a tranpose of a $m \times p$ matrix is $p \times m$ matrix made by rewriting the columns as rows and rows as columns. Thus $(AB)^T = \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_{m1}b_{11} + ..a_{mn}b_{n1}) \\ \vdots & \ddots & \vdots \\ (a_{11}b_{1p} + ... + a_{1n}b_{np} & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$..

By the same definition we can rewrite $A^T = \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_m} \\ | & & | \end{bmatrix}$ where $A^T \in \mathbb{R}^{n\times m}$ and $B^T = \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_n} \\ | & & | \end{bmatrix}$ where $B^T \in \mathbb{R}^{p\times n}$. Thus $B^TA^T = \begin{bmatrix} | & & | \\ B^T\vec{a_1} & \cdots & B^T\vec{a_m} \\ | & & | \end{bmatrix}$ Therefore expanding this using the same property of vector multiplication that we did above

$$\begin{bmatrix} | & & | \\ B^T\vec{a_1} & \cdots & B^T\vec{a_m} \\ | & & | \end{bmatrix} = \begin{bmatrix} (b_{11}a_{11} + ... + b_{n1}a_{1n}) & \cdots & (b_{1p}a_{11} + ... + b_{np}a_{1n} \\ \vdots & \ddots & \vdots \\ (b_{11}a_{m1} + ..b_{n1}a_{mn}) & \cdots & (b_{1p}a_{m1} + ... + b_{np}a_{mn}) \end{bmatrix}.$$

As each element of B^TA^T is computed through the multiplication and addition of real numbers, and as addition and multiplication of real numbers is commutative then it follows that

$$B^TA^T = \begin{bmatrix} (b_{11}a_{11} + ... + b_{n1}a_{1n}) & \cdots & (b_{1p}a_{11} + ... + b_{np}a_{1n} \\ \vdots & \ddots & \vdots \\ (b_{11}a_{m1} + ..b_{n1}a_{mn}) & \cdots & (b_{1p}a_{m1} + ... + b_{np}a_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_{11}b_{1p} + ... + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + ..a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$$

$= (AB)^T$.

Therefore as each element of $(AB)^T$ is equal to the elements of B^TA^T, $(AB)^T = B^TA^T$. □

(c) *Proof.* Let $A = \begin{bmatrix} - & \vec{a_1} & - \\ & \vdots & \\ - & \vec{a_m} & - \end{bmatrix}$ and $\begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_p} \\ | & & | \end{bmatrix}$ where $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ where $n = n$. Therefore by the definition of matrix products given above we know that $AB = \begin{bmatrix} | & & | \\ A\vec{b_1} & \cdots & A\vec{b_p} \\ | & & | \end{bmatrix}$. Expanding this out we see that $AB = \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_11b_{1p} + ... + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + ..a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$.

Now looking at $\vec{a_1}B$ of $\begin{bmatrix} - & \vec{a_1}B & - \\ & \vdots & \\ - & \vec{a_m}B & - \end{bmatrix}$ we can expand

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_p} \\ | & & | \end{bmatrix}$$

$= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + ..a_{1n}b_{n1}) & \cdots & (a_{11}b_{1p} + a_{12}b_{2p} + ..a_{1n}b_{np}) \end{bmatrix}$.Continuing this for row $\vec{a_2}$ through $\vec{a_m}$ we find that $\begin{bmatrix} - & \vec{a_1}B & - \\ & \vdots & \\ - & \vec{a_m}B & - \end{bmatrix}$

$= \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_11b_{1p} + ... + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + ..a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$. As this matrix has all the same elements as AB we can then say that $\begin{bmatrix} - & \vec{a_1}B & - \\ & \vdots & \\ - & \vec{a_m}B & - \end{bmatrix}$

$= \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_1 1b_{1p} + ... + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + ..a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$

$= \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_{n1}) & \cdots & (a_1 1b_{1p} + ... + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + ..a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + ... + a_{mn}b_{np}) \end{bmatrix}$

$= \begin{bmatrix} | & & | \\ A\vec{b_1} & \cdots & A\vec{b_p} \\ | & & | \end{bmatrix}$

$= AB.$ □

(d) *Proof.* From part c we know that $AB = \begin{bmatrix} - & \vec{a_1}B & - \\ & \vdots & \\ - & \vec{a_m}B & - \end{bmatrix}$ such that each element in the first row of the matrix AB can by the multiplication of a $1 \times p$ row vector by one such $n \times 1$ column vector of B. Therefore continuing this process for each element of AB, then $AB = \begin{bmatrix} (a_{11}b_{11} + ... + a_{1n}b_n 1) & \cdots & (a_{11}b_{1p} + \cdots + a_{1n}b_{np}) \\ \vdots & \ddots & \vdots \\ (a_{n1}b_{11} + ... + a_{pn}b_{n1}) & \cdots & (a_{n1}b_{1p} + .. + a_{nn}b_{np}) \end{bmatrix}$. Thus this is then equal to $AB = \begin{bmatrix} (a_{11}b_{11}) & \cdots & (a_{11}b_{1p}) \\ \vdots & \ddots & \vdots \\ (a_{n1}b_{11}) & \cdots & (a_{n1}b_{1p} \end{bmatrix} + ... + \begin{bmatrix} (a_{1n}b_n 1) & \cdots & (a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ (a_{pn}b_{n1}) & \cdots & (a_{nn}b_{np}) \end{bmatrix}$.

Note that $\begin{bmatrix} (a_{11}b_{11}) & \cdots & (a_{11}b_{1p}) \\ \vdots & \ddots & \vdots \\ (a_{n1}b_{11}) & \cdots & (a_{n1}b1p \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \end{bmatrix} + ... + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \begin{bmatrix} b_{n1} & \cdots & b_{np} \end{bmatrix} = a_1b_1 + ... + a_nb_n = \sum_{k=1}^{n} a_kb_k$.

Therefore $AB = \sum_{k=1}^{n} a_kb_k$ □

6. (a) *Proof.* We want to show that $\forall n \in \mathbb{N}$ $A^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n-1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$.

For the induction base $n = 1$, we have $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = A^1 =$

$\begin{bmatrix} 1 & 1 & \frac{1}{2}1(1-1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. For the inductive step, let $k \in \mathbb{N}$. Assume for the inductive step that $A^k = \begin{bmatrix} 1 & k & \frac{1}{2}k(k-1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$. Note that $A^k = AA^{k-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k-1 & \frac{1}{2}(k-1)(k-1-1) \\ 0 & 1 & k-1 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore it follows that $A^{k+1} = AA^k$. Thus as we suppose A^k is true we can say that $A^k = \begin{bmatrix} 1 & k & \frac{1}{2}k(k-1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$.

Then $A^{k+1} = AA^k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & \frac{1}{2}k(k-1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$. Computing the matrix product we find that

$A^{k+1} = \begin{bmatrix} (1*1+1*0+0*) & (1*k+1*1+0*0) & (1*\frac{1}{2}k(k-1)+(k*1)+0*1) \\ (0*1+1*0+0*1) & (0*k+1*1+1*0) & (0*\frac{1}{2}k(k-1)+1*k+1*1) \\ (0*1+0*0+1*0) & (0*k+0*1+1*0) & (0*\frac{1}{2}k(k-1)+0*k+1*1) \end{bmatrix}$

$= \begin{bmatrix} 1 & k+1 & \frac{1}{2}k^2+\frac{1}{2}k \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{bmatrix}$. Thus we find that

$\begin{bmatrix} 1 & k+1 & \frac{1}{2}k^2+\frac{1}{2}k \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{bmatrix} = A^{k+1} = \begin{bmatrix} 1 & k+1 & \frac{1}{2}(k+1)(k+1-1) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & k+1 & \frac{1}{2}k^2+\frac{1}{2}k \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{bmatrix}$. Thus $A^{k+1} = \begin{bmatrix} 1 & k+1 & \frac{1}{2}k^2+\frac{1}{2}k \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{bmatrix}$ and the inductive step holds.

Therefore as the base case and the inductive step is true. By induction we know then know for $\forall n \in \mathbb{N}$, $A^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n-1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$.

$\square$

7. (a) The composition $R_\alpha \circ R_\beta$ geometrically is first, the linear transformation of R_β where this is the rotation counter clockwise around the origin by β. This new location is then the starting point at for the linear transformation R_α which is the rotation about the origin by α. Therefore $R_\alpha R_\beta$ is rotating points in $\mathbb{R}^2$ to be first rotated by R_β where its new point is then rotated by R_α. This transformation in effect is rotating all possible points in $\mathbb{R}^2$ by the composition of the two transformations, in effect rotating the points of $\mathbb{R}^2$.

   (b) One way to conceptualize the composition of $R_\alpha \circ R_\beta$ is as a single transformation is to find one such matrix that models the change. As we have seen geometrically, this linear transformation first rotates a point counterclockwise by R_β followed by rotating the new point found from this transformation by R_α. Geometrically this is the same as rotating counterclockwise by $\beta+\alpha$. Therefore we can determine that the matrix of the composition would just be the sum of the two compositions. Note that the counterclockwise rotating about the some arbitrary θ is modeled by the matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ as stated in Theorem 2.23 from Linear Application with Applications, therefore the matrix of the composition of $R_\alpha \circ R_\beta$ can be $\begin{bmatrix} \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix}$

   (c) Another way to conceptualize this composition as a linear transformation and determine its matrix is to think of it as the product of two matrices, $\alpha\beta$ as defined in Theorem 2.14 in the Theory of Linear Algebra. Firstly let us define the matrices for α and β. As above a general linear transformation of a rotating by a given theta counterclockwise can be written as $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ as defined in Theorem 2.23 from Linear Application with Applications. Therefore we can say that $\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ and $\beta = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}$ Therefore $\alpha\beta = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}$. Computing the product of these two matrices we find that
$\alpha\beta = \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix}$.
Thus the matrix of the composition of $R_\alpha \circ R_\beta$ can be written as $\begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix}$.

   (d) As we have found in part b and c there are different ways to find the standard matrix of the composition of $R_\alpha \circ R_\beta$. While they make look different in fact they are same matrix. First let us rewrite the matrix from part c to align like elements. Therefore we have

$$\begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \end{bmatrix}.$$
At this point we can notice that these are in fact the trig identities for the sum of two angles for both sin and cos. As $\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \sin(\alpha+\beta)$ and $\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \cos(\alpha+\beta)$. Thus we can rewrite this matrix as $\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$. Therefore it is clear to see that $\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$ $= \begin{bmatrix} \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix}$ as the sum of two angles are commutative. Therefore we know that either way of deriving the standard matrix are equivalent.

8. (a) *Proof.* Let A, B be diagonal $n \times n$ matrices. Therefore by definition we can write $A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & \ddots & \\ & & & b_{nn} \end{bmatrix}$.
From the definitions give in Homework 2 for matrix products we can write $AB = \begin{bmatrix} | & & | \\ A\vec{b}_1 & \cdots & A\vec{b}_n \\ | & & | \end{bmatrix}$. Note that for an arbitrary $\vec{b}_i$ in B where $1 \leq i \leq n$ $A\vec{b}_i = a_{ii}b_{ii}$ as that is the only pair of elements where one is not a 0. Thus we know that AB only has elements being multiplied on its diagonals, such that

$$AB = \begin{bmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & \\ & & \ddots & \\ & & & a_{nn}b_{nn} \end{bmatrix}.$$

Using the same approach as above we find that $BA = \begin{bmatrix} | & & | \\ B\vec{a}_1 & \cdots & B\vec{a}_n \\ | & & | \end{bmatrix}$.
Similiar to above, for an arbitrary $\vec{a}_i$ in A where $1 \leq i \leq n$ $B\vec{a}_i = a_{ii}b_{ii}$. Therefore as above it follows that $BA = \begin{bmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & \\ & & \ddots & \\ & & & a_{nn}b_{nn} \end{bmatrix}$.

Thus as $\begin{bmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & \\ & & \ddots & \\ & & & a_{nn}b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & \\ & & \ddots & \\ & & & a_{nn}b_{nn} \end{bmatrix}$, we can conclude that $AB = BA$ when they are both diagonal matrices of the same length. $\square$

(b) False.
Counterexample: let A be a diagonal matrix 2×2 matrix, and let B be a non diagonal 2×2 where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Therefore computing the the matrix product we find that AB is $\begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$. On the contrary, computing the matrix product of BA we find that $BA = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$. Thus as $\begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.It follows that $AB \neq BA$.

9. (a) *Proof.* Let $c \in R$ and suppose $x, y \in C$ where $x = a + bi$ and $y = c + di$ Therefore we know that by the definition of the function γ that $\gamma(x+y) = \overline{x+y} = (a+c) - (b+d)i$. Likewise $\gamma(x) + \gamma(y) = \overline{x} + \overline{y} = a + bi - c + di$. Combining like terms we find $a + c - (b + d)i$. Thus $(a+c)-(b+d)i = (a+c)-(b+d)i$. Therefore $\gamma(x+y) = \gamma(x)+\gamma(y)$, and therefore gamma holds for vector addition.

Likewise, $c\gamma(x) = c\overline{x} = c(a - bi) = ac - bci$, and $\gamma(cx) = \overline{cx} = ac - bci$. Notice that this is the same the mapping from $c\gamma(x)$. Thus $c\gamma(x) = c\overline{x} = c\overline{x} = ac - bci = ac - bci = \gamma(cx)$. Therefore γ holds for scalar multiplication. As γ holds for both vector addition and scalar multiplication we can γ is a linear transformation.

Let $c + di = \overline{w}$ where $c, d \in \mathbb{R}$ and $\overline{w} \in \mathbb{C}$. Suppose that $w = c - di$ where $w \in \mathbb{C}$. Therefore $\gamma(w) = \overline{w}$, and thus it follows by the definition of injectivity that γ is injective as for all $\overline{w} \in \mathbb{C}$ there exists an $w \in C$ such that $\gamma(c)$.

Suppose $v = a + bi$ and $v' = a' + b'i$ where $v \neq v'$ meaning $a \neq a'$ or $b \neq b'$. Therefore $\gamma(v) = \overline{v} \overset{\dagger}{=} a - bi$ and $\gamma(v') = \overline{v'} \overset{(\dagger)}{=} a' - b'i$. Where $(\dagger)$ follows from the definition of conjugation given in the handout Complex Numbers. Therefore as we when $v \neq v'$ $a \neq a'$ or $b \neq b'$ in the case that $a = a'$, $-b \neq -b'$ and likewise in the case $b = b'$, $a \neq a'$. Therefore we can conclude that γ is injective as $v \neq v'$ such that $\gamma(v) \neq \gamma(v')$.

As γ is both surjective and injective we can therefore say that γ is bijective. Therefore as γ is bijective and a linear transformation we can therefore say that conjugation defined by γ is an isomorphism. The inverse of γ would then just be γ as when $x \in C$ where $x = a+bi$ and $a, b \in R$. Thus $\gamma(\gamma(a+bi)) = \gamma(a-bi) = \gamma(a+bi)$. Thus, γ has an inverse that is in fact is itself.

$\square$

(b) *Proof.* Let $c \in R$ and suppose $x, y \in C$ Therefore we know that by the definition of the function of μ that $\mu(x+y) = i(x+y)$ where i is an imaginary number. Distributing the i we then find that this is $ix + iy$. Then given $\mu(x) + \mu(y)$ we find this equal to $ix + iy$. Therefore as $\mu(x) + \mu(y) = ix + iy = ix + iy = \mu(x+y)$. Therefore μ preserves vector addition. Give $c\mu(x)$ we find that this is equal to $c(ix) = cix$. Similarly, we find that $\mu(cx) = i(cx) = cix = c\mu(x)$. Therefore it stands that μ preserves scalar multiplication. Thus, as μ preserves both vector addition as well as scalar multiplication we know that μ must be a linear transformation.

Let $x = ai - b$ where $x \in \mathbb{C}, a, b \in \mathbb{R}$. Therefore as there exists complex number denoted $a+bi$ that then given $\mu(a+bi)$ we find that this is equal to $i(a+bi) = ai + bi^2$. Recall from the the handout complex numbers that $i^2 = -1$. Thus $ai + bi^2 = ai - b$. Thus for some arbitrary complex number β in the target there exists some complex number α in the source such that $\mu(\alpha) = \beta$. Thus μ is surjective. Let $v = a + bi, w = a' + b'i$ define two complex numbers v, w where $a, b \in \mathbb{R}$ and $v \neq w$ such that either $a \neq a'$ or $b \neq b'$. It follows that $\mu(v) = iv = i(a+bi) = ai + bi^2$ and that $\mu(w) = iw = i(a+bi) = a'i + b'i^2$. Therefore as we know $v \neq w$ meaning $a \neq a'$ or $b \neq b'$, we can say that $\mu(w) \neq \mu(v)$ as in the case that $a = a'$ is the same for $v \neq w$ then $b \neq b'$. This follows according in the case that $b = b'$. Thus $w \neq v, \mu(w) \neq \mu(v)\ \forall v, w \in \mathbb{C}$. Therefore we can conclude that μ is injective. As μ is both injective and surjective we can conclude that it must be bijective. As we also know that μ is a linear transformation as it preserves vector addition and scalar multiplication, we can say μ is an isomorphism. The inverse of μ would then be $\mu^{-1} : C \to C, \mu^{-1}(v) = -1i$. Thus when $v = a + bi$ where $v \in C, a, b \in \mathbb{R}$. Then we can write that $\mu(\mu^{-1}(a+bi)) \to \mu(-i(a+bi) \to \mu(-ai+b) \to i(-ai+b) \to a+bi$. Thus μ^{-1} is the inverse of μ. $\square$

(c) *Proof.* let $c \in \mathbb{R}$ and $x, y \in \mathbb{C}$. Therefore we know that by the definition of the function f that $f(x+y) = \frac{1+i}{4}((1-i)(x+y)+(1+i)\bar{(}x+y))$. Therefore as we have shown that conjugation is a linear transformation in part a we know that $\overline{x+y} = \overline{x} + \overline{y}$. Thus we have $\frac{1+i}{4}((1-i)(x+y)+(1+i)(\overline{x}+\overline{y}))$, distributing this we have $f(x+y) = \frac{1+i}{4}((1-i)(x)+(1-i)(y)+(1+i)\overline{x}+(1+i)\overline{y})$. Given $f(x)+f(y)$ we find this equal to $\frac{1+i}{4}((1-i)(x)+(1+i)\overline{x})+\frac{1+i}{4}((1-i)(y)+(1+i)\overline{y})$. Factoring $\frac{1+i}{4}$ we see that $f(x+y) = \frac{1+i}{4}((1-i)(x)+(1-i)(y)+(1+i)\overline{x}+(1+i)\overline{y}) = \frac{1+i}{4}((1-i)(x)+(1-i)(y)+(1+i)\overline{x}+(1+i)\overline{y}) = f(x)+f(y)$. Therefore we can conclude that f must preserve vector addition.

Likewise for $f(cx)$ we can see that this is equal to $\frac{1+i}{4}((1-i)(xc) + (1+i)\overline{cx})$. Note that as $c \in \mathbb{R}$ that $\overline{c} = c$. Thus factoring out the we can rewrite this as $c\frac{1+i}{4}((1-i)(x) + (1+i)\overline{x})$. Similiarly for $c(f(x)$ we find that this this is equal to $c(\frac{1+i}{4}(1-i)(x)+(1+i)\overline{x})$. Therefore $c(f(x)) = c\frac{1+i}{4}((1-i)(x) + (1+i)\overline{x}) = c\frac{1+i}{4}((1-i)(x) + (1+i)\overline{x}) = f(cx)$.

Thus f preserves scalar multiplication. Therefore as f preserves scalar multiplication and vector addition we know that f must be a linear transformation. f is not bijective however, as f is not injective. For example let $z = 1+0i$ such that $f(z) = \frac{1+i}{4}((1-i)(1)+(1+i)(1)) = \frac{1+i}{4}(2) = \frac{1+i}{2}$. Similarly given $w = 0+i$ we see that $f(w) = \frac{1+i}{4}((1-i)(i) + (1+i)(i)) = \frac{1+i}{4}((1+i) + (1-i)) = \frac{1+i}{4}(2) = \frac{1+i}{2}$. Thus $f(v) = f(w)$ while $v = w$. Therefore f is not injective. As f is not a bijective linear transformation f can not an isomorphism. □

(d) Counterexample: Let $x, y \in \mathbb{C}$ where $x = 2+5i$ and $y = 1+4i$. Therefore $\eta(x) + \eta(y) = |x| + |y|$. From the handout, complex numbers, we know that the nonegative number $r = \sqrt{a^2+b^2}$ where $a, b \in \mathbb{R}$ is $|z|$. Therefore $|x| = \sqrt{2^2+5^2}$ and $|y| = \sqrt{1^2+4^2}$. Therefore we have $|x|+|y| = \sqrt{29}+\sqrt{17}$. For $\eta(x+y)$ we must first add these complex numbers together. Doing this we find that $x+y = 3+9i = c$, where $c \in \mathbb{C}$. Therefore $\eta(c) = |c|$. As above we find that $|c| = \sqrt{3^2+9^2}$ and therefore $|c| = \sqrt{90}$. As $\sqrt{90} \neq \sqrt{29} + \sqrt{17}$ we then know that $\eta(x+y) \neq \eta(x) + \eta(y)$ and thus that vector addition is not preserved. Therefore as vector addition is not preserved we know that η can not be a linear transformation. Similarly we know that η is not bijective as it not injective, consider $a, b \in \mathbb{C}$ where $a = 1+4i$ and $b = 4+1i$. Therefore $\eta(a) = |a| = \sqrt{1^2+4^2} = \sqrt{17}$, and $\eta(b) = |b| = \sqrt{(1^2+4^2} = \sqrt{17}$. Therefore as $b \neq a$ while $\eta(b) = \eta(a)$ we know that η is in fact not injective. As f is not a bijective linear transformation η can not an isomorphism.

(e) *Proof.* Let $a, b, c, d, z \in \mathbb{R}$. Therefore we know by the definition of T that $T(\begin{bmatrix} a \\ b \end{bmatrix})+T(\begin{bmatrix} c \\ d \end{bmatrix} = a+bi+c+di$. Factoring out the i we find that this is equal to $a+c+(b+d)i$ Likewise $T(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}) = T(\begin{bmatrix} a+c \\ b+d \end{bmatrix} = a+c+(b+d)i$. Therefore $T(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}) = a+c+(b+d)i = a+c+(b+d)i = T(\begin{bmatrix} c \\ d \end{bmatrix}$ Thus T preserves vector addition. $cT(\begin{bmatrix} c \\ d \end{bmatrix}) = z(a+bi)$. Distributing out the z we find that this is equal to $za+zbi$. Likewise, $T(z\begin{bmatrix} c \\ d \end{bmatrix}) = za+zbi$. Therefore $cT(\begin{bmatrix} c \\ d \end{bmatrix}) = za+zbi = za+zbi = T(z\begin{bmatrix} c \\ d \end{bmatrix})$. Therefore T must preserve scalar multiplication. Thus as T preserves both vector addition and scalar multiplication we can conclude that T must be a linear transformation.

Let $v \in \mathbb{C}$ where $v = x+yi$ where $x, y \in R$. Therefore $T(\begin{bmatrix} x \\ y \end{bmatrix})$ will result in $x+yi$. Thus for all elements in $\mathbb{C}$ there exists some mapping to it in $\mathbb{R}^2$. Thus T is surjective.
Now, let $x'y' \in R$ and the column vectors in $\mathbb{R}^{\nvDash}$, $\begin{bmatrix} x' \\ y' \end{bmatrix} \neq \begin{bmatrix} x \\ y \end{bmatrix}$ such that $x \neq x'$ or $y \neq y'$ Therefore $T(\begin{bmatrix} x \\ y \end{bmatrix}) = x+yi$ and $T(\begin{bmatrix} x' \\ y' \end{bmatrix}) = x'+y'i$. Given if $\begin{bmatrix} x' \\ y' \end{bmatrix} \neq \begin{bmatrix} x \\ y \end{bmatrix}$ if $x = x'$ then $y \neq y'$ and thus $x+yi \neq x'+y'i$. In the case that $y = y'$ then $x \neq x'$ and thus again $x+yi \neq x'+y'i$. Therefore given that two column vectors are not equal in the source, then T will map them to two elements that are not equal in the target space. Thus T is injective. As T is injective and surjective it follows by the definition of bijectivity that T is a bijection. Furthermore as T is a linear transformation and bijective we can conclude that T is an isomorphism. $\square$

(f) From part a we found that γ is an isomorphism meaning that it iself is a bijective linear transformation. Similiarly we found in part e that T is an isomorphism. Therefore we know from Proposition 2.11 from the Theory of Linear Algebra that the inverse map T^{-1} must also be linear. Therefore by Theorem 2.12 from the Theory of Linear Algebra we know that the composition of linear transformations is linear. Thus $T^{-1} \circ f \circ T$ must be a linear transformation. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be such a linear transformation, and let $x, y \in \mathbb{R}$. Therefore $L(\begin{bmatrix} x \\ y \end{bmatrix})$
$= T^{-1}(\gamma(T(\begin{bmatrix} x \\ y \end{bmatrix})$
$= T^{-1}(\gamma(x+yi))$
$= T^{-1}(x-yi)$
$= \begin{bmatrix} x \\ -y \end{bmatrix}$. Therefore using the Key thereom to compute the matrix of

L of the transformation $L(\vec{x})$ we find $L = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Geometrically the transformation L is reflecting the point across the x axis.

(g) From part b we found that μ is an isomorphism meaning that it iself is a bijective linear transformation. Similiarly we found in part e that T is an isomorphism. Therefore we know from Proposition 2.11 from the Theory of Linear Algebra that the inverse map T^{-1} must also be linear. Therefore by Theorem 2.12 from the Theory of Linear Algebra we know that the composition of linear transformations is linear. Thus $T^{-1} \circ \mu \circ T$ must be a linear transformation. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be such a linear transformation. Therefore $L(\begin{bmatrix} x \\ y \end{bmatrix})$
$= T^{-1}(\mu(T(\begin{bmatrix} x \\ y \end{bmatrix})$
$= T^{-1}(\mu(x + yi))$
$= T^{-1}(xi - y)$
$= \begin{bmatrix} -y \\ x \end{bmatrix}$. Therefore using the Key thereom to compute the matrix of L of the transformation $L(\vec{x})$ we find $L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Geometrically the transformation L is reflecting the point across the line $y = x$ and the x value by -1. Thus a point such as $(1, 2)$ would be $(-2, 1)$ after applying the transformation.

(h) From part c we found that f is an linear transformation. Similiarly we found in part e that T is an isomorphism. Therefore we know from Proposition 2.11 from the Theory of Linear Algebra that the inverse map T^{-1} must also be linear. Therefore by Theorem 2.12 from the Theory of Linear Algebra we know that the composition of linear transformations is linear. Thus $T^{-1} \circ f \circ T$ must be a linear transformation. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be such a linear transformation, and let $x, y \in \mathbb{R}$. Therefore $L(\begin{bmatrix} x \\ y \end{bmatrix}) = T^{-1}(f(T(\begin{bmatrix} x \\ y \end{bmatrix})$
$= T^{-1}(f(x + yi))$
$= T^{-1}(\frac{1+i}{4}((1 - i)(x + yi) + (1 + i)(x - yi)))$
$= T^{-1}(\frac{1+i}{4}(2x + 2y))$
$= T^{-1}(\frac{2x+2y+2xi+2yi}{4})$
$= T^{-1}(\frac{1}{2}((x + y) + (x + y)i))$
$= \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$Therefore using the Key thereom to compute the matrix of L of the transformation $L(\vec{x})$ we find $L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Geometrically the transformation L is collapsing all points onto the line $y = x$.

10. (a) *Proof.* Suppose that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} \in \mathbb{R}^k$.

We can therefore write $\vec{v}$ as a linear combination such $v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_k\vec{e_k}$. Furthermore $T(\vec{v}) = T(v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_k\vec{e_k})$. As we know T is a linear transformation it follows that it must preserve vector addition and scalar multiplication therefore
$T(v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_k\vec{e_k})$
$= T(v_1\vec{e_1}) + T(v_2\vec{e_2}) + \cdots + T(v_k\vec{e_k})$
$= v_1T(\vec{e_1}) + v_2T(\vec{e_2}) + \cdots + v_kT(\vec{e_k})$.
From our hypothesis that $T(\vec{e}_i) = L(\vec{e}_i)$, $\forall i \in \{1 \cdots k\}$ it follows that $v_1T(\vec{e_1}) + v_2T(\vec{e_2}) + \cdots + v_kT(\vec{e_k}) = v_1L(\vec{e_1}) + v_2L(\vec{e_2}) + \cdots + v_kL(\vec{e_k})$
Furthermore, as L is also a linear transformation it preserves vector addition and scalar multiplication such that it follows
$v_1L(\vec{e_1}) + v_2L(\vec{e_2}) + \cdots + v_kL(\vec{e_k})$
$= L(v_1\vec{e_1}) + L(v_2\vec{e_2}) + \cdots + L(v_k\vec{e_k})$
$= L(v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_k\vec{e_k})$
$= L(\vec{v})$.
Therefore $L(\vec{v} = T(\vec{v})$, $\forall \vec{v} \in \mathbb{R}^k$ □

(b) *Proof.* Suppose that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} \in \mathbb{R}^k$, and define a linear transformation $T : \mathbb{R}^k \to \mathbb{R}^m$. Note that we can write a linear combination of $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + ... + v_k\vec{e_k})$. Thus $T(\vec{v}) = T(v_1\vec{e_1} + v_2\vec{e_2} + ... + v_k\vec{e_k})$
$\overset{\dagger}{=} T(v_1\vec{e_1}) + T(v_2\vec{e_2}) + ... + T(v_k\vec{e_k})$
$= v_1T(\vec{e_1}) + v_2T(\vec{e_2}) + ... + v_kT(\vec{e_k})$. Where † follows as a linear transformation is defined by the preservation of vector addition and scalar multiplication. Therefore there is a linear transformation T such that $T(\vec{e_i} = f(\vec{e_i})$ for each $i \in \{1, ..., k\}$. Suppose there also exist another linear transformation $S : \mathbb{R}^k \to \mathbb{R}^m$ where $S(\vec{e_i} = f(\vec{e_i})$ for each $i \in \{1, ..., k\}$. Therefore $S(\vec{v})$ where v is as defined above is equal to $S(v_1\vec{e_1} + v_2\vec{e_2} + ... + v_k\vec{e_k})$
$= S(v_1\vec{e_1}) + S(v_2\vec{e_2}) + ... + S(v_k\vec{e_k})$
$= v_1S(\vec{e_1}) + v_2S(\vec{e_2}) + ... + v_kS(\vec{e_k})$
$= v_1f(\vec{e_1}) + v_2f(\vec{e_2}) + ... + v_kf(\vec{e_k})$
$\overset{\dagger}{=} v_1T(\vec{e_1}) + v_2T(\vec{e_2}) + ... + v_kT(\vec{e_k})$
$= T(v_1\vec{e_1}) + T(v_2\vec{e_2}) + ... + T(v_k\vec{e_k}$
$= T(v_1\vec{e_1} + v_2\vec{e_2} + ... + v_k\vec{e_k} = T(\vec{v}))$. Where † follows from the fact that $S(\vec{e_i}) = f(\vec{e_i}) = T(\vec{e_i})$ for each $i \in \{1, ..., k\}$. Therefore $T(\vec{v}) = S(\vec{v})$ for all $v \in \mathbb{R}^k$. Thus there exists a unique linear transformation T such that $T(\vec{e_i} = f(\vec{e_i})$ for each $i \in \{1, ..., k\}$. □

item

(a) False.

Counterexample, suppose that $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ where $A+B$ follows as an invertable matrix. Therefore, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$. Note that B is not invertable as $det = \frac{1}{0(4)-0} = \frac{1}{0}$ and thus by definition B is not invertable. Similarly for A, $det = \frac{1}{(2)0-0} = \frac{1}{0}$ and thus it is also not invertable. However its sum is. Therefore this is clearly not true.

(b) Suppose that A, B are invertable. Therefore there exists an inverse matrix A^{-1} and B^{-1}. Thus to test if AB is invertable we can show that AB has an inverse matrix such that $A^{-1}B - -1AB = I_n$. Thus $A^{-1}B - -1BA = A^{-1}I_nA = A^{-1}A = I_n$ and likewise $BAA^{-1}B - -1 = B^{-1}I_nB = B^{-1}B = I_n$ Thus $A^{-1}B^{-1} = AB^{-1}$ and therefore AB must be invertable. This proof can be furthered referenced from Theorem 2.4.7 from Linear algebra with applications.

(c) False. Counterexample. Suppose that $A, B \in \mathbb{R}^{2\times 2}$ are invertable where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$, and let $\vec{v} = 0_{2\times 2}$. Therefore $A\vec{v} = 0$ and $B\vec{v} = 0$ such that $A\vec{v} = B\vec{v}$ but $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = B$. Therefore this does not hold.

(d) Suppose that A is invertable. Therefore there exists a unique inverse matrix A^{-1} such that $AA^{-1} = I_n$. Further note that $A^n = A_1A_2A_3...A_n$ or A being multiplied together A times therefore by part B we know that the product of two matrices are invertable. Thus AA is invertable, and $AA \times A$ is also invertable as A and AA are. Therefore it follows that A^n must also be invertable as there exists a inverse matrix equal to $A_1^{-1}A_2^{-1}..A_n^{-1}$ such that A^n multiplied by this is the identity matrix. Therefore we have shown the contrapositive is true and thus the original statement holds.

(e) Counterexample. Suppose that $A.B$ are invertable matrices and that $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Therefore it follows as A is invertable that it has an unique inverse. $A^{-1} \stackrel{\dagger}{=} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Where $\dagger$ follows from the textbook definition of inverse for 2×2 matrices. Thus $ABA^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$. Thus $ABA^{-1} \neq B$.

(f) Suppose that A is an invertable matrix such that $A\vec{x} = 0$. From definition 2.4.2 from linear algebra with applications A is invertable if the linear transformation is invertable denoted $A\vec{x}$ is invertable such that there exist a inverse transformation T^{-1} and inverse matrix A^{-1} where the inverse is $\vec{x} = A^{-1}\vec{y}$. In this case $\vec{y} = 0$. Therefore as shown by theorem 2.44 $\vec{x} = A^{-1}\vec{y}$ such that $\vec{x} = 0$, and thus $A\vec{x} = 0$ as $\vec{x} = 0$.

(g) Suppose that A is invertable and $c \in \mathbb{R}$ where $c \neq 0$. Thus as A is invertable there must exists an unique inverse A^{-1} such that $AA^{-1} = I_n$. Further suppose that given cA that there is another matrix $\frac{1}{c}A^{-1}$, where it follows that as c and $\frac{1}{c} \in \mathbb{R}$ that they must commute and we could then rewrite this as $\frac{1}{c}cAA^{-1} = 1AA^{-1} = AA^{-1} = I_n$. Therefore we can conclude that $\frac{1}{c}A^{-1}$ is the inverse matrix of cA and thus cA must be invertable

11. (a) True, Note that V, W are two subspaces of $\mathbb{R}^n$ Furthermore, let $x, y \in V \cap W$. Therefore we know that $x \in V$ and $x \in W$ by what it means to be in the intersection of two sets. Similarly $y \in V$ and $y \in W$. As V is a subspace we know that it is closed under addition and thus the sum of two elements within it are also within the set. Thus $x + y \in V$. Similarly for W as it is also a subspace where $x, y \in W$ we can conclude that $x + y \in W$. Thus as $x + y \in V$ and $x + y \in W$ we can conclude that $x + y \in V \cap W$. Therefore we can conclude that $V \cap W$ is closed under addition. Furthermore as V, W are sub spaces we know they must preserve scalar multiplication. Thus given $c \in \mathbb{R}$ we know that $cx \in V$ and $cx \in W$. Therefore as $cx \in V$ and $cx \in W$. Therefore it is also in the intersection and thus $cx \in V \cap W$. Furthermore as V, W are both sub spaces we know that they $0_v \in V$ and $0w \in W$. Therefore as 0 is both in V and W we can conclude that $0 \in V \cap W$. Therefore we can say that $V \cap W$ contains 0. As we have shown above that $0 \in V \cap W$, its closed under addition and preserves scalar multiplication. We can then conclude that $V \cap W$ is a subspace for $\mathbb{R}^n$.

(b) False.
Counterexample:
Suppose that V is the subspace of the line on the y-axis, and let W be the subspace of the line on the x-axis. Suppose that $V \cup W$ is a subspace which would include all points in either on the x-axis or the y-axis. Therefore if $V \cup W$ is a subspace we know that it must be must contain 0, closed under addition, and preserves scalar multiplication. However this is not true suppose we have $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore the sum of these would be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However this is not a point on either the x-axis or the y-axis, and thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V \cup W$. Therefore this is not

closed under addition and therefore our hypothesis must be wrong. $V \cup W$ must not be always be a subspace.

12. (a) $\boxed{\text{Un}}$`titled Page (1)-page-001.jpg`

(b) Counterexample:

Let $A = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}$ and Let $B = \begin{bmatrix} b_1 & \\ & b_2 \end{bmatrix}$ where $a_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that $A = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Furthermore let $b_1 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

and therefore $B = \begin{bmatrix} b_1 & \\ & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Therefore $AB = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} \begin{bmatrix} b_1 & \\ & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & \\ & a_2 b_2 \end{bmatrix}$

where $a_1 b_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

$a_2 b_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Therefore $AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. However $BA = \begin{bmatrix} b_1 & \\ & b_2 \end{bmatrix} \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} =$ $\begin{bmatrix} b_1 a_1 & \\ & b_2 a_2 \end{bmatrix}$ where $b_1 a_1 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$ and

$b_2 a_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ Therefore $BA = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Thus $AB \neq BA$ therefore if A and B are diagonal block matrices $AB \neq BA$.

(c) Suppose that A is a block matrix where each block matrix A_i is invertable where $i \in \{1, ...n\}$ As A_i is invertable by problem 3 from jamboard 7 we know there is a unique inverse matrix A_i^{-1} for each A_i. Thus there exists a block matrix B where B contains the inverse matrices for all A_i. Therefore $B = \begin{bmatrix} A_1^{-1} & & & \\ & A_2^{-1} & & \\ & & \ddots & \\ & & & A_n^{-1} \end{bmatrix}$ Therefore

$$AB = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix} \begin{bmatrix} A_1^{-1} & & & \\ & A_2^{-1} & & \\ & & \ddots & \\ & & & A_n^{-1} \end{bmatrix}$$ which thus equals $$\begin{bmatrix} A_1A_1^{-1} & & & \\ & A_2A_2^{-1} & & \\ & & \ddots & \\ & & & A_nA_n^{-1} \end{bmatrix} = \begin{bmatrix} I_n & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}$$. This follows as the block diagonal matrix of all identity matrices is an identity matrix itself. Therefore $B = A^-1$ the inverse matrix of A as $AB = I_n$. Therefore by Theorem 2.4.8 from Linear Algebra with applications we can conclude that A, A^{-1} are both invertable.

13. (a) The kernel of T_A is defined to be $s \in S$ such that $T(s) = \vec{0}$. In terms of this question, from the definiton of S we know that $\vec{s} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ such that $A\vec{s} = \vec{0}$ Therefore we can conclude that $a = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ where $a, b \in \mathbb{R}$ as any non zero elements in the first column would result in a $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin ker(T_A)$. Suppose that $s = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It follows then from homework 2, 2.a that any transformation will map 0 to 0. Therefore we can conclude that $s(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0$ such that $s \in S$. Therefore we can conclude that the the zero element is in S. Furthermore suppose that $v = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ and $w = \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix}$ where $v, w \in S$. Therefore $v + w$ is equal to the sum of the two matrices, and therefore $v+w = \begin{bmatrix} 0+0 & a+x \\ 0+0 & b+y \end{bmatrix}$. It follows that $v+w(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0$. Therefore $v+w$ is also in s. Therefore we can conclude that S is closed under addition. Lastly let $k \in R$. Therefore $kv = \begin{bmatrix} k0 & ka \\ 0 & kb \end{bmatrix} = \begin{bmatrix} 0 & ka \\ 0 & kb \end{bmatrix}$. Therefore the matrix vector product of this $kv(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0$. Therefore $kv \in S$ and we can conclude that S must preserve scalar multiplication. Therefore, as S contains the zero element, is closed under addition, and preserves scalar multiplication we can conclude that S must be a subspace of V.

(b) Not a subspace. A subspace must preserve scalar multiplication. For example, suppose $v \in S$ where $v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $k = \frac{1}{2}$. Thus $kv = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ but $kv \notin S$ as the elements of kv are not integers. Therefore S does not preserve scalar multiplication and thus not a subspace.

(c) Not a subspace.
Counterexample: Suppose that $v = 0 + 1i$ and $w = 0 + 1i$ where $v, w \in S$ as $|v| = 1$ and $|w| = 1$. A subspace is closed under addition and thus the sum of two elements of the set must also be in the set. Therefore for $v+w$ it follows that $v+w = 0+2i$, and thus $|v+w| = 2$, which is not less than or equal to one. Therefore $v+w \notin S$. Therefore this is not closed under addition and thus not a subspace.

(d) Suppose that $a = 0, b = 0$ where $a + bi \in S$ therefore as$0 = 3(0) \rightarrow 0 = 0$ we know that $0+0i \in S$ such that $0 \in S$. Therefore $0_s \in S$ holds for the definition for what it means to be a subspace. Furthermore suppose $a + bi \in S$, and $k \in \mathbb{R}$. as it follows by definition of set S that $3a \mid b$ such that $3ak \mid bk$. Therefore we can then conclude that $ka + kbi \in S$ and thus S preserves scalar multiplication. Lastly suppose that $c + di \in S$ Therefore by definition of S, $3a = b$ and $3c = d$. It follows that $b + d = 3(a) + 3(c) \rightarrow b + d = 3(a + c)$, and therefore $(a + bi + c + di) \in S$. Therefore S is closed under addition. As S contains the zero vector, is closed under addition, and preserves scalar multiplication we can conclude that S is a subspace of V

(e) Let S = the set of all polynomials where $f(3) = 0$ such that we can denote such polynomials with $27a+9b+3c+d = 0$ where $a, b, c, d \in \mathbb{R}$. Therefore when $a, b, c, d = 0$ we find that $27(0) + 9(0) + 3(0) + 0 = 0 \rightarrow 0 = 0$. Thus we can conclude that $0_s \in S$ Furthermore let $k \in \mathbb{R}$ such that the scalar multiple of a polynomial in S is denoted as $k(27a + 9b + 3c + d) = 0 \implies k(0) = 0$ as polynomials in S are defined by $27a+9b+3c+d) = 0$ Thus we can conclude that all scalar multiples of an element in S are also in S such that S preserves scalar multiplication. Lastly, further suppose that $27w + 9x + 3y + z$ also denotes a element in S. Therefore the sum of these two arbitrary polynomials we find $27a + 27w + 9b + 9x + 3c + 3y + d + z \implies (27a + 9b + 3c + d) + (27w + 9x + 3y + z) \overset{\dagger}{\rightarrow} 0 + 0 = 0 \implies 0 = 0$. Where † follows as $27a + 9b + 3c + d = 0$ and $27w + 9x + 3y + z = 0$. Therefore as $0 = 0$ we can conclude that the sum is also in S and therefore S is closed under addition. As S contains the zero element, is closed under addition, and preserves scalar multiplication. We can conclude that S must be a subspace of V.

14. (a) Let S and T be linear transformations where $S : V \rightarrow U$, and $T : U \rightarrow W$. Suppose that $v \in V$ is an element of the kernel of S. Where the kernel of S is where $S(\vec{v}) = 0_u$. Furthermore note that the $ker(T)$ is where $T(\vec{u}) = 0_w$ for some $u \in U$. Therefore it follows that $T \circ S(v) = T(S(\vec{v})) = T(\vec{0}_u) \overset{\dagger}{=} 0_w$. Where † follows from homework 2 problem 2.a where we found that given T is a linear transformation, $T(\vec{0}) = 0$. Thus we can conclude that $\vec{0_u} \in ker(T)$

and thus that $S(\vec{v}) \in ker(T)$ such that $v \in ker(T \circ S)$. Therefore we can conclude that $ker(S) \subseteq ker(T \circ S)$.

(b) Suppose that $x \in Im(T \circ S)$ such that $Im(T \circ S) \subseteq W$. $T \circ S$ is the mapping from $V \to W$ where $u \in U$ such that $\exists v \in V$ where $S(v) = u$. Therefore this implies that $T(S(v)) = w$ where $w \in W$. Thus $Im(T \circ S(v))$ contains all $w \in W$ where $\in V$ such that $T(S(v)) = w$. Also note that $\forall u \in U$, $T(u) = w$. Thus as $x \in Im(T \circ S)$ we can conclude that $x \in Im(T)$. Thus $Im(T \circ S) \subseteq Im(T)$.

(c) Suppose that $T \circ S$ is the zero mapping. Therefore by definition $T(S(\vec{v})) = 0_w \forall v \in V$.Further suppose that $u \in Im(S)$. Therefore $\exists v_1 \in V$ such that $S(v_1) = u$. Thus $T(u) = T(S(v_1))$. From our hypothesis that $T \circ S$ is the zero mapping we can then conclude $T(u) = T(S(v_1)) = 0_w$. Therefore $u \in ker(T)$. Thus $Im(S) \subseteq ker(T)$,

Let us now suppose that $Im(S) \subseteq ker(T)$. Therefore $\forall u \in Im(S)$ $T(u) = 0_w$ Therefore by the definition of an image given by jamboard 8, $u \in Im(S)$ if there exists some $v \in V$ such that $S(v) = u$. Generally, if $u = S(v)$ for some $v \in V$ then $u \in ker(T)$, such that $T(u) = 0_w$. Therefore it follows that $T(S(v)) = 0_w$, and it follows that $S(v) \in Im(S)$ for all $v \in V$ as $Im(S)$ is a subset of $ker(T)$ and thus all of the image must be contained within the kernel.Therefore we know $T \circ S$ must be the zero mapping.

Therefore as we have shown both conditional statements are true we can conclude that $Im(S) \subseteq ker(T)$ if and only if $T \circ S$ is the zero mapping.

(d) For part a suppose that $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $ker(S)$ would be such vectors $v \in \mathbb{R}^2$ such that $S\vec{v} = 0$. The only matrix which results in the zero vector given a the matrix is the identity matrix is the zero vector thus when $v_1 = 0^2_R$. On the other hand given a $v_2 \in \mathbb{R}^2$ where $v_2 = \begin{bmatrix} 0 \\ a \end{bmatrix}$ and $a \in \mathbb{R}$ then $S(v_2) = v_2$ and $T \circ S(v_2) = \vec{0}$. Thus $v_2 \in ker(T \circ S)$. Therefore it is clear to see that $ker(S) \subset ker(T \circ S)$ For part B suppose that $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $Im(S)$ would be R as S would map to all matrices in the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$ where $a \in \mathbb{R}$. Therefore it will map to all points on the axis. Therefore $im(T \circ S)$ will also map to all points on the x-axis as T is just the identity matrix which will output the same as the input. However, $im(T)$ be all of $\mathbb{R}^{\nvDash}$ as $T : \mathbb{R}^{\nvDash} \to \mathbb{R}^2$

where T is the identity matrix such that the input will be the same as the output, and therefore can include all points in $\mathbb{R}^2$ including the x-axis but not limited to the x-axis. Thus $im(T \circ S) \subset im(T)$.

Lastly for part C suppose that $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $T\vec{x} = 0$ for all $\vec{x}$. Therefore given $T(S(\vec{x}))$ we can conclude that this is also the zero mapping as $im(S) \subset R^2$ and T maps all of R^2 to 0. Therefore it also follows that $ker(T) = \mathbb{R}^2$. And thus it holds that $im(S) \subset ker(T)$. Thus the equivalence statement holds for this example.

15. (a) Suppose that $L(v_i) = T(v_i) \forall i \in \{1...k\}$. Further suppose that $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix} \in \mathbb{R}^k$.We can therefore write $\vec{w}$ as the linear combination of the basis vectors $v_1, v_2, v_3, v_4, ..., v_k \in \mathbb{R}^k$. Therefore $\vec{w} = w_1v_1 + w_2v_2 + ... + w_kv_k$. As T is a linear transformation it must preserve vector addition and scalar multiplication. Therefore $T(\vec{w})$
$= T(\ w_1v_1 + w_2v_2 + ... + w_kv_k)$
$= T(\ w_1v_1) + T(w_2v_2) + ... + T(w_kv_k)$
$= w_1T(v_1) + w_2T(v_2) + ... + w_kT(v_k)$
$\stackrel{\dagger}{=} w_1L(v_1) + w_2L(v_2) + ... + w_kL(v_k)$
$= (w_1Lv_1) + L(w_2v_2) + ... + L(w_kv_k)$
$= L(w_1Lv_1 + w_2v_2 + ... + w_kv_k)$
$L(\vec{w})$. Where $\dagger$ follows from our hypothesis that $L(v_i) = T(v_i)$. Thus we have $T(\vec{w}) = L(\vec{w}) \forall w \in \mathbb{R}^k$. Therefore we have shown that the contrapositive of the statement is true and thus we know that the initial statement $L(\vec{w}) \neq T(\vec{w})$ for some vector $w \in \mathbb{R}^k$ such that $L(\vec{i}) \neq T(\vec{v_i})$ for some $i \in \{1, ..., k\}$ must also be true.

(b) Counterexample Suppose $k = 3, m = 2$ Therefore L is a linear transformation where $\mathbb{R}^3 \to \mathbb{R}^2$. Furthermore suppose we have a set $\{e_1, e_2, e_3\} \in \mathbb{R}^3$. As L is a linear transformation from a larger vector space to a smaller vector space we know that it cannot be injective. Therefore, $\{L(\vec{e_1}), L(\vec{e_2}), L(\vec{e_3})\}$ must not be independent as it will contain 3 points in $\mathbb{R}^2$ leading to one being redundant. Therefore as this is not linearly independent, it is not a basis of $\mathbb{R}^2$

16. (a) Let $D : P_n \to \mathbb{R}$ sending $f \to f'(0)$. Thus let $\alpha = a_1 + a_2x + a_3x^2 + ..a_{n+1}x^n$ where $a_1, a_2, ..a_{n+1} \in \mathbb{R}$. Thus as D sends f to $f'(0)$ then it $D(\alpha)$ is equal to $a_2 + 2a_3(0) + ...na_{n+1}(0) = a_2$. Likewise let $\beta = b_1 + b_2x + b_3x^2 + b_{n+1}x^n \in P_n$ where $b_1, b_2, ...b_{n+1} \in \mathbb{R}$. Therefore $D(\beta)$ is equal to $b_2 + 2b_3(0) + ... + nb_{n+1}(0) = b_2$. Therefore $D(\alpha) = a_2$ and $D(\beta) = b_2$. Thus $D(\alpha) + D(\beta) = a_2 + b_2$. Let us also compute $D(\alpha + \beta)$ which is $D(a_1 + a + 2x + ...a_{n+1}x^n + b_1 + b_2x + ... + b_{n+1}x^n)$

Thus $D(a_1 + a + 2x + ... a_{n+1}x^n + b_1 + b_2x + ... + b_{n+1}x^n) \overset{\dagger}{=} a_2 + b + n + 2(0)(a_1 + b_1) + ... + n(0)(a_{n+1} + b_{n+1}) = a_2 + b_2$, where † follows as in general D sends f to $f'(0)$. Therefore $D(\alpha + \beta) + a_2 + b_2 = a_2 + b_2 = D(\alpha) + T(\beta)$. Therefore we can conclude that D must be closed under addition. Furthermore let us suppose that $k \in \mathbb{R}$. Therefore $kD(\alpha) = k(a_2)$. Furthermore, $D(k\alpha) = D(k(a_1 + a_2x + ... + a_{n+1}x^n)) = D(ka_1 + ka_2x + .. + ka_{n+1}x^n) \overset{\dagger}{=} ka_2 + 2ka_3(0) + ...kn(a_{n+1}(0) = ka_2$, where † follows as in general D sends f to $f'(0)$. Thus we have $kD(\alpha) = ka_2 = ka_2D(k\alpha)$. Therefore we can conclude that D must be closed under scalar multiplication. As we have shown that D is closed under both scalar multiplication and closed under addition we can conclude that D must be a linear transformation.

(b) Let $\alpha = a_1 + a_2x + a_3x^2 .. + a_{n+1}x^n \in P_n$ where $a_1, a_2, ..., a_{n+1} \in \mathbb{R}$. Furthermore note that the kernel of D is all $\vec{x} \in P_n$ where $D(\vec{x}) = 0_R$. Also not that D sends α to $a_1 \in \mathbb{R}$. Therefore when $a_2 = 0$ then D sends α to 0 and then α must be in the kernel of D. Notice that this does not depend on the values $a_1, a_3, ...a_{n+1}$. Thus in general we can conclude that an element of $u \in ker(D)$ can be given as a linear combination such that $u = a_1 + a_2x + a_3x^2 .. + a_{n+1}x^n \ \forall a_1, a_3, ..a_{n+1} \in \mathbb{R}$ given that $a_2 = 0$. Thus we can write the basis of $ker(T)$ as $\{1, x, x^2, ...x^n\}$.

(c) Let $\alpha = a_1 + a_2x + a_3x^2 .. + a_{n+1}x^n \in P_n$ where $a_1, a_2, ..., a_{n+1} \in \mathbb{R}$. Note that the image of D contains all elements of $\mathbb{R}$ where there exists a valid mappings from $P_n \to \mathbb{R}$ where D sends $f \to f'(0)$. Written arbitrarily with $alpha$, D sends $alpha$ to $a_2 + 2(a_3)(0) + ... + na_{n+1}(0) = a_2$. Thus we can conclude that in general $Im(D)$ will contains all values of a_2, as all other elements are eliminated when $x = 0$. In other words if $u = Im(D)$ then there exists the linearly independent linear combination a_2x such that $a_2x = u$. Therefore we can conclude that the basis of $Im(D) = \{x\}$.

17. (a) Suppose that $B = (v_1, v_2, v_3, ...v_n)$ is an ordered basis of the vector space V. Furthermore let $C = (v_2 - v_1, v_3 - v_2, ..., v_n - v_{n-1}, v_n)$. Therefore the linear combination of this is $a_1(v_2 - v_1) + a_2(v_3 - v_2) + ... + a_{n-1}(v_{n-1} - v_n) + a_nv_n$ where $a_1, ...a_n \in \mathbb{R}$. Factoring this out we have $a_1v_2 + -a_1v_1 + a_2v_3 - a_2v_2 + ... + a_{n-1}v_{n-1} - a_{n-1}v_n + a_nv_n$ We can now factor out like terms in this case $v_1, ...v_n$. Therefore we can rewrite this as $-a_1v_1 + v_2(a_1 - a_2) + v_3(a_2 - a_3) + ... + v_{n-1}(a_{n-2} - a_{n-1}) + v_n(a_{n-1} - a_n)$. There as $a_1, a_2, ...a_n \in \mathbb{R}$ it follows that $(a_1 - a_2), (a_2 - a_3), ...(a_{n-1} - a_n) \in \mathbb{R}$ therefore it is clear to see that $-a_1v_1 + v_2(a_1 - a_2) + v_3(a_2 - a_3) + ... + v_{n-1}(a_{n-2} - a_{n-1}) + v_n(a_{n-1} - a_n) = c_1v_1 + c_2v_2 + ...c_nv_n$ where $c_1, ...c_n \in \mathbb{R}$ such that $c_1 = -a_1, c_2 = (a_1 - a_2), ...c_n = (a_{n-1} - a_n)$ Therefore as these are the same linear combination with the same constants, it follows that $C = B$ therefore C must be a basis of V.

(b) First note that the set $\{x^n - x^{n-1} \mid n = 1, 2, ..100\}$ is equal to $\{x-1, x^2-x, x^3-x^2, ..., x^{100}-x^{99}\}$. Let us call this set C. From Jamboard 10 question 3 we know that we $B = \{1xx^2, x^3, ..., x^{100}\}$ be a basis for P_{100}. Also recall part A we showed that given a basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, ..., \vec{v_n}\}$ that $\{\vec{v_2} - \vec{v_1}, ...\vec{v_n} - \vec{v_{n-1}}, \vec{v^n}\}$ is also a basis. Therefore given that B is a basis we know that also $\{x-1, x^2-x, ..., x^{100}-x^{99}, x^{100}\}$ is a basis. Therefore notice that C is subset of a basis of P_{100}, $\{x-1, x^2-x, ..., x^{100}-x^{99}, x^{100}\}$. As a basis for P_{100} must both be linearly independent and spans P_{100}, we know that $\{x-1, x^2-x, ..., x^{100}-x^{99}, x^{100}\}$ must be linearly independent. Therefore by Jamboard 11 problem $7c$ since C is a subset of a linearly independent set we then know that it itself is linearly independent.

18. (a) The zero vector in the vector space $F(\mathbb{R}, \mathbb{R})$ would be the function that is 0 for all $\mathbb{R}$. Therefore $z(x) = 0_{\mathbb{R}}$, $\forall x \in \mathbb{R}$.

(b) Suppose that $c_1 \sin(x) + c_2 \sin(2x) + ...c_{n+1} \sin(2^n x) = 0$. Further suppose for the first base case, that $x = \frac{\pi}{2}$, thus we can show that the first element of the set, $\sin(x)$ when $n = 1$ must be zero. First, notice that we can rewrite this relation when $x = \frac{\pi}{2}$ as $c_1 \sin(\frac{\pi}{2}) + c_2 \sin(2\frac{\pi}{2}) + ...c_{n+1} \sin(2^n \frac{\pi}{2}) = 0 \to c_1(1) + c_2(0) + ... + c_{n+1}(0) = 0$. Therefore for this relation to hold $c_1 = 0$. Then each term would equal 0 and thus the sum would also be 0. Therefore this relation holds. Next suppose that we have $x = \frac{\pi}{4}$. Similarly substituting this in for x we find that $c_1 \sin(\frac{\pi}{4}) + c_2 \sin(2\frac{\pi}{4}) + ...c_{n+1} \sin(2^n \frac{\pi}{4}) = 0 \to c_1(\frac{\sqrt{2}}{2}) + c_2(1) + ... + c_{n+1}(0) = 0$. From above we know that $c_1 = 0$. Thus we know that the first element would be 0. Therefore we can rewrite this relation as $0 + c_2(1) + ... + c_{n+1}(0) = 0$. Therefore it is clear to see that $c_2 = 0$ for this relation to hold. Therefore suppose in general that $x = \frac{\pi}{2^{n+1}}$ where $n \geq 0$. Thus we have $c_1 \sin(\frac{\pi}{2^{n+1}}) + c_2 \sin(2\frac{\pi}{2^{n+1}}) + ...c_{n+1} \sin(2^n \frac{\pi}{2^{n+1}}) = 0$. As we have shown the bases cases, then in general we notice that fact that all terms before c_{n+1} have been found to be 0. We have finitely shown this for $n = 0$ that $n = 1$ will be zero. this follows in general that $c_1, ..., c_n = 0$ and thus we can rewrite this relation as $0\sin(\frac{\pi}{2^{n+1}}) + 0\sin(2\frac{\pi}{2^{n+1}}) + ...c_{n+1} \sin(2^n \frac{\pi}{2^{n+1}}) = 0 \to 0 + 0 + ... + c_{n+1} \sin(2^n \frac{\pi}{2^{n+1}} = 0 \xrightarrow{\dagger} 0 + 0 + c_{n+1} \sin(\frac{\pi}{2}) = 0 \to 0 + 0 + c_{n+1}(1) = 0$. Where $\dagger$ follows as $\frac{2^n \pi}{2^{n+1}} = \frac{\pi}{2}$. Thus it is clear that $c_{n+1} = 0$ for this relation to hold. Therefore in general we have shown that $c_1 = c_2 = ... = c_{n+1} = 0$. Therefore the only relation is the trivial one, and thus this set is linearly independent.

(c) The set $\{\sin(x), \cos(x), \sin(x + \frac{\pi}{3})$ can be shown to be linearly dependent. To be linearly dependent we would have to show that one such element of the set is a linear combination of some of the other elements, In this case let $a_1 = \frac{1}{2}$, $a_2 = \frac{\sqrt{3}}{2}$. Thus we can write this as a linear combination with a_1, a_2 be constants. Thus

$\frac{1}{2}\sin(x)+\frac{\sqrt{3}}{2}\cos(x)$. Now note that this is equivalent to $\cos(\frac{\pi}{3}\sin(x)+$ $+\sin(\frac{\pi}{3})\cos(x)$ as $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Therefore it is now clear to see that this linear combination is equal to angle sum identity that we used on homework 3. In general this we know that $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$. Thus we can rewrite $x, \frac{\pi}{3}$ in this form $\sin(x)\cos(\frac{\pi}{3} + +\cos(x)\sin(\frac{\pi}{3})$, as multiplications of real numbers is commutative. Thus we can see that this is in fact the sum angle identity and that the linear combination $\frac{1}{2}\sin(x)+\frac{\sqrt{3}}{2}\cos(x) = \sin(x)\cos(\frac{\pi}{3} + +\cos(x)\sin(\frac{\pi}{3}) = \sin(x+\frac{\pi}{3}$. Therefore we have shown that we can write $\sin(x+\frac{\pi}{3}$ as a combination of the other elements in the set, and thus it is clear that this is set is a linearly dependent subset of $F(\mathbb{R},\mathbb{R})$.

19. (a) Assume for the sake of contradiction that $u_i = v_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Therefore it follows that $T(u_i) = T(v_j)$. Note that v_j is in the kernel of T such that $T(v_j) = 0_v$. Furthermore, we know that $T(u_i)$ is in the basis of the image of T. Therefore 0_v must also be in the basis of the image of T. However, this is in false, as if this set contained the 0_v it would no longer be linear independent, and thus not a basis. Therefore we know that our initial assumption must be false, and that in fact $u_i \neq v_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

    (b) Suppose there exists $c_1, c_2, ...c_m, k_1, ..., k_n \in \mathbb{R}$ such that $c_1u_1+c_2u_2+ ... + c_mu_m + k_1v_1 + ...k_nv_n = 0_u$. Applying T to this relation we find that $T(c_1u_1 + c_2u_2 + ... + c_mu_m + k_1v_1 + ...k_nv_n) = T(0_u) \rightarrow T(c_1u_1 + c_2u_2 + ... + c_mu_m + k_1v_1 + ...k_nv_n) = 0_v$. Further as T is a linear transformation we know that it is closed under vector addition and we can rewrite this as $T(c_1u_1 + c_2u_2 + ... + c_mu_m) + T(k_1v_1 + ...k_nv_n) = 0_v$. Recall that $\{v_1, ..., v_n\}$ is a basis for the kernel of T. Therefore $T(k_1v_1+...k_nv_n) = 0_v$. Thus we can rewrite this relation as $T(c_1u_1+c_2u_2+...+c_mu_m)+0_v = 0_v \rightarrow T(c_1u_1+c_2u_2+...+c_mu_m) = 0_v$. As T is a linear transformation we again know that is closed under addition and scalar multiplication thus we could rewrite this relation as $c_1T(u_1) + c_2T(u_2) + ... + c_mT(u_m) = 0_v$. Further recall that $\{T(u_1), ..T(u_m)\}$ forms the basis of the image of T and therefore as it forms a basis it must be linear independent. For it to be linear independent it must have only the trivial relation, and thus we can conclude that $c_1 = c_2 = ... = c_m = 0$. Looking back at the original relation $c_1u_1+c_2u_2+...+c_mu_m+k_1v_1+...k_nv_n = 0_u$ since we know that $c_1 = c_2 = ... = c_m = 0$ we can rewrite this as $k_1v_1+...k_nv_n = 0_u$. Note now that we know that $\{v_1, v_2, ..v_n\}$ is a basis. Therefore we know that this this set must be linear independent, and thus the only relationship on it is trivial solution. Therefore we can conclude that $k_1 = k_2 = ... = k_n = 0$. Thus as $c_1 = c_2 = ... = c_m = k_1 = k_2 = ... = k_n = 0$ we can conclude that the only solution to this relation is the trivial one, and thus it is linearly independent.

(c) Let $w \in U$ thus we know that $w = a_1u + 1 + a_2u_2 + ... + a_mu_m + b_1v_1 + ... + b_nv_n$ where $a_1, ..a_m, b_1, ..b_n \in \mathbb{R}$. Consider $T(w)$. $T(w)$ is in the image of T. Recall that $T(u_1), ..., T(u_m)$ form a basis for $Im(T)$. Thus $T(w)$ can be written as a linear combination of $T(u_1), ..., T(u_m)$ such that there exists $c_1, c_2, ...c_m \in \mathbb{R}$. Thus $T(w) = c_1T(u_1) + c_2T(u_2) + ... + c_mT(u_m)$. As T is a linear combination thus it must preserve vector addition and therefore we can conclude that $T(w) = T(c_1u_1 + c_2u_2 + ... + c_mu_m)$. Therefore we also say that $T(w) - T(c_1u_1 + c_2u_2 + ... + c_mu_m) = 0_v$. As state above as T is a linear combination we can combine these two transformations into a single one as T preserves vector addition. Thus we have that $T(w - c_1u_1 - c_2u_2 - ... - c_mu_m) = 0_v$. Therefore it is clear that this $w - c_1u_1 - c_2u_2 - ... - c_mu_m \in ker(T)$. Also recall that the basis of the kernel of T is the set $\{v_1, ..v_n\}$. Therefore as $w - c_1u_1 - c_2u_2 - ... - c_mu_m \in ker(T)$ we know that there must exists $k_1, ..k_n \in \mathbb{R}$ such that $w - c_1u_1 - c_2u_2 - ... - c_mu_m = k_1v_1 + k_2v_2 + ... + k_nv_n \to w = c_1u_1 + c_2u_2 + .. + c_mu_m + k_1v_1 + k_2v_2 + ... + k_nv_n$. Therefore it is clear that we can write w as a linear combination of $\{u_1, ..u_m, v_1, ..v_n\}$ and thus it spans this set.

(d) As we have shown in part b, c that $\{u_1, ..u_m, v_1, ..v_n\}$ is both linearly independent and spans u we can conclude that this is a basis for U Recall that the dimension of a vector space is equal to the number of elements in its basis thus $dim(U) = m + n$. Further recall that every vector space has a basis. We know that the $im(T)$ has one such basis defined by $\{T(u_1), T(u_2), ..T(u_m)\}$ Thus $dim(im(T)) = m$ or the number of elements in this basis. Similarly we know that $\{v_1, ...v_n\}$ is one such basis of $ker(T)$ and thus $dim(ker(T)) = n$. Therefore putting this together it follows that $\dim(U) = \dim(\ker(T)) + \dim((T))$.

20. (a) Suppose that $\vec{v_1} \in V$ and $\vec{w_1} \in W$ therefore it follows that $\vec{v_1} + \vec{w_1} \in V + W$. Furthermore suppose that $\vec{v_2} \in V$, and $\vec{w_2} \in W$, thus it also follows that $\vec{v_2} + \vec{w_2} \in V + W$. As V is a subspace of $\mathbb{R}^n$ we know that it is closed under addition, and therefore as $\vec{v_1}, \vec{v_2} \in V$ we know that also $\vec{v_1} + \vec{v_2} \in V$. Likewise as W is a subspace of $\mathbb{R}^n$ we know that it is also closed under addition, and that again as $\vec{w_1}, \vec{w_2} \in W$, that $w_1 + \vec{w_2} \in W$. Therefore as $\vec{v_1} + \vec{v_2} \in V$ and $\vec{w_1} + \vec{w_2} \in W$, we can conclude that $\vec{v_1} + \vec{v_2} + \vec{w_1} + \vec{w_2} \in V + W$. Thus $V + W$ is closed under addition. Next, let $\lambda \in \mathbb{R}$. Thus as $\vec{v_1} \in V$ we know that again, $\lambda\vec{v_1} \in V$ as V is a subspace of $\mathbb{R}^n$, and a subspace must preserve scalar multiplication. Likewise as $\vec{w_1} \in W$ and W is also a subspace of $\mathbb{R}^n$ we can conclude that $\lambda\vec{w_1} \in W$. Therefore $\lambda\vec{v_1} \in V$ and $\lambda\vec{w_1} \in W$ we can conclude that $\lambda\vec{v_1} + \lambda_{w1} \in V + W$. Therefore we can deduce that $V + W$ preserves scalar multiplication. Lastly as V, W are subspaces of $\mathbb{R}^n$ we know that they both contain $0_{\mathbb{R}^n}$ and thus as $0_{\mathbb{R}^n} \in V$ and $0_{\mathbb{R}^n} \in W$ we can conclude that $0_{\mathbb{R}^n} + 0_{\mathbb{R}^n} \in V + W$ and note that

this is the same as $0_{\mathbb{R}^n} \in V + W$, as the sum of two zero vectors is also the zero vector. Thus $V + W$ also contains the zero vector. Therefore as $V + W$ contains the zero vector, closed under addition, and preserves scalar multiplication we can conclude that $V + W$ must be a subspace of $\mathbb{R}^n$.

(b) Let $\lambda \in \mathbb{R}$ and let $(\vec{v_1}, \vec{w_1}) \in V \times W$. Therefore as $T(\vec{v_1}, \vec{w_1}) = \vec{v_1} - \vec{w_1}$. Furthermore $\lambda T(\vec{v_1}, \vec{w_1}) = \lambda(\vec{v_1} - \vec{w_1}) = \lambda\vec{v_1} - \lambda\vec{w_1}$ and $T(\lambda\vec{v_1}, \lambda\vec{w_1}) = \lambda v_1 - \lambda w_1$. Therefore $\lambda T(\vec{v_1}, \vec{w_1}) = \lambda\vec{v_1} - \lambda\vec{w_1} = \lambda\vec{v_1} - \lambda\vec{w_1} = T(\lambda\vec{v_1}, \lambda\vec{w_1})$. Thus we can conclude that T preserves scalar multiplication. Furthermore let $(\vec{v_2}, \vec{w_2}) \in V \times W$. Thus $T(\vec{v_2}, \vec{w_2}) = \vec{v_2} - \vec{w_2}$. Therefore $T(\vec{v_1}, \vec{w_2}) + T(\vec{v_2}, \vec{w_2}) = \vec{v_1} - \vec{w_1} + \vec{v_2} - \vec{w_2}$, and $T(\vec{v_1} + \vec{v_2}, \vec{w_1} + \vec{w_2}) = (\vec{v_1} + \vec{v_2}) - (\vec{w_1} + \vec{w_2}) = \vec{v_1} + \vec{v_2} - \vec{w_1} - \vec{w_2} = \vec{v_1} - \vec{w_1} + \vec{v_2} - \vec{w_2}$. Thus $T(\vec{v_1} + \vec{v_2}, \vec{w_1} + \vec{w_2}) = \vec{v_1} - \vec{w_1} + \vec{v_2} - \vec{w_2} = \vec{v_1} - \vec{w_1} + \vec{v_2} - \vec{w_2} = T(\vec{v_1}, \vec{w_2}) + T(\vec{v_2}, \vec{w_2})$, Therefore we can conclude that T preserved vector addition and thus as T preserves vector addition and scalar multiplication we can conclude that T is a linear transformation.

(c) Let $U \in V + W$. Therefore we know by definition of the subspace $V + W$ that $u = v_1 + w_1$ such that $v_1 \in V$ and $w_1 \in W$. Furthermore $T((\vec{v_1}, -\vec{w_1})) = v_1 + w_1$, where $(\vec{v_1}, -\vec{w_1}) \in V \times W$ such that $v_1 \in V$ by assumption and $-w_1 \in W$ as W is a subspace and thus preserves scalar multiplication such that as $-1 \cdot w_1 \in W$ as $w_1 \in W$ by assumption. Thus we know that for all $u \in V + W$ there exists some element $(\vec{v_1}, -\vec{w_1}) \in V \times W$ such that $T((\vec{v_1}, -\vec{w_1})) = u$. Therefore we can conclude that T is surjective.

Furthermore let $(\vec{v'}, \vec{w'}) \in \ker(T)$. In general for $(\vec{v'}, \vec{w'}) \in \ker(T)$ then $T(\vec{v'}, \vec{w'}) = 0_{\mathbb{R}^n}$, in the case of T describe above where $T(\vec{v}, \vec{w}) = \vec{v} - \vec{w}$. Therefore if$T(\vec{v'}, \vec{w'}) = 0_{\mathbb{R}^n}$ then $\vec{v'} - \vec{w'} = 0_{\mathbb{R}^n}$ and thus $\vec{v'} = \vec{w'}$. Therefore $\ker(T) = \{(\vec{v'}, \vec{v'}) : \vec{v'} \in V, \vec{v'} \in W\}$. Next let $S : \ker(T) \to V \cap W$ such that $(\vec{v'}, \vec{v'}) \to \vec{v'}$. Therefore let $(\vec{v_2}, \vec{v_2}) \in \ker(T)$. Thus $S(\vec{v_2}, \vec{v_2}) = \vec{v_2}$. Furthermore let $(\vec{v_3}, \vec{v_3}) \in \ker(T)$ thus similarly $S(\vec{v_3}, \vec{v_3}) = \vec{v_3}$. Therefore $S(\vec{v_2}, \vec{v_2}) + S(\vec{v_3}, \vec{v_3}) = \vec{v_2} + \vec{v_3}$. $S(\vec{v_2} + \vec{v_3}, \vec{v_2} + \vec{v_3})$ would then equal $\vec{v_2} + \vec{v_3}$. Thus $S(\vec{v_2}, \vec{v_2}) + S(\vec{v_3}, \vec{v_3}) = \vec{v_2} + \vec{v_3} = \vec{v_2} + \vec{v_3} = S(\vec{v_2} + \vec{v_3}, \vec{v_2} + \vec{v_3})$. Therefore we can conclude that S must be closed under addition. Furthermore let $\lambda \in \mathbb{R}$ therefore $\lambda(\vec{v_2}, \vec{v_2}) = \lambda\vec{v_2}$ and $S(\lambda\vec{v_2}, \lambda\vec{v_2}) = \lambda\vec{v_2}$. Thus $\lambda(\vec{v_2}, \vec{v_2}) = \lambda\vec{v_2} = \lambda\vec{v_2} = S(\lambda\vec{v_2}, \lambda\vec{v_2})$. Therefore we can also conclude that S is closed under scalar multiplication. Therefore as S is closed under vector addition and scalar multiplication S must a linear transformation.

Furthermore suppose that $v_4 \in V \cap W$ such that $v_4 \in V$ and $v_4 \in W$. Thus $S((v_4, v_4)) = v_4$, and thus for an arbitrary v_4 in $V \cap W$ there exists an element $(\vec{v_4}, \vec{v_4}) \in \ker(T)$ such that $S((\vec{v_4}, \vec{v_4})) = \vec{v_4}$. Therefore we can conclude that S is surjective. Lastly let $(\vec{v_5}, \vec{v_5}) \in \ker(T)$ thus if $S((\vec{v_5}, \vec{v_5})) = S((\vec{v_4}, \vec{v_4}))$ then we know from S that $\vec{v_4} = \vec{v_5}$ as S maps $(\vec{v'}, \vec{v'})$ to $\vec{v'}$. Therefore S must be injective. Therefore as S is both injective and surjective we know that it must be bijective. Furthermore as S is a bijective linear transformation we can then conclude that S is an isomorphism and thus $\ker(T)$ must be isomporhic to $V \cap W$.

(d) Let $(v_1, v_2, ..., v_d)$ be a basis for V therefore $dim(V) = d$. Furthermore let $(w_1, w_2, ..., w_n)$ be a basis for W therefore $dim(W) = n$. Lastly let $b = \{v_1, v_2, ...v_d, w_1, w_2, ..., w_n\}$ be a set of vector in $V \times W$. Suppose that $a_1v_1 + ...a_dv_d + b_1w_1 + ...b_nw_n = 0_{V\times W}$, where $a_1, ..a_d, b_1, ..b_n \in \mathbb{R}$. As $(v_1, v_2, ..., v_d)$ is a basis of V we know that it must then span V and must be linearly independent, therefore the only relation on it is the trivial one such that $a_1 = ...a_d = 0$. Likewise as $(w_1, w_2, ..., w_n)$ is a basis for W we know similarly that it spans W and is linearly independent. Therefore the only relation on it is the trivial one and thus $b_1 = ... = b_n = 0$. Therefore we can conclude that the only relation on $a_1v_1 + ...a_dv_d + b_1w_1 + ...b_nw_n = 0_{V\times W}$ is the trivial one and thus it must also be linearly independent. Furthermore suppose that $u \in V \times W$ therefore $u = v_i + w_i$ where $v_i \in V$ and $w_i \in W$. Thus we know that $v_i = a_1v_1 + ...a_dv_d$ as $(v_1, v_2, ..., v_d)$ is a basis of V and thus $(v_1, v_2, ..., v_d)$ spans v_i. Furthermore we know that $w_i = b_1w_1 + ...b_nw_n$ as $(w_1, w_2, ..., w_n)$is a basis for W and therefore spans W. Therefore it is clear that $u = v_i + w_i = a_1v_1 + ...a_dv_d + b_1w_1 + ...b_nw_n$ and therefore $a_1v_1 + ...a_dv_d + b_1w_1 + ...b_nw_n$ spans $V + W$. Thus $(v_1 + ...v_d, w_1 + ..., w_n)$ is a basis for $V + W$ as it spans $V+W$ and is linearly independent. Thus $\dim(V+W) = V+W$. Therefore $\dim(V + W) = \dim(V) + \dim(W)$.

(e) Let T be a linear transformation from $V \times W \to V + W$. Further recall the rank-nullity theorem, which states that $\dim(source(T)) = \dim(\ker(T)) + \dim(Im(T))$. Note that the source of T is equal to $V \times X$, and thus $\dim(source(T)) = \dim(V \times X) \overset{\dagger}{=} \dim(V) + \dim(V)$. Where † follows directly from part d. Thus we can rewrite the original equation as $\dim(V) + \dim(V) = \dim(\ker(T)) + \dim(Im(T))$. Also recall that in part c we showed that T is surjective therefore we can conclude that $Im(T) = V + W$. Therefore $\dim(Im(T)) = \dim(V + W)$. Further rewriting our initial equation we see now that $\dim(V) + \dim(V) = \dim(\ker(T)) + \dim(V + W)$. As we have shown in part C we further know that $V \cap W \cong \ker(T)$, therefore from worksheet 12 problem 3 we know they they must have the same dimension. Therefore $\dim(V \cap W) = \dim(\ker(T))$. Thus we can lastly substitute this into the equation, and thus we have that $\dim(V) + \dim(V) = \dim(V \cap W) + \dim(V + W)$.

(f) Assume for contradiction that two $\mathbb{R}^5$ can intersect at a point. From part e we know that $\dim(V) + \dim(V) = \dim(V \cap W) + \dim(V +$

W). Let us call the first $\mathbb{R}^{\ntrianglelefteq}$ subspace A and the second B. Thus if two $\mathbb{R}^5$ sub spaces intersected at a point we know that $\dim(V \cap W) = 1$. Furthermore we know that $\dim(A) = 5$ and $\dim(B) = 5$. Furthermore from d we know that $\dim(V + W) = \dim(V) + \dim(W)$ therefore $\dim(A + B) = \dim(A) + \dim(B)$ thus $\dim(A + B) = 10$. Therefore we have that $5 + 5 = 10 + 1$ which is clearly not true, therefore we know that two subspaces in $\mathbb{R}^5$ cannot intersect at a point.

21. (a) Suppose that $Q \in \mathbb{R}^{n\times n}$ where Q is invertable. Recall that generally for matrices A, B, $(AB)^T = B^T A^T$ as shown in Homework 2 problem 5. Thus as Q is invertable we know that $QQ^{-1} = I_n$. Therefore $(QQ^{-1})^T = I_n^T \to (QQ^{-1})^T = I_n \xrightarrow{\dagger} {Q^{-1}}^T Q^T = I_n$. Where † follows from the fact that $(AB)^T = B^T A^T$. Therefore we can conclude that Q^T is invertable where $(Q^T)^{-1} = (Q^{-1})^T$

    (b) Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric. Therefore we know that $A^T = A$. Furthermore let $Q \in \mathbb{R}^{n\times n}$and recall from homework 2 that for matrices A, B, $(AB)^T = B^T A^T$. Therefore, $(QAQ^T)^T = Q^T(AQ^T)^T = Q^T A^T {Q^T}^T = QA^T Q^T \stackrel{\dagger}{=} QAQ^T$.Where † follows as A is symmetric and thus $A = A^T$. Thus QAQ^T is also symmetric.

    (c) Let $A, B \in V$, and let $S_Q : V \to V$ be the mapping defined by $S_Q(A) = QAQ^T$. Therefore $S_Q(A) + S_Q(B) = QAQ^T + QBQ^T = Q(A + B)Q^T = S_Q(A + B)$. Furthermore let $k \in \mathbb{R}$ therefore $kS_Q(A) = kQAQ^T \stackrel{\dagger}{=} QkAQ^T = S_Q(kA)$. Where † follows as scalar multiplication is commutative with matrix multiplication. Therefore as S_Q is closed under vector addition and scalar multiplication is is thus a linear transformation.

    (d) Suppose that S_Q is an Isomorphism. Therefore we know that S_Q must be a bijective linear transformation, and therefore surjective. Therefore for I_n in the target there exists some A in the source such that $S_Q(A) = I_n$ as we know S_Q is surjective. Therefore, we know that $QAQ^T = I_n$ for some A in the source. Thus as $Q(AQ^T) = I_n$ it is clear to see that Q is invertable.

    Suppose that Q is invertible. Therefore from part a we know that if Q is invertable then Q^T is invertable and that $(Q^T)^{-1} = (Q^-1)^T$. Suppose that $A \in ker(S_Q)$. Therefore $QAQ^T = 0$. Multiplying both sides by $(Q^T)^{-1}$ we have $QAQ^T(Q^T)^{-1} = 0(Q^T)^{-1} \xrightarrow{\dagger} QAI_n = 0 \to QA = 0$. Where † follows as if Q is invertable we have shown in part a that Q^T is also invertable such that $Q^T(Q^T)^{-1} = I_n$. Thus $QA = 0$ and therefore $ker(S_Q) = \{\vec{0}\}$. Thus by definition S_Q must be injective as $ker(S_Q) = \{\vec{0}\}$. Furthermore, as $S_Q : V \to V$, we know that $dim(Source(S_Q) = V$, and we have shown above that S_Q is injective and therefore $dim(Ker(S_Q)) = 0$. Thus by the rank nullity theorem $V = dim(Im(S_Q)) + 0$. Therefore we can conclude that $dim(Im(S_Q) = V$, and thus S_Q is surjective as the dimension of its image is equal to the source. Therefore we have shown that S_Q is bijective as it is both injective and surjective. Along with the fact that S_Q is a linear transformation found in part b, we can therefore conclude that S_Q is an isomorphsim.

Therefore as we have shown that if Q is invertable then S_Q is an isomorphism, and that if S_Q is an isomorphism that Q is invertable then we can conclude that S_Q is an isomorphism if and only if Q is invertable.

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22. (a) Let $S_Q : V \to V$ be the mapping defined by $S_Q(A) = QAQ^T$, and let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = Q^T$. Therefore $S_Q(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix}$. This is therefore an explicit formula for $S_Q(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix})$, given $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) As above, let $S_Q : V \to V$ be the mapping defined by $S_Q(A) = QAQ^T$, and let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Furthermore let $\mathcal{B}$ be the basis for V given in the hypothesis, and let $\begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{22} \end{bmatrix} \in V$. Therefore $\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}_{\mathcal{B}} = \mathcal{L}_{\mathcal{B}}(\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix}$. Next, let us calculate $[S_Q(\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix})]_{\mathcal{B}} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}$. Furthermore, looking for the matrix $[S_Q]_{\mathcal{B}}$ that takes $\begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}$, we can find the columns of such matrix by using the Generalized Key Theorem. It follows from definition of the Generalized Key Theorem then $[S_Q]_{\mathcal{B}} =$ $\begin{bmatrix} [S_Q(\vec{b_1})]_{\mathcal{B}} & [S_Q(\vec{b_2})]_{\mathcal{B}} & [S_Q(\vec{b_3})]_{\mathcal{B}} \end{bmatrix}$ where $[S_Q(\vec{b_1})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[S_Q(\vec{b_2})]_{\mathcal{B}} =$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $[S_Q(\vec{b_3})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus $[S_Q]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $[S_Q]_{\mathcal{B}} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix})_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}$ and $[S_Q(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix})]_{\mathcal{B}} \stackrel{\dagger}{=}$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}$. Where † follows from part a where we found an explicit formula $S_Q(A)$ for an arbitrary $A \in V$. Thus $[S_Q]_{\mathcal{B}} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix})_{\mathcal{B}} = [S_Q(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix})]_{\mathcal{B}}$

(c) Let $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Q^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Therefore $S_Q(A) = QAQ^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $= \begin{bmatrix} a^2a_{11} + 2aba_{12} + b^2a_{22} & aca_{11} + ada_{12} + bca_{12} + bda_{22} \\ aca_{11} + ada_{12} + bca_{12} + bda_{22} & c^2a_{11} + 2cda_{12} + d^2a_{22} \end{bmatrix}$. Thus $[S_Q(A)]_{\mathcal{B}} = \begin{bmatrix} a^2a_{11} + 2aba_{12} + b^2a_{22} \\ aca_{11} + ada_{12} + bca_{12} + bda_{22} \\ c^2a_{11} + 2cda_{12} + d^2a_{22} \end{bmatrix}$. Furthermore like in part B we know that $\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}_{\mathcal{B}} = \mathcal{L}_{\mathcal{B}}(\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix}$. using the Generalized key Theorem we can find the $\mathcal{B}$-matrix $[S_Q]_{\mathcal{B}}$ where $[S_Q]_{\mathcal{B}} = \begin{bmatrix} \\ [S_Q(\vec{b_1})]_{\mathcal{B}} & [S_Q(\vec{b_2})]_{\mathcal{B}} & [S_Q(\vec{b_3})]_{\mathcal{B}} \\ \\ \end{bmatrix}$. Therefore using $S_Q(A)$ from above we can calculate $S_Q(\vec{b_1}), S_Q(\vec{b_2})$, and $S_Q(\vec{b_3})$. Therefore $[S_Q(\vec{b_1})]_{\mathcal{B}} = \begin{bmatrix} a^2 & ac \\ ac & c^2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} a^2 \\ ac \\ c^2 \end{bmatrix}$ which is the first column of $[S_Q]_{\mathcal{B}}$. Likewise $[S_Q(\vec{b_2})]_{\mathcal{B}} = \begin{bmatrix} 2ab & ad + bc \\ ad + bd & 2cd \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2ab \\ ad + bd \\ 2cd \end{bmatrix}$ is the second column, and $[S_Q(\vec{b_3})]_{\mathcal{B}} = \begin{bmatrix} b^2 & bd \\ bd & d^2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} b^2 \\ bd \\ d^2 \end{bmatrix}$ as the third. Thus $[S_Q]_{\mathcal{B}} = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bd & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$ where $[S_Q]_{\mathcal{B}} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a^2a_{11} + 2aba_{12} + b^2a_{22} \\ aca_{11} + ada_{12} + bca_{12} + bda_{22} \\ c^2a_{11} + 2cda_{12} + d^2a_{22} \end{bmatrix}$.

(d) Let $Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = Q_0^T$. Therefore for some $A \in V$ $S_{Q_0}(A) = \vec{0}$. Thus, $dim(Im(S_{Q_0})) = 0$. Furthermore let $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, therefore from part a, given an arbitrary $A \in V$ we know that $S_{Q_1}(A) =$

$\begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $dim(Im(S_{Q_1})) = 1$. Lastly let $Q_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = Q_3^T$ therefore $S_Q(A) = Q_3 A Q_3^T = \begin{bmatrix} a_{22} & a_{12} + a_{21} \\ a_{12} + a_{21} & a_{12} + a_{11} + a_{12} + a_{21} + a22 \end{bmatrix}$. Therefore it follows that for Q_3 $\dim(Im(S_{Q_3}) = 3$.

(e) Assume for contradiction that $\dim(Im(S_Q) = 2$ where $Q : V \to V$. Therefore we can conclude that Q is not invertible. Therefore given $a, c\lambda \in \mathbb{R}$ we know that $Q = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix}$ or $Q = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$. Further suppose that $A \in V$ where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. Therefore when $Q = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix}$, $S_Q(A) = QAQ^T = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix} = \begin{bmatrix} a(aa_1 + \lambda aa_3) + c(aa_2 + \lambda aa_4 & \lambda a(aa_1 + \lambda aa_3) + \lambda c(aa_2 + \lambda aa_4) \\ a(ca_1 + \lambda ca_3) + c(ca_2 + \lambda c + a_4) & \lambda a(ca_1 + \lambda ca_3) + \lambda c(ca_2 + \lambda ca_4) \end{bmatrix}$ Therefore given $Q = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix}$ $\dim(Im(S_Q)) = 4$ which is not true given our hypothesis. Furthermore, now suppose that $Q = \begin{bmatrix} & \lambda a & a \\ \lambda c & c & \end{bmatrix}$. Therefore $S_Q(A) = QAQ^T = \begin{bmatrix} \lambda a & a \\ \lambda c & c \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \lambda a & a \\ \lambda c & c \end{bmatrix} = \begin{bmatrix} \lambda a(\lambda aa_1 + aa_3) + \lambda c(\lambda aa_2 + aa_4) & a(\lambda aa_1) + c(\lambda aa_2 + aa_4) \\ \lambda a(\lambda ca_1 + ca_3) + \lambda c(\lambda ca_2 + ca_4) & a(\lambda ca_1 + ca_3) + c(\lambda ca_2 + ca_4) \end{bmatrix}$ As each of these elements are unique it follows that when $Q = \begin{bmatrix} & a \\ \lambda c & c \end{bmatrix}$ $dim(Im(S_Q)) = 4$ which is not true given our hypothesis. Therefore as this contradicts both Q it follows that our assumption must be false and $dim(Im(Sq)) \neq 2$.

23. (a) Let $B = \{e^t, te^t, te^t, t^2e^tt^3e^t\}$. Suppose that $c_1e^t + c_2te^t + c_3t^2e^t + c_4t^3e^t = 0$. First let $t = 0$ therefore $c_1(1) + c_2(0) + c_3(0) + c_4(0) = 0$. Therefore it is clear that $c_1 = 0$. Furthermore, as e^t is never 0 we can divide by e^t on both sides rewriting this relation as $c_1 + c_2t + c_3t^2 + c_4t^4 = \frac{0}{e^t}$. Therefore it is clear that relation is of powers of t, and we know that the powers of t are linearly independent. Therefore it is clear for this relation to hold $c_1 = c_2 = c_3 = c_4 = 0$, therefore this relation is trivial. Given in the hypothesis it is said that B spans C^∞. Therefore as $\{e^t, te^t, te^t, t^2e^tt^3e^t\}$ is linearly independent and spans C^∞ then $\{e^t, te^t, te^t, t^2e^tt^3e^t\}$ is a basis.

(b) First note that given $f = c_1e^t + c_2te^t + c_3t^2e^t + c_4t^3e^t$, or a arbitrary linear combination of the basis we found in part a. Therefore $f' = c_1e^t + c_1te^t + c_2t^2e^t + 2c_2te^t + c_3t^3e^t + 3c_3t^2e^t + c_4t^4e^t + 4c_4t^3e^t \implies c1e^t + (c_1 + 2c_2)te^t + (c_2 + 3c_3)t^2e^t + (c_3 + 4c_4)t^3e^t$. Therefore from rewriting it it is clear to see that f' can be made from a linear combination of the basis given in part a. Therefore f' in in the span of the basis and thus $f' \in V$.

(c) To find $\mathcal{B}$-matrix $[D]_{\mathcal{B}}$ of D. We first know that $D : V \to V$ where $D(f) = f' \forall f \in V$.
Therefore B-matrix $[D]_{\mathcal{B}} = \big[[D(e^t)]_B \quad [D(te^t)]_B \quad [D(t^2e^t)]_B \quad [D(t^3e^t)]_B\big]$.
It follows that $D(e^t) = e^t$. Therefore writing this in terms of the basis B we have $[D(e^t)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. $D(te^t) = te^t + e^t$, and thus $[D(te^t)] = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ as $te^t + e^t = e^t(1) + te^t(1) + t^2e^t(0) + t^3e^t(0)$.
$D(t^2e^t) = t^2e^t + 2te^t$ Therefore we can write this as a linear combination of the basis such that $t^2e^t + 2te^t = e^t(0) + te^t(2) + t^2e^t(1) + t^3e^t(0)$.
Thus $[D(t^2e^t)] = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Lastly, $D(t^3e^t) = t^3e^t + 3t^2e^t$, and we can then write this as a linear combination of our basis given in part a such that $t^3e^t + 3t^2e^t = e^t(0) + te^t(0) + t^2e^t(3) + t^3e^t(1)$Thus $[D(t^3e^t)] = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$. Therefore we have computed each of the columns of the B-matrix of D. Combining this we find that $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(d) I propose that $[D]_B^{-1}$ is $\begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. It follows that $[D]_B^{-1}[D]_B =$
$\begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_n$.

(e) Let $[D]_B^{-1}$ be $\begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which we found in part d. Furthermore let $x = te^t - t^2e^t + 7t^3e^t$. Therefore $[x]_B$ is the coordinates of the linear combination of the basis found in part a such that the linear combination is equal to x. Therefore $te^t - t^2e^t + 7t^3e^t =$

$e^t(0) + te^t(1) + t^2e^t(-1) + t^3e^t(7)$. Thus $[x]_b = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 7 \end{bmatrix}$ Therefore the anti derivative of x can be computed by multiplying $[D]_B[x]_B = \begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -45 \\ 45 \\ -22 \\ 7 \end{bmatrix}$. Writing this as a linear combination of the basis above, we find $e^t(-45) + te^t(45) + t^2e^t(-22) + t^3e^t(7) = -45e^t + 45te^t - 22t^2e^t + 7t^3e^t$. Thus $-45e^t + 45te^t - 22t^2e^t + 7t^3e^t$ is the anti derivative of $te^t - t^2e^t + 7t^3e^t$.

24. (a) Let F_λ be the subset of S consisting of all sequences $(x_0, x_1, x_2, ...)$ satisfying $x_{n+2} = x_{n+1} - \lambda x_n, \forall n \geq 0$. Therefore when $\lambda = -1$ and $x_0 = 0, x_1 = 1$, $x_2 = 1 - -1(0) = 1$. Therefore $x_3 = 1 + 1 = 2$. Continuing with this we find that in general each term n where $n \geq 0$ follows the pattern of $x_{n+2} = x_{n+1} + x_n$ which is the relation of the Fibonacci sequence. Continuing with elements in this general case we find that this forms the sequence $(0, 1, 1, 2, 3, 5...) \in \mathcal{F}_\lambda$ which is the Fibonacci sequence.

(b) Fix an arbitrary λ, furthermore let z be the zero sequence in S where $z = (0, 0, 0....)$. Thus as $0 = 0 - \lambda(z_0)\forall n \geq 0$ is true, then we can conclude that $z \in \mathcal{F}_\lambda$. Therefore the zero sequence can be found in $\mathcal{F}_\lambda$. Furthermore suppose that $x = (x_0, x_1, x_2, ...) \in \mathcal{F}_\lambda$ and $y = (y_0, y_1, y_2, ...) \in \mathcal{F}_\lambda$. It follows that $x+y = (x_0+y_o, x_1+y_1, x_2+y_2, ...)$. Therefore $(x + y)_{n+2} - (x + y)_{n+1} - \lambda(x + y)_n \forall n \geq 0$ Furthermore, as $x, y \in \mathcal{F}_\lambda$ then $x_{n+2} = x_{n+1} - \lambda x_n, \forall n \geq 0$, and $y_{n+2} = y_{n+1} - \lambda y_n \forall n \geq 0$ then $x_{n+2} + y_{n+2} = x_{n+1} - \lambda x_n + y_{n+1} - \lambda y_n = y_{n+1} + x_{n+1} - \lambda(x_n + yn)$ Therefore it is clear that $x_{n+2} + y_{n+2} = y_{n+1} + x_{n+1} - \lambda(x_n + yn) = (x+y)_{n+1} - \lambda(x+y)_n = (x+y)_{n+2}$. Therefore $\mathcal{F}_\lambda$ is closed under vector addition. Furthermore let $k \in \mathbb{R}$ Thus $k(x) = k(x_0, x_1, x_2, ...) = (kx_0, kx_1, kx_2, ...)$ such that $k(x_{n+2}) = k(x_n + 1) - k(\lambda x_n)$, and $k\underline{x} \in \mathcal{F}_\lambda$ where $k\underline{x} = ((kx)_0, (kx)_1, (kx)_2, ...)$ such that $(kx)_{n+2} = (kx)_{n+1} - \lambda(kx)_n$. It is that $(kx)_{n+2} = kx_{n+2}$ such that $(kx)_{n+2} = (kx)_{n+1} - \lambda(kx)_n = k(x_n + 1) - k(\lambda x_n) = k(x_{n+2})$. Therefore $\mathcal{F}_\lambda$ is closed under scalar multiplication. Therefore as $\mathcal{F}_\lambda$ contains $\underline{0}$, closed under addition, and preserves scalar multiplication $\mathcal{F}_\lambda$ is a subspace of S.

(c) Let $\underline{s}, \underline{t} \in \mathcal{F}_\lambda$ where $\underline{s} = (1, 0, s_2, s_3, ...)$ and $\underline{t} = (0, 1, t_2, t_2, ...)$. Suppose that $c_1\underline{s} + c_2\underline{t} = \underline{(}0)$ where $c_1, c_2 \in \mathbb{R}$. Then this is equal to $c_1(s_0, s_1, s_2, ...) + c_2(t_0, t_1, t_2, t_3, ...) = \underline{0}$
$\implies (c_1s_o, c_1s_1, ...) + (c_2t_0, c_2t_1, ...) = \underline{0}$
$\implies)c_1s_0 + c_2t_0, c_1s_1 + c_2t_1, ...) = \underline{0}$ Since we know that $s_0 = 1, s_1 = 0, t_0 = 0, t_1 = 1$, substituting this into the relation above we find that $(c_1(1) + c_2(0), c_1(0) + c_2(1), ...) = \underline{0}$

$\implies (c_1, c_2, ...) = \underline{0}$
$\implies (c_1, c_2, ...) = (0, 0, ...)$. Therefore $c_1 = c_2 = 0$ and the relation is true for only the trivial solution. Therefore $(\underline{s}, \underline{t})$ are linearly independent.

Suppose that $\underline{x} \in \mathcal{F}_\lambda$, then $\underline{x} = (x_0, x_1, x_2, ..)$. Therefore given the set $\{\underline{s}, \underline{t}\}$, we can write $\underline{x}$ as a linear combination of the set such that $\underline{x} = a\underline{s} + b\underline{t}$. Suppose that $a = x_0$, and $b = x_1$. Therefore $\underline{x} = x_0(\underline{s} + x_1(\underline{t})$
$\implies \underline{x} = x_0(1, 0, ...) + x_1(0, 1, ...)$
$\implies \underline{x} = (x_0, 0, ..) + (0, x_1, ...)$
$\implies \underline{x} = (x_0, x_1, ...)$. Therefore as an arbitrary element $\underline{x} \in \mathcal{F}_\lambda$ can be found as a linear combination of $\{\underline{s}, \underline{t}\}$, we can conclude that $\{\underline{s}, \underline{t}\}$ spans $\mathcal{F}_\lambda$. Therefore, as $\{\underline{s}, \underline{t}\}$ spans $\mathcal{F}_\lambda$ and $\{\underline{s}, \underline{t}\}$ is linearly independent we can conclude that $\{\underline{s}, \underline{t}\}$ is a basis of $\mathcal{F}_\lambda$. Therefore as $\{\underline{s}, \underline{t}\}$ is a basis of $\mathcal{F}_\lambda$, then the number of elements in the basis is the dimension. Thus $\dim(\mathcal{F}_\lambda) = 2$, as the basis shown above for $\mathcal{F}_\lambda$ contains two elements.

(d) Begin by noting that the geometric sequence in $\mathcal{F}_\lambda$ is of the form $(1, r, r^2, r^3, ...)$. Furthermore fix an arbitrary λ. To be within $\mathcal{F}_\lambda$ the condition $x_{n+2} = x_{n+1} - \lambda x_n$ must be met. Therefore if such a sequence exists then $r^2 - r + \lambda = 0$. Using the quadratic formula we can compute for what λ this is true. Therefore $r = \frac{1 \pm \sqrt{1-4\lambda}}{2}$. This is only in $\mathbb{R}$ when $1 - 4\lambda \geq 0 \implies \lambda < \frac{1}{4}$, Thus when $-\infty < \lambda < \frac{1}{4}$, $r \in \mathbb{R}$ where r satisfies the condition for a geometric sequence to be in $\mathcal{F}_\lambda$. Therefore when $-\infty < \lambda < \frac{1}{4}$, $\mathcal{F}_\lambda$ contains the geometric sequence.

(e) Suppose that $\lambda < \frac{1}{4}$ then from part d we know that $r = \frac{1 \pm \sqrt{1-4\lambda}}{2}$ wh . Thus $r_+ = \frac{1+\sqrt{1-4\lambda}}{2}$, and $r_- = \frac{1-\sqrt{1-4\lambda}}{2}$. Therefore we have the set of two geometric sequences of r_-, r_+,
$\{(1, r_-, r_-^2, r_-^3, ...), (1, r_+, r_+^2, r_+^3, ...)\}$.
Suppose that $c_1(1, r_-, r_-^2, r_-^3, ...) + c_2(1, r_+, r_+^2, r_+^3, ...) = \underline{0}$
$\implies (c_1, c_1 r_-, c_1 r_-^2, ...) + (c_2, c_2 r_+, c_2 r_+^2, ...) = \underline{0}$
$\implies (c_1 + c_2, c_1 r_- + c_2 r_+, c_1 r_-^2 + c_2 r_+^2, ...) = \underline{0}$
$\implies (c_1 + c_2, c_1 r_- + c_2 r_+, c_1 r_-^2 + c_2 r_+^2, ...) = (0, 0, ...)$. Therefore we know that $c_1 + c_2 = 0 \implies c_1 = -c_2$. and that $c_1 r_- + c_2 r_+ = 0 \implies c_1 r_- = -c_2 r_+$. Therefore as $c_1 = -c_2, c_1 r_- = -c_2 r_+ \implies c_1 r_- = c_1 r_+$. Thus if $c_1, c_2 \neq 0$ then $r_- = r_+$. However from our definition of r_- and r_+ given above we know this is not true, thus this contradicts the fact that $c_1, c_2 \neq 0$. Therefore $c_1, c_2 = 0$ and thus $c_1(1, r_+, r_+^2, r_+^3, ...) + c_2(1, r_-, r_-^2, r_-^3, ...) = \underline{0}$ is the trivial relation. Thus $\{(1, r_-, r_-^2, r_-^3, ...), (1, r_+, r_+^2, r_+^3, ...)\}$ is linearly independent. As we have shown that the dimension of $\mathcal{F}_\lambda$ is 2 in part c and we know that $\{(1, r_-, r_-^2, r_-^3, ...), (1, r_+, r_+^2, r_+^3, ...)\}$ is linearly independent we can conclude that in fact $\{(1, r_-, r_-^2, r_-^3, ...), (1, r_+, r_+^2, r_+^3, ...)\}$ is a

basis of $\mathcal{F}_\lambda$ when $\lambda < \frac{1}{4}$.

(f) From part e we showed that $\{(1, r_-, r_-^2, r_-^3, ...), (1, r_+, r_+^2, r_+^3, ...)\}$ forms of a basis of $\mathcal{F}_\lambda$ when $\lambda < \frac{1}{4}$ and $r_+ = \frac{1+\sqrt{1-4\lambda}}{2}$, and $r_- = \frac{1-\sqrt{1-4\lambda}}{2}$. Further note that in part a we find Fibonacci sequence is in $\mathcal{F}_{-1}$, where $\lambda = -1$. Thus when $\lambda = -1, r_+ = \frac{1+\sqrt{1-4(-1)}}{2} = \frac{1+\sqrt{5}}{2}$, and $r_- = \frac{1-\sqrt{1-4(-1)}}{2} = \frac{1-\sqrt{5)}}{2}$. Thus the basis for $\mathcal{F}_{-1}$ given the basis we found in part e is $\{(1, \frac{1+\sqrt{5}}{2}, (\frac{1+\sqrt{5}}{2})^2, ...), (1, \frac{1-\sqrt{5}}{2}, (\frac{1-\sqrt{5}}{2})^2, ...)\}$.

(g) From part f we found that that $\{(1, \frac{1+\sqrt{5}}{2}, (\frac{1+\sqrt{5}}{2})^2, ...), (1, \frac{1-\sqrt{5}}{2}, (\frac{1-\sqrt{5}}{2})^2, ...)\}$ is a basis consisting of geometric sequences for $\mathcal{F}_{-1}$. Therefore we know that the Fibonacci sequence, $(0, 1, 1, 3, 5..) = a(1, \frac{1+\sqrt{5}}{2}, (\frac{1+\sqrt{5}}{2})^2, ...) + b(1, \frac{1-\sqrt{5}}{2}, (\frac{1-\sqrt{5}}{2})^2, ...)$ where $a, b \in \mathbb{R}$. Combining the sequences we find that $(0, 1, 1, 3, 5..) = (a, a\frac{1+\sqrt{5}}{2}, a(\frac{1+\sqrt{5}}{2})^2, ...) + (b, b\frac{1-\sqrt{5}}{2}, b(\frac{1-\sqrt{5}}{2})^2, ...)$ $\implies (0, 1, 1, 3, 5..) = (a+b, a\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2}, a(\frac{1+\sqrt{5}}{2})^2 + b(\frac{1-\sqrt{5}}{2})^2, ...)$. Therefore looking at the first elements we know that $a + b = 0 \implies a = -b$, furthermore from looking at the second elements of sequences we know that $\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2} = 1 \implies \frac{a}{2} + \frac{a\sqrt{5}}{2} + \frac{b}{2} + \frac{-b\sqrt{5}}{2} = 1$
$\implies \frac{a}{2} + \frac{a\sqrt{5}}{2} + \frac{-a}{2} + \frac{a\sqrt{5}}{2} = 1$
$\implies \frac{2a\sqrt{5}}{2} = 1$
$\implies a\sqrt{5} = 1$
$\implies a = \frac{1}{\sqrt{5}}$. Furthermore as $a = -b$ then $-b = -\frac{1}{\sqrt{5}}$. Thus we found that in terms of the basis given in f we can write $(0, 1, 1, 3...) = \frac{1}{5}(1, \frac{1+\sqrt{5}}{2}, (\frac{1+\sqrt{5}}{2})^2, ...) + \frac{-1}{5}(1, \frac{1-\sqrt{5}}{2}, (\frac{1-\sqrt{5}}{2})^2, ...)$. Thus the nth term of the fibonacci sequence can be written as $\frac{1}{5}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{5}(\frac{1-\sqrt{5}}{2})^n \overset{\dagger}{=} \frac{\frac{(1+\sqrt{5})}{2}^n - \frac{(1-\sqrt{5})}{2}^n}{\sqrt{5}}$. Where † follows from factoring out $\frac{1}{5}$. Thus we have shown that the nth element of the Fibnonacci sequence is $\frac{1}{5}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{5}(\frac{1-\sqrt{5}}{2})^n = \frac{\frac{(1+\sqrt{5})}{2}^n - \frac{(1-\sqrt{5})}{2}^n}{\sqrt{5}}$.

25. (a) First note that $S_{c\to a} = L_a \circ L_c^{-1}$. Furthermore, that $S_{b\to a} = L_a \circ L_b^{-1}$ and $S_{c\to b} = L_b \circ L_c^{-1}$. Therefore it follows that $S_{b\to a} \circ S_{c\to b} = L_a \circ L_b^{-1} \circ L_b \circ L_c^{-1} = L_a \circ id_w \circ L_c^{-1} = L_a \circ L_c^{-1} = S$. Thus $S_{c\to a} = S_{b\to a} \circ S_{c\to b}$.

(b) As in part C we know that generally $S_{c\to a} = L_a \circ L_c^{-1}$, $S_{b\to c} = L_c \circ L_b^{-1}$, and $S_{A\to B} = L_b \circ L_a^{-1}$. Therefore it follows that $S_{c\to a} \circ S_{b\to c} \circ S_{a\to b}$
$= L_a \circ L_c^{-1} \circ L_c \circ L_b^{-1} \circ L_b \circ L_a^{-1}$
$= L_a \circ L_c^{-1} \circ L_c \circ id_w \circ L_a^{-1}$
$= L_a \circ id_w \circ L_c \circ id_w \circ L_a^{-1}$
$= L_a \circ L_a^{-1}$
$= id_w$. Therefore it that $S_{c\to a} \circ S_{b\to c} \circ S_{c\to a} = id_w$, that $S_{c\to a} \circ S_{b\to c} \circ S_{c\to a} = I_n$.

26. (a) Let $V = span(\vec{e_1})$ and $W = span(\vec{e_1}, \vec{e_2})$. Therefore it follows that $dim(V) = 1$ and $dim(W) = 2$ By definition $V^\perp = \{v' \in \mathbb{R}^3 | v' \cdot v = 0, \forall v \in V\}$. Let $x \in V^{\{\perp\}}$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Therefore it follows by definition that $x \cdot e_1 = 0$ and thus $x_1 = 0$. Therefore x_2, x_3 can be anything and thus we can use to find a basis for $W^\perp$ is $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. Similarly let $y \in W^\perp$ where $W^\perp$ is defined as being orthogonal to every basis of W, let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. As W $spans(\vec{e_1}, \vec{e_2})$ it follows from definition that $y \cdot e_1 = 0$ such that $y_1 = 0$, and similarly $y \cdot e_2$ such that $y_2 = 0$, therefore it leaves that y_3 can be any real number and thus the basis of $W^\perp = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. This idea can be further seen through the use of visual aid shown below this problem. Untitled Page (2).pdf

(b) Let V be an arbitrary vector space. From theorem 5.1.8 from the textbook we know that $\dim(V) + \dim(V^\perp = n$ where $V^\perp = \{v' \in \mathbb{R}^n | v' \cdot v = 0, \forall v \in V\}$ as defined in 5.1.7. Furthermore that $V^\perp$ is kernel of the orthogonal projection onto V. Therefore it is clear that V is the image of the orthogonal projection onto V. It follows then $\dim(V) = n - \dim(V^\perp)$. Furthermore $V^{\perp^\perp} = \{u \in \mathbb{R}^n | u \cdot v' = 0, \forall v' \in V^\perp\}$ and note that similarly $V^{\perp^\perp}$ is the image of the orthogonal projection onto V. Therefore using 5.1.8 as above we see that $\dim(V^{\perp^\perp}) = n - \dim(V^\perp) = \dim(V)$. Thus $\dim(V^{\perp^\perp}) = \dim(V)$.

(c) Suppose that $x \in V$, therefore from our jamboard 16 we know that $V^\perp = \{v' \in \mathbb{R}^n | v' \cdot v = 0, \forall v \in V\}$. Furthermore $V^{\perp^\perp} = \{u \in \mathbb{R}^n | u \cdot v' = 0, \forall v' \in V^\perp\}$. Notice then that an element is in $V^{\perp^\perp}$ if it is orthogonal to all elements of $V^\perp$. From above we know that x is, therefore $x \in V^{\perp^\perp}$, this along with the fact that $\dim(V) = \dim(V^{\perp^\perp})$ which we showed in part b we can conclude that $V = V^{\perp^\perp}$.

(d) Suppose that $V \subset W$, and let $x \in V$. Therefore it follows that $x \in W$. Furthermore let $W^\perp - \{w \in \mathbb{R}^n | w \cdot v = 0, \forall v \in V\}$, suppose that $u \in W^\perp$ therefore by definition $\forall x \in V$, $u \cdot x = 0$, and thus by definition $u \in V^\perp$, therefore $W^\perp \subset V^\perp$.

Secondly suppose that $W^\perp \subset V^\perp$. We showed directly above that if $A \subset B$, then $B^\perp \subset A \perp$. Let $A = W^\perp$ and let $B = V^\perp$ Therefore it follows as if $A \subset B$, then $B^\perp \subset A \perp$ as shown above, and thus $V^{\perp^\perp} \subset W^{\perp^\perp}$. From part C we can know that $V^{\perp^\perp} = V$ and $W^{\perp^\perp} = W$ therefore if $W^\perp \subset V^\perp \in V^\perp$ then $V \subset W$.

As we have shown both directions of this proof we can conclude that $V \subset W$ if and only if $W^{\perp} \subset V^{\perp}$.

27. (a) Let $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$ and let $V = span \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Furthermore let $\vec{w} \in V$ where $\vec{w} = \begin{bmatrix} 3w_1 \\ 2w_1 \end{bmatrix}$, and $w_1 \in \mathbb{R}$. Therefore it follows that $A\vec{w} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} 3w_1 \\ 2w_1 \end{bmatrix} = \begin{bmatrix} -18w_1 - 60w_1 \\ -90w_1 + 38w_1 \end{bmatrix} = \begin{bmatrix} 3(-6w_1 - 20w_1) \\ 2(-45w_1 + 19w_1) \end{bmatrix}$. It is then clear that $\begin{bmatrix} 3(-6w_1 - 20w_1) \\ 2(-45w_1 + 19w_1) \end{bmatrix} \in V$, and as $\vec{w}$ is an arbitrary element of V it follows that $\forall \vec{v} \in V$, $A\vec{v} \in V$

(b) Let $V^{\perp} = \{\vec{w} | w \cdot v = 0, v \in V\}$. I propose that $\begin{bmatrix} -2 & 3 \end{bmatrix}$ is a basis for $V^{\perp}$. Thus as the basis spans, we know that for all $c_1 \in \mathbb{R}$ that there exists a matrix $c \in V^{\perp}$ where $c = \begin{bmatrix} -2c_1 \\ 3c_1 \end{bmatrix}$. As we know that $V = span(\begin{bmatrix} 3 \\ 2 \end{bmatrix})$ then we know for all $b_1 \in \mathbb{R}$ that there exists some $\vec{b} = \begin{bmatrix} 3b_1 \\ 2b_1 \end{bmatrix} \in V$. Thus when $b_1 = c_1$ we see that $ba = 0$ and therefore it holds that this is a basis for $V^{\perp}$. Furthermore notice $Ab = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} -2c_1 \\ 3c_1 \end{bmatrix} = \begin{bmatrix} 12c_1 + 90c_1 \\ 60c_1 + 57c_1 \end{bmatrix} = \begin{bmatrix} -2(-6c_1 + -45c_1) \\ 3(20c_1 + 19c_1) \end{bmatrix}$. Therefore it is clear that you can also write this is a linear combination of the basis, and thus $Ac \in V^{\perp}$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$. Let B be a basis of $\mathbb{R}^2$ where $B = \{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. As this basis contains two elements and is linearly independent, this can be seen through inspection that neither of these elements are a multiple of each other, that this is then a basis for $\mathbb{R}^2$. Therefore $[T]_B = [[T(\vec{b_1})]_B, [T(\vec{b_2})]_B]$ and it follows that $T(b_1] = A\vec{b_1} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -78 \\ -52 \end{bmatrix}$ Thus it follows that $[T(\vec{b_1})]_B = \begin{bmatrix} -26 \\ 0 \end{bmatrix}$ as $-26 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ -52 \end{bmatrix}$. Secondly $T(\vec{b_2}) = A\vec{b_2} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ 117 \end{bmatrix}$ Thus it follows that $[T(\vec{b_1})]_B = \begin{bmatrix} 0 \\ 39 \end{bmatrix}$ as $0 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 39 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ 117 \end{bmatrix}$. Therefore $[T]_B = \begin{bmatrix} -26 & 0 \\ 0 & 39 \end{bmatrix}$.

(d) Let $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$ be a 2×2 matrix. Let $n = 1$ then it is clear that

$\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^1 = \begin{bmatrix}\gamma_1^1 & 0\\ 0 & \gamma_2^1\end{bmatrix}$. Suppose that $\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^k = \begin{bmatrix}\gamma_1^k & 0\\ 0 & \gamma_2^k\end{bmatrix}$, then we can show that $\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^{k+1} = \begin{bmatrix}\gamma_1^{k+1} & 0\\ 0 & \gamma_2^{k+1}\end{bmatrix}$ by the fact that $\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^{k+1} = \begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^k \begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix} = \begin{bmatrix}\gamma_1^k & 0\\ 0 & \gamma_2^k\end{bmatrix}\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix} = \begin{bmatrix}\gamma_1^{k+1} & 0\\ 0 & \gamma_2^{k+1}\end{bmatrix}$, and thus $\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^{k+1} = \begin{bmatrix}\gamma_1^{k+1} & 0\\ 0 & \gamma_2^{k+1}\end{bmatrix}$. Therefore through the inductive process we have shown that $\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}^n = \begin{bmatrix}\gamma_1^n & 0\\ 0 & \gamma_2^n\end{bmatrix}$ $\forall n \geq 1$. Thus given $[T]_B$ found in part b where $[T]_B = \begin{bmatrix}-26 & 0\\ 0 & 39\end{bmatrix}$, it follows from our proof above $[T^{10}]_B = [T]_B^{10}$ and thus $[T^{10}]_B = \begin{bmatrix}-26^{10} & 0\\ 0 & 39^{10}\end{bmatrix}$.

(e) Let us first define the change of matrix $S_{B\to E}$ which is equal to $[[\vec{b_2}]_E, [\vec{b_2}]_E]$ Let B refer to the B basis from part c and let E be the standard basis. Therefore it follows that $[\vec{b_1}]_E = \begin{bmatrix}3\\2\end{bmatrix}$ as $3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$. Secondly $[\vec{b_2}]_E = \begin{bmatrix}-2\\3\end{bmatrix}$ as $-2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}$. Therefore $S_{B\to E} = \begin{bmatrix}3 & -2\\ 2 & 3\end{bmatrix}$. Therefore as this is 2×2 matrix we can easily computer the inverse $S_{B\to E}^{-1} = \begin{bmatrix}\frac{3}{12} & \frac{2}{13}\\ \frac{-2}{13} & \frac{3}{13}\end{bmatrix}$, Furthermore look below to problem $4a$ and note that B^k is similar to C^k meaning that that $B^k = S^{-1}CS$ where S is some change of basis function. Therefore it follows from Theorem 4.3.5 and our proof from problem $4a$ that when $C = [T]_B^1 0$ and $S = S_{B\to E}$ such that $[T]_E = S^{-1}[T]_B S = \begin{bmatrix}\frac{3}{12} & \frac{2}{13}\\ \frac{-2}{13} & \frac{3}{13}\end{bmatrix}\begin{bmatrix}-26^{10} & 0\\ 0 & 39^{10}\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}\frac{3}{12} & \frac{2}{13}\\ \frac{-2}{13} & \frac{3}{13}\end{bmatrix}\begin{bmatrix}(-26)^{10}3 & -2(-26)^{10}\\ 39^{10}2 & 39^{10}3\end{bmatrix}$ $= \frac{1}{13}\begin{bmatrix}9((-26)^{10}) + 4(39^{10}) & 6(-26^{10}) + 6(39^{10})\\ 6((-26)^{10}) + 6(39^{10}) & 9(-26^{10}) + 4(39^{10})\end{bmatrix}$. Thus $[T]_E^{10} = \frac{1}{13}\begin{bmatrix}9((-26)^{10}) + 4(39^{10}) & 6(-26^{10}) + 6(39^{10})\\ 6((-26)^{10}) + 6(39^{10}) & 9(-26^{10}) + 4(39^{10})\end{bmatrix}$ and since from part D we showed that $[T^{10}]_E = [T]_E^{10}$ it follows that $[T^{10}]_E = \frac{1}{13}\begin{bmatrix}9((-26)^{10}) + 4(39^{10}) & 6(-26^{10}) + 6(39^{10})\\ 6((-26)^{10}) + 6(39^{10}) & 9(-26^{10}) + 4(39^{10})\end{bmatrix}$.

28. (a) Let B, C be defined as in the question header. B is Similiar to C as it follows from definition 4.3.5 from the textbook that $B = S^{-1}CS$ where S is the change of basis function between C and B. Furthermore for B^n note that this is equal to $B_1 \circ B_2 \circ B_3 ... \circ B_n$
$= S_1^{-1} \circ C_1 \circ S_1 \circ S_2^{-1} \circ C_2 \circ S_2 \circ S_3^{-1} \circ C_3 \circ S_3 ... \circ S_n^{-1} \circ C_n \circ S_n$

$= S_1^{-1} \circ C_1 \circ I_n \circ C_2 \circ I_n ... \circ C_n \circ S_n$
$= S_1^{-1} \circ C_1 \circ C_2 ... \circ C_n \circ S$
$= S^{-1}C^n \circ S$. Thus $\forall n \geq 1$ B^n is similiar to C^n.

(b) False.
Counterexample, let B, C be bases of the vector space P_1. Fix $B = (1, x)$ and $C = (x, 1)$. Furthermore let $T : P_1 \to P_1$ where T is defined by f'. Therefore $[T]_B = [[T(1)]_B, [T(x)]_B] = [[0]_B, [1]_B] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $[T]_C = [[T(x)]_C, [T(1)]_C] = [[1]_C, [0]_C] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Notice that $ker([T]_C) = \{\begin{bmatrix} 0 \\ x \end{bmatrix} | x \in \mathbb{R}\}$ and $ker([T]_B) = \{\begin{bmatrix} 0 \\ y \end{bmatrix} | y \in \mathbb{R}\}$. Therefore it is clear that $ker([T]_C) \neq ker([T]_A)$.

(c) Let $B = \{b_1, b_2, ...b_n\}$ and $C = \{c_1, c_2, ...c_n\}$. Let $T : V \to V$. Let $\gamma : ker([T]_B) \to ker([T]_C)$ where γ is defined as the change of basis matrix $S_{B\to C}$. Let $u \in ker([T]_B)$ such that $[T]_B[u]_B = 0 \overset{\dagger}{\Longrightarrow}$ $(S_{B\to C}^{-1}[T]_C S_{B\to C})[u]_B = 0$
$\overset{\pi}{\Longrightarrow} (S_{B\to C}^{-1}[T]_C) S_{B\to C})[u]_B = 0$
$\Longrightarrow [T]_B S_{B\to C} = 0$. Therefore it is clear that γ is well defined linear transformation between $ker([T]_B)$ and $ker([T]_C)$. Therefore as γ is a well defined and an invertable linear transformation by definition, we can conclude it is an ismorphism. As isomorphism preserve dimensions we can then conclude that $\dim([T]_B) = \dim([T]_C)$.

29. (a) Let $T : U \to W$, and let $B = (u_1, u_2, ...u_k)$ be a basis for the source U, and $C = (w_1, ...w_d)$ be a basis for W. Furthermore let $L_B : U \to \mathbb{R}^k$, $L_C : W \to \mathbb{R}^d$, and lastly $T' : \mathbb{R}^k \to \mathbb{R}^d$ where the condition that $T' \circ L_b = L_c \circ T$ holds. I propose that $T' = L_C \circ T \circ L_B^{-1}$ which is a linear transformation as it is a composition of linear transformations which thus must be a linear transformation itself. This transformation is from from $\mathbb{R}^\urcorner \to \mathbb{R}^d$ and holds the condition that $T' \circ L_B = L_C \circ T =$ as $L_C \circ T \circ L_B^{-1} \circ L_B = L_C \circ T \circ id = L_C \circ T$.

(b) Let $T : U \to W$, and let $B = (u_1, u_2, ...u_k)$ be a basis for the source U and $C = (w_1, ...w_d)$ be a basis for W. Furthermore let $L_B : U \to \mathbb{R}^k$, $L_C : W \to \mathbb{R}^d$, and lastly $T' : \mathbb{R}^k \to \mathbb{R}^d$ where the condition that $T' \circ L_b = L_c \circ T$ holds. From part a we found that there exists some T' such that $T' \circ L_b = L_c \circ T$. Let $[T]_{(B,C)}$ be the standard matrix of T' by the key theorem. This means for $T'(\vec{x}) = [T]_{(B,C)}\vec{x}$, $\forall \vec{x} \in \mathbb{R}^k$. We can alternatively write what we found from part a to be that $\forall u \in U$ $T'(L_B(u)) = L_C(T(u))$. It follows that
$T'(L_B(u)) = L_C(T(u))$
$\Longrightarrow T'(L_B(u)) = [T(u)]_C$
$\Longrightarrow T'([u]_B) = [T(u)]_C$
$\overset{\dagger}{\Longrightarrow} [T]_{(B,C)}[U]_B = [T(u)]_C$. Where $\dagger$ follows from our definition of

$[T]_{(B,C)}$ as the standard matrix of T' above. Thus, $[T]_{(B,C)}[U]_B = [T(u)]_C$

(c) As we saw in part b, $[T]_{(B,C)}$ is the standard matrix of the linear transformation L' where $T' : \mathbb{R}^k \to \mathbb{R}^d$, from the key theorem we know that $[T]_{(B,C)}\vec{e_1}$ is the first column of $[T]_{B,C)}$ where $\vec{e_1} \in R^k$, furthermore in general it follows that $[T]_{(B,C)}\vec{e_j}$ is the jth column of $[T]_{(B,C)}$ where $\vec{e_j} \in \mathbb{R}^k$. It follows from part a and b that $[T]_{(B,C)}$
$= T'(\vec{e_j})$
$= L_C(T'(L_B^{-1}(e_j)))$
$\stackrel{\dagger}{=} L_C(T'(u_j))$.
$= [T(\vec{u_j})]_C$. Where † follows as we know that $L_B(u_j) = \vec{e_j}$ and furthermore that $L_B^{-1}L_B(u_j) = u_j$. Thus, as $B = (u_1, ...u_k)$ there are k many us and thus k columns of $[T]_{(B,C)}$ where the jth column is equal to $[T(u_j)]_C$ such that each column is a column matrix of the coordinates of C, and thus each column has d rows. Therefore it follows that $[T]_{(B,C)}$ is a $d \times k$ matrix.

30. (a) I propose that $C = \{1, x, \frac{x^2}{2!}, \frac{x^3}{3!}\}$ is a basis for P_3,note that as each term is of a different polynomial degree and reasonably spans P_3 C could be a basis for P_3. Note that $T : V \to P_3$ is defined as assigning $f \in V$ to the third degree Taylor polynomial $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$ polynomial approximation. Furthermore let $f_1 = \sin(x)$, $f_2 = \cos(x)$, $f_3 = e^x$. Therefore $T(f_1) = \sin(0) + \cos(0)x + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3$. Furthermore it was given from the hypothesis that $[T(f_1)]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Therefore writing this as a linear combination of the basis C we find this is equal to $0(1) + 1(x) + 0(\frac{x^2}{2!}(+1(\frac{x^3}{3!}) = T(f_1)$. Similarly $[T(f_2)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, therefore writing this as a linear combination of the basis C we find that $(1)(1) + 0(x) + -1(\frac{x^2}{2!}) + 0(\frac{x^3}{3!}) = 1 + 0 + -\frac{x^2}{2!} + 0 = \cos(0) - \sin(0)x + \frac{-\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 = T(f_2)$. Lastly it was given that $[T(f_3)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ therefore writing this as a linear combination of the basis C we find that $(1)(1) + 1(x) + 1(\frac{x^2}{2!}) + 1(\frac{x^3}{3!}) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = e^0 + e^0x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 = T(f_2)$. Therefore it is clear that such a given C is a basis for P_3 that holds true for conditions on the 3 functions shown above.

(b) Let C be the basis defined in a, and let $B = \{b_1, b_2, b_3\} = \{\sin(x) + \cos(x), \sin(x) - \cos(x), e^x + \sin(x)$. Further note that from problem 5 we found that $[T]_{(B,C)}[u]_B = [T]_C$. Therefore by the key theorem it follows that $[T]_{(B,C)}[b_1]_B$ is the first column of $[T]_{(B,C)}$ and from problem five we know that this is equal to $[T(b_1)]_C$. Further note that $[T(b_1)]_C = [T(\sin(x) + \cos(x)]_C \stackrel{\dagger}{=} [T(\sin(x)]_C + [T(\cos(x)]_C$. Where $\dagger$ follows as $[T]_C$ is a linear transformation. From part a we know that $[T(\sin(x)]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, and $[T(\cos(x))]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. Therefore adding together we find $[T(b_1)]_C = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ which is the first column of $[T]_{(B,C)}$. Continuing with this we find that the second column of $[T]_{(B,C})$ is equal to $[T(b_2)]_C$ as we showed with the key theorem. $[T(b_2)]_C = [T(\sin(x) - \cos(x)]_C = [T(\sin(x)]_C - T(\cos(x)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Therefore this is the second column of $[T]_{(B,C})$. Lastly, $[T]_{(B,C)}b_3 = [T(b_3)]_C$, it follows that $[T(b_3)]_C = [T(e^x + sin(x)]_C = [T(e^x)]_C + [T(\sin(x))]_C$. From part b we know $[T(e^x)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $[T(\sin(x)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ as shown above therefore $[T(b_3)]_C = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and is the third column of $[T]_{(B,C)}$. Thus

$$[T]_{(B,C)} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

31. (a) First note that $S_{c \to a} = L_a \circ L_c^{-1}$. Furthermore, that $S_{b \to a} = L_a \circ L_b^{-1}$ and $S_{c \to b} = L_b \circ L_c^{-1}$. Therefore it follows that $S_{b \to a} \circ S_{c \to b} = L_a \circ L_b^{-1} \circ L_b \circ L_c^{-1} = L_a \circ id_w \circ L_c^{-1} = L_a \circ L_c^{-1} = S$. Thus $S_{c \to a} = S_{b \to a} \circ S_{c \to b}$.

(b) As in part C we know that generally $S_{c \to a} = L_a \circ L_c^{-1}$, $S_{b \to c} = L_c \circ L_b^{-1}$, and $S_{A \to B} = L_b \circ L_a^{-1}$. Therefore it follows that $S_{c \to a} \circ S_{b \to c} \circ S_{a \to b}$
$= L_a \circ L_c^{-1} \circ L_c \circ L_b^{-1} \circ L_b \circ L_a^{-1}$

$= L_a \circ L_c^{-1} \circ L_c \circ id_w \circ L_a^{-1}$
$= L_a \circ id_w \circ L_c \circ id_w \circ L_a^{-1}$
$= L_a \circ L_a^{-1}$
$= id_w$. Therefore it that $S_{c\to a} \circ S_{b\to c} \circ S_{c\to a} = id_w$, that $S_{c\to a} \circ S_{b\to c} \circ S_{c\to a} = I_n$.

32. (a) Let $V = span(\vec{e_1})$ and $W = span(\vec{e_1}, \vec{e_2})$. Therefore it follows that $dim(V) = 1$ and $dim(W) = 2$ By definition $V^\perp = \{v' \in \mathbb{R}^3 | v' \cdot v = 0, \forall v \in V\}$. Let $x \in V^\{\perp\}$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Therefore it follows by definition that $x \cdot e_1 = 0$ and thus $x_1 = 0$. Therefore x_2, x_3 can be anything and thus we can use to find a basis for $W^\perp$ is $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. Similarly let $y \in W^\perp$ where $W^\perp$ is defined as being orthogonal to every basis of W, let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. As W $spans(\vec{e_1}, \vec{e_2})$ it follows from definition that $y \cdot e_1 = 0$ such that $y_1 = 0$, and similarly $y \cdot e_2$ such that $y_2 = 0$, therefore it leaves that y_3 can be any real number and thus the basis of $W^\perp = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. This idea can be further seen through the use of visual aid shown below this problem. Untitled Page (2).pdf

(b) Let V be an arbitrary vector space. From theorem 5.1.8 from the textbook we know that $\dim(V) + \dim(V^\perp = n$ where $V^\perp = \{v' \in \mathbb{R}^n | v' \cdot v = 0, \forall v \in V\}$ as defined in 5.1.7. Furthermore that $V^\perp$ is kernel of the orthogonal projection onto V. Therefore it is clear that V is the image of the orthogonal projection onto V. It follows then $\dim(V) = n - \dim(V^\perp)$. Furthermore $V^{\perp^\perp} = \{u \in \mathbb{R}^n | u \cdot v' = 0, \forall v' \in V^\perp\}$ and note that similarly $V^{\perp^\perp}$ is the image of the orthogonal projection onto V. Therefore using 5.1.8 as above we see that $\dim(V^{\perp^\perp}) = n - \dim(V^\perp) = \dim(V)$. Thus $\dim(V^{\perp^\perp}) = \dim(V)$.

(c) Suppose that $x \in V$, therefore from our jamboard 16 we know that $V^\perp = \{v' \in \mathbb{R}^n | v' \cdot v = 0, \forall v \in V\}$. Furthermore $V^{\perp^\perp} = \{u \in \mathbb{R}^n | u \cdot v' = 0, \forall v' \in V^\perp\}$. Notice then that an element is in $V^{\perp^\perp}$ if it is orthogonal to all elements of $V^\perp$. From above we know that x is, therefore $x \in V^{\perp^\perp}$, this along with the fact that $\dim(V) = \dim(V^{\perp^\perp})$ which we showed in part b we can conclude that $V = V^{\perp^\perp}$.

(d) Suppose that $V \subset W$, and let $x \in V$. Therefore it follows that $x \in W$. Furthermore let $W^\perp - \{w \in \mathbb{R}^n | w \cdot v = 0, \forall v \in V\}$, suppose that $u \in W^\perp$ therefore by definition $\forall x \in V$, $u \cdot x = 0$, and thus by definition $u \in V^\perp$, therefore $W^\perp \subset V^\perp$.

Secondly suppose that $W^\perp \subset V^\perp$. We showed directly above that if $A \subset B$, then $B^\perp \subset A \perp$. Let $A = W^\perp$ and let $B = V^\perp$ Therefore it follows as if $A \subset B$, then $B^\perp \subset A \perp$ as shown above, and thus $V^{\perp^\perp} \subset W^{\perp^\perp}$. From part C we can know that $V^{\perp^\perp} = V$ and $W^{\perp^\perp} = W$ therefore if $W^\perp \subset V^\perp \in V^\perp$ then $V \subset W$.

As we have shown both directions of this proof we can conclude that $V \subset W$ if and only if $W^\perp \subset V^\perp$.

33. (a) Let $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$ and let $V = span \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Furthermore let $\vec{w} \in V$ where $\vec{w} = \begin{bmatrix} 3w_1 \\ 2w_1 \end{bmatrix}$, and $w_1 \in \mathbb{R}$. Therefore it follows that $A\vec{w} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} 3w_1 \\ 2w_1 \end{bmatrix} = \begin{bmatrix} -18w_1 - 60w_1 \\ -90w_1 + 38w_1 \end{bmatrix} = \begin{bmatrix} 3(-6w_1 - 20w_1) \\ 2(-45w_1 + 19w_1) \end{bmatrix}$. It is then clear that $\begin{bmatrix} 3(-6w_1 - 20w_1) \\ 2(-45w_1 + 19w_1) \end{bmatrix} \in V$, and as $\vec{w}$ is an arbitrary element of V it follows that $\forall \vec{v} \in V$, $A\vec{v} \in V$

(b) Let $V^\perp = \{\vec{w} | w \cdot v = 0, v \in V\}$. I propose that $\begin{bmatrix} -2 & 3 \end{bmatrix}$ is a basis for $V^\perp$. Thus as the basis spans, we know that for all $c_1 \in \mathbb{R}$ that there exists a matrix $c \in V^\perp$ where $c = \begin{bmatrix} -2c_1 \\ 3c_1 \end{bmatrix}$. As we know that $V = span(\begin{bmatrix} 3 \\ 2 \end{bmatrix})$ then we know for all $b_1 \in \mathbb{R}$ that there exists some $\vec{b} = \begin{bmatrix} 3b_1 \\ 2b_1 \end{bmatrix} \in V$. Thus when $b_1 = c_1$ we see that $ba = 0$ and therefore it holds that this is a basis for $V^\perp$. Furthermore notice $Ab = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} -2c_1 \\ 3c_1 \end{bmatrix} = \begin{bmatrix} 12c_1 + 90c_1 \\ 60c_1 + 57c_1 \end{bmatrix} = \begin{bmatrix} -2(-6c_1 + -45c_1) \\ 3(20c_1 + 19c_1) \end{bmatrix}$. Therefore it is clear that you can also write this is a linear combination of the basis, and thus $Ac \in V^\perp$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$. Let B be a basis of $\mathbb{R}^2$ where $B = \{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. As this basis contains two elements and is linearly independent, this can be seen through inspection that neither of these elements are a multiple of each other, that this is then a basis for $\mathbb{R}^2$. Therefore $[T]_B = [[T(\vec{b_1})]_B, [T(\vec{b_2})]_B]$ and it follows that $T(b_1] = A\vec{b_1} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -78 \\ -52 \end{bmatrix}$ Thus it follows that $[T(\vec{b_1})]_B = \begin{bmatrix} -26 \\ 0 \end{bmatrix}$ as $-26 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ -52 \end{bmatrix}$. Secondly $T(\vec{b_2}) = A\vec{b_2} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ 117 \end{bmatrix}$ Thus it follows that $[T(\vec{b_1})]_B = \begin{bmatrix} 0 \\ 39 \end{bmatrix}$ as $0\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 39\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ 117 \end{bmatrix}$. Therefore $[T]_B = \begin{bmatrix} -26 & 0 \\ 0 & 39 \end{bmatrix}$.

(d) Let $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$ be a 2×2 matrix. Let $n = 1$ then it is clear that $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^1 = \begin{bmatrix} \gamma_1^1 & 0 \\ 0 & \gamma_2^1 \end{bmatrix}$. Suppose that $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^k = \begin{bmatrix} \gamma_1^k & 0 \\ 0 & \gamma_2^k \end{bmatrix}$, then we can show that $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^{k+1} = \begin{bmatrix} \gamma_1^{k+1} & 0 \\ 0 & \gamma_2^{k+1} \end{bmatrix}$ by the fact that $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^{k+1} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^k \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_1^k & 0 \\ 0 & \gamma_2^k \end{bmatrix} \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_1^{k+1} & 0 \\ 0 & \gamma_2^{k+1} \end{bmatrix}$, and thus $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^{k+1} = \begin{bmatrix} \gamma_1^{k+1} & 0 \\ 0 & \gamma_2^{k+1} \end{bmatrix}$. Therefore through the inductive process we have shown that $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^n = \begin{bmatrix} \gamma_1^n & 0 \\ 0 & \gamma_2^n \end{bmatrix} \forall n \geq 1$. Thus given $[T]_B$ found in part b where $[T]_B = \begin{bmatrix} -26 & 0 \\ 0 & 39 \end{bmatrix}$, it follows from our proof above $[T^{10}]_B = [T]_B^{10}$ and thus $[T^{10}]_B = \begin{bmatrix} -26^{10} & 0 \\ 0 & 39^{10} \end{bmatrix}$.

(e) Let us first define the change of matrix $S_{B \to E}$ which is equal to $[[\vec{b_2}]_E, [\vec{b_2}]_E]$ Let B refer to the B basis from part c and let E be the standard basis. Therefore it follows that $[\vec{b_1}]_E = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as $3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Secondly $[\vec{b_2}]_E = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ as $-2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Therefore $S_{B \to E} = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$. Therefore as this is 2×2 matrix we can easily computer the inverse $S_{B \to E}^{-1} = \begin{bmatrix} \frac{3}{12} & \frac{2}{13} \\ \frac{-2}{13} & \frac{3}{13} \end{bmatrix}$, Furthermore look below to problem 4a and note that B^k is similar to C^k meaning that that $B^k = S^{-1}CS$ where S is some change of basis function. Therefore it follows from Theorem 4.3.5 and our proof from problem 4a that when $C = [T]_B^1 0$ and $S = S_{B \to E}$ such that $[T]_E = S^{-1}[T]_B S = \begin{bmatrix} \frac{3}{12} & \frac{2}{13} \\ \frac{-2}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} -26^{10} & 0 \\ 0 & 39^{10} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{12} & \frac{2}{13} \\ \frac{-2}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} (-26)^{10}3 & -2(-26)^{10} \\ 39^{10}2 & 39^{10}3 \end{bmatrix}$
$= \frac{1}{13} \begin{bmatrix} 9((-26)^{10}) + 4(39^{10}) & 6(-26^{10}) + 6(39^{10}) \\ 6((-26)^{10}) + 6(39^{10}) & 9(-26^{10}) + 4(39^{10}) \end{bmatrix}$. Thus
$[T]_E^{10} = \frac{1}{13} \begin{bmatrix} 9((-26)^{10}) + 4(39^{10}) & 6(-26^{10}) + 6(39^{10}) \\ 6((-26)^{10}) + 6(39^{10}) & 9(-26^{10}) + 4(39^{10}) \end{bmatrix}$ and since from part D we showed that $[T^{10}]_E = [T]_E^{10}$ it follows that $[T^{10}]_E =$

$\frac{1}{13}\begin{bmatrix} 9((-26)^{10})+4(39^{10}) & 6(-26^{10})+6(39^{10}) \\ 6((-26)^{10})+6(39^{10}) & 9(-26^{10})+4(39^{10}) \end{bmatrix}$.

34. (a) Let B, C be defined as in the question header. B is Similiar to C as it follows from definition 4.3.5 from the textbook that $B = S^{-1}CS$ where S is the change of basis function between C and B. Furthermore for B^n note that this is equal to $B_1 \circ B_2 \circ B_3 ... \circ B_n$
$= S_1^{-1} \circ C_1 \circ S_1 \circ S_2^{-1} \circ C_2 \circ S_2 \circ S_3^{-1} \circ C_3 \circ S_3 ... \circ S_n^{-1} \circ C_n \circ S_n$
$= S_1^{-1} \circ C_1 \circ I_n \circ C_2 \circ I_n ... \circ C_n \circ S_n$
$= S_1^{-1} \circ C_1 \circ C_2 ... \circ C_n \circ S$
$= S^{-1}C^n \circ S$. Thus $\forall n \geq 1$ B^n is similiar to C^n.

(b) False.
Counterexample, let B, C be bases of the vector space P_1. Fix $B = (1, x)$ and $C = (x, 1)$. Furthermore let $T : P_1 \to P_1$ where T is defined by f'. Therefore $[T]_B = [[T(1)]_B, [T(x)]_B] = [[0]_B, [1]_B] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $[T]_C = [[T(x)]_C, [T(1)]_C] = [[1]_C, [0]_C] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Notice that $ker([T]_C) = \{\begin{bmatrix} 0 \\ x \end{bmatrix} |x \in \mathbb{R}\}$ and $ker([T]_B) = \{\begin{bmatrix} 0 \\ y \end{bmatrix} |y \in \mathbb{R}\}$. Therefore it is clear that $ker([T]_C) \neq ker([T]_A)$.

(c) Let $B = \{b_1, b_2, ...b_n\}$ and $C = \{c_1, c_2, ...c_n\}$. Let $T : V \to V$. Let $\gamma : ker([T]_B) \to ker([T]_C)$ where γ is defined as the change of basis matrix $S_{B\to C}$. Let $u \in ker([T]_B)$ such that $[T]_B[u]_B = 0 \overset{\dagger}{\Longrightarrow}$
$(S_{B\to C}^{-1}[T]_C S_{B\to C})[u]_B = 0$
$\overset{\pi}{\Longrightarrow} (S_{B\to C}^{-1}[T]_C)S_{B\to C})[u]_B = 0$
$\Longrightarrow [T]_B S_{B\to C} = 0$. Therefore it is clear that γ is well defined linear transformation between $ker([T]_B)$ and $ker([T]_C)$. Therefore as γ is a well defined and an invertable linear transformation by definition, we can conclude it is an ismorphism. As isomorphism preserve dimensions we can then conclude that $\dim([T]_B) = \dim([T]_C)$.

35. (a) Let $T : U \to W$, and let $B = (u_1, u_2, ...u_k)$ be a basis for the source U, and $C = (w_1, ...w_d)$ be a basis for W. Furthermore let $L_B : U \to \mathbb{R}^k$, $L_C : W \to \mathbb{R}^d$, and lastly $T' : \mathbb{R}^k \to \mathbb{R}^d$ where the condition that $T' \circ L_b = L_c \circ T$ holds. I propose that $T' = L_C \circ T \circ L_B^{-1}$ which is a linear transformation as it is a composition of linear transformations which thus must be a linear transformation itself. This transformation is from from $\mathbb{R}^{\urcorner} \to \mathbb{R}^d$ and holds the condition that $T' \circ L_B = L_C \circ T =$ as $L_C \circ T \circ L_B^{-1} \circ L_B = L_C \circ T \circ id = L_C \circ T$.

(b) Let $T : U \to W$, and let $B = (u_1, u_2, ...u_k)$ be a basis for the source U and $C = (w_1, ...w_d)$ be a basis for W. Furthermore let $L_B : U \to \mathbb{R}^k$, $L_C : W \to \mathbb{R}^d$, and lastly $T' : \mathbb{R}^k \to \mathbb{R}^d$ where the condition that $T' \circ L_b = L_c \circ T$ holds. From part a we found that there exists some T' such that $T' \circ L_b = L_c \circ T$. Let $[T]_{(B,C)}$ be the standard matrix of

T' by the key theorem. This means for $T'(\vec{x}) = [T]_{(B,C)}\vec{x}$, $\forall \vec{x} \in \mathbb{R}^k$. We can alternatively write what we found from part a to be that $\forall u \in U$ $T'(L_B(u)) = L_C(T(u))$. It follows that
$T'(L_B(u)) = L_C(T(u))$
$\implies T'(L_B(u)) = [T(u)]_C$
$\implies T'([u]_B) = [T(u)]_C$
$\overset{\dagger}{\implies} [T]_{(B,C)}[U]_B = [T(u)]_C$. Where † follows from our definition of $[T]_{(B,C)}$ as the standard matrix of T' above. Thus, $[T]_{(B,C)}[U]_B = [T(u)]_C$

(c) As we saw in part b, $[T]_{(B,C)}$ is the standard matrix of the linear transformation L' where $T' : \mathbb{R}^k \to \mathbb{R}^d$, from the key theorem we know that $[T]_{(B,C)}\vec{e_1}$ is the first column of $[T]_{B,C)}$ where $\vec{e_1} \in R^k$, furthermore in general it follows that $[T]_{(B,C)}\vec{e_j}$ is the jth column of $[T]_{(B,C)}$ where $\vec{e_j} \in \mathbb{R}^k$. It follows from part a and b that $[T]_{(B,C)}$
$= T'(\vec{e_j})$
$= L_C(T'(L_B^{-1}(e_j)))$
$\overset{\dagger}{=} L_C(T'(u_j)$.
$= [T(\vec{u_j})]_C$. Where † follows as we know that $L_B(u_j) = \vec{e_j}$ and furthermore that $L_B^{-1}L_B(u_j) = u_j$. Thus, as $B = (u_1, ...u_k)$ there are k many us and thus k columns of $[T]_{(B,C)}$ where the jth column is equal to $[T(u_j)]_C$ such that each column is a column matrix of the coordinates of C, and thus each column has d rows. Therefore it follows that $[T]_{(B,C)}$ is a $d \times k$ matrix.

36. (a) I propose that $C = \{1, x, \frac{x^2}{2!}, \frac{x^3}{3!}\}$ is a basis for P_3,note that as each term is of a different polynomial degree and reasonably spans P_3 C could be a basis for P_3. Note that $T : V \to P_3$ is defined as assigning $f \in V$ to the third degree Taylor polynomial $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$ polynomial approximation. Furthermore let $f_1 = \sin(x)$, $f_2 = \cos(x)$, $f_3 = e^x$. Therefore $T(f_1) = \sin(0) + \cos(0)x + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3$. Furthermore it was given from the hypothesis that $[T(f_1)]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Therefore writing this as a linear combination of the basis C we find this is equal to $0(1) + 1(x) + 0(\frac{x^2}{2!}(+1(\frac{x^3}{3!}) = T(f_1)$. Similarly $[T(f_2)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, therefore writing this as a linear combination of the basis C we find that $(1)(1) + 0(x) + -1(\frac{x^2}{2!}) + 0(\frac{x^3}{3!}) = 1 + 0 + -\frac{x^2}{2!} + 0 = \cos(0) - \sin(0)x + \frac{-\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 = T(f_2)$. Lastly it was given that

$[T(f_3)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ therefore writing this as a linear combination of the basis C we find that $(1)(1)+1(x)+1(\frac{x^2}{2!})+1(\frac{x^3}{3!}) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!} = e^0+e^0x+\frac{e^0}{2!}x^2+\frac{e^0}{3!}x^3 = T(f_2)$. Therefore it is clear that such a given C is a basis for P_3 that holds true for conditions on the 3 functions shown above.

(b) Let C be the basis defined in a, and let $B = \{b_1, b_2, b_3\} = \{\sin(x) + \cos(x), \sin(x) - \cos(x), e^x + \sin(x)\}$. Further note that from problem 5 we found that $[T]_{(B,C)}[u]_B = [T]_C$. Therefore by the key theorem it follows that $[T]_{(B,C)}[b_1]_B$ is the first column of $[T]_{(B,C)}$ and from problem five we know that this is equal to $[T(b_1)]_C$. Further note that $[T(b_1)]_C = [T(\sin(x) + \cos(x)]_C \stackrel{\dagger}{=} [T(\sin(x)]_C + [T(\cos(x)]_C$. Where $\dagger$ follows as $[T]_C$ is a linear transformation. From part a we know that $[T(\sin(x)]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, and $[T(\cos(x))]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. Therefore adding together we find $[T(b_1)]_C = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ which is the first column of $[T]_{(B,C)}$. Continuing with this we find that the second column of $[T]_{(B,C})$ is equal to $[T(b_2)]_C$ as we showed with the key theorem. $[T(b_2)]_C = [T(\sin(x) - \cos(x)]_C = [T(\sin(x)]_C - T(\cos(x)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Therefore this is the second column of $[T]_{(B,C)}$. Lastly, $[T]_{(B,C)}b_3 = [T(b_3)]_C$, it follows that $[T(b_3)]_C = [T(e^x + sin(x)]_C = [T(e^x)]_C + [T(\sin(x))]_C$. From part b we know $[T(e^x)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $[T(\sin(x)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ as shown above therefore $[T(b_3)]_C = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and is the third column of $[T]_{(B,C)}$. Thus

$$[T]_{(B,C)} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

37. (a) By our hypothesis we know that $\|T(\vec{x}+\vec{y}\|^2 = \|\vec{x}+\vec{y}\|^2$. Therefore from our understanding of dot product we know that this is the same as $T(\vec{x}+\vec{y})\cdot T(\vec{x}+\vec{y}) = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \implies (T(\vec{x})+T(\vec{y}))\cdot(T(\vec{x})+T(\vec{y})) = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$. Expanding this we find this is the same as $T(\vec{x})\cdot T(\vec{x}) + 2T(\vec{x})\cdot T(\vec{y}) + T(\vec{y})\cdot T(\vec{y})$. Therefore it follows that we know that $T(\vec{x})\cdot T(\vec{x}) = \|T(\vec{x})\|^2 = \|x\|^2 = x\cdot x$, and then likewise for y. Therefore we can simplify the equality above to $2T(\vec{x})\cdot T(\vec{y}) = 2(x\cdot y)$. Thus dividing by 2 on both sides we find that $T(\vec{x})\cdot T(\vec{y}) = x\cdot y$.

(b) FOr $T(v_1), ...T(v_k)$ to be orthornormal we know that $T(v_i)\cdot T(v_j) = 0$ when $i \neq j$ where $1 \leq i \leq k$ and $1 \leq j \leq k$. Therefore as $(v_1, ...v_k)$ is an orthornormal basis we know by definition that $v_i \cdot v_j = 0$ when $i \neq j$ and equal to 1 when $i = j$. Therefore as in part a we found that $v_i \cdot v_j = T(v_i)\cdot T(v_j)$ we can then conclude that this is orthonormal as we know that $v_i \cdot v_j$ is orthonormal. Further suppose that $c_1T(v_1)+c_2T(v_2)+...+c_kT(v_k) = 0$ where $c_1, c_2, ...c_k \in \mathbb{R}$. Let $i \in \mathbb{R}$ where $1 \leq i \leq k$. Therefore dot producting both sides of this equality by $T(v_i)$ we have $T(v_i)\cdot(c_1T(v_1)+c_2T(v_2)+...+c_kT(v_k)) = T(v_i)\cdot 0$
$\implies T(v_i)\cdot(c_1T(v_1)+c_2T(v_2)+...+c_kT(v_k)) = 0$
$\implies T(v_i)\cdot c_1T(v_1) + T(v_i)\cdot c_2T(v_2) + ... + T(v_i)\cdot c_kT(v_k) = 0$
$\implies c_1(T(v_i)\cdot T(v_1)) + c_2(T(v_i)\cdot T(v_2)) + ... + c_k(T(v_i)\cdot T(v_k)) = 0$
$\implies c_1(v_i\cdot v_1) + c_2(v_i\cdot v_2) + ... + c_k(v_i\cdot v_k) = 0$
$\implies c_1(0)+c_2(0)+...c_i(1)+...c_k(0) = 0$. Therefore we can conclude that c_i must be 0. As i is an arbitrary variable between 1 and k inclusive we then know that $c_1, ...c_k$ must all be 0. Therefore the only relation is the trivial one and thus $T(v_1), T(v_2), ...T(v_k)$ must be linearly independent.

Furthermore extend $v_1, ..., v_k$ to a basis $v_1, ..., v_k, u_1, ..., u_j$ where this forms a basis for $\mathbb{R}^n$. Applying the graham Schmidt method we can turn this basis into an orthonormal basis for $\mathbb{R}^n$ $v_1, ..., v_l, y_1, ...y_j$. Next let us take an arbitrary vector $x \in im(T)$. Therefore we can x as a linear combination of the orthonormal basis found above such that $x = c_1v_1 + ...c_kv_k + d_1y_1 + ...d_jy_j$. Therefore as we know that $im(T) = T(v)\forall v \in \mathbb{R}^n$ we then know $T(c_1v_1 + ...c_kv_k + d_1y_1 + ...d_jy_j) \in im(T)$. Therefore because of linearity we can rewrite this as $c_1T(v_1) + ... + c_kT(v_k) + d_1T(y_1) + ...d_jT(y_j)$. Therefore as we know $y_1, ...y_j$ are orthonormal we then know that they are in $W^\perp$ and thus $T(y_i) = 0$ such that we can rewrite this linear combination as $c_1T(v_1) + ... + c_kT(v_k)$. Therefore we can write an arbitrary element of $im(T)$ as a linear combination of $T(v_1), ...T(v_k)$ therefore we know that $T(v_1), ...T(v_k)$ spans $im(T)$.

Thus as $T(v_1), ...T(v_k)$ is orthonormal and further both linearly independent and spans ImT we can conclude that $T(v_1), ...T(v_k)$ is an orthonormal basis for $im(T)$.

(c) Let $(w_1, ...w_l)$ be an orthonormal basis for $im(T)^{\perp}$. Furthermore from part b we know that $(T(v1), ...T(v_k))$ is an orthonormal basis for $im(T)$. Therefore it follows that $(T(v1), ...T(v_k), w_1, ...w_l)$ is orthnormal if $T(v_i) \cdot w_j = 0$ where $1 \leq i \leq k$, and $1 \leq j \leq l$. As $T(v_i) \in (T)$ and $w_j \in im(T)^{\perp}$ then it follows by definition of $im(T)^{\perp}$ that $T(v_i) \cdot w_j = 0$. Therefore we can conclude that this set is orthogonal. Furthermore remember from worksheet 16 problem $3c$ that an orthonormal set is always linearly independent. Therefore we can conclude that $(T(v1), ...T(v_k), w_1, ...w_l)$ is linearly indepdent as we have shown above that it is orthonormal. Lastly, as $(T(v1), ...T(v_k))$ is an orthonormal basis for $im(T)$ and $(w_1, ...w_l)$ be an orthonormal basis for $im(T)^{\perp}$, we know that $dim(im(T) = dim(T(v1), ...T(v_k))$ and $dim(im(T)^{\perp}) = dim((w_1, ...w_l))$. Therefore we have know that $dim((T(v1), ...T(v_k), w_1, ...w_l)) = dim(im(T)) + dim(im(T)^{\perp})$. Therefore by theorem $5.1.8.c$ we know that $dim((T(v1), ...T(v_k), w_1, ...w_l)) = dim(im(T)) + dim(im(T)^{\perp}) = n$. Further notice that $dim(\mathbb{R}^n) = n$. Therefore as $dim(\mathbb{R}^n) = dim((T(v1), ...T(v_k), w_1, ...w_l))$, and we know that $(T(v1), ...T(v_k), w_1, ...w_l)$ is linearly indepedent we can further conclude that it must also span $\mathbb{R}^n$. Thus $(T(v1), ...T(v_k), w_1, ...w_l)$ is a basis for $\mathbb{R}^n$.

38. (a) The asociated standard matrix of ref_l can be found by the key theorem where $ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ and thus ref_L matrix is $\begin{bmatrix} ref_L(e_1) & ref_L(e_2) \end{bmatrix}$. Therefore as $ref_L(e_1)$ s a unit vector, there exists an angle 2μ such that $ref_L(e_1) = \begin{bmatrix} \cos(2\mu) \\ \sin(2\mu) \end{bmatrix}$. Note that any angle between the $x-axis$ and any line $\mathcal{L}$ can be thought to live between $\frac{-\pi}{2}, \frac{\pi}{2}$. Similarly to above we know that $ref_L(e_2)$ is a unit vector so it must also live on the unit circle and therefore we know there must exist some 2μ, furthermore we know that the that for $ref_L(e_2)$ the angle between e_2 and L would be $\frac{\pi}{2} - \mu$ from what we know about and from observation, therefore we have $ref_L(e_2) = \begin{bmatrix} \cos(\frac{\pi}{2} - 2\mu \\ \sin(\frac{\pi}{2} - 2\mu) \end{bmatrix}$. Further recall the identity that $\sin(\frac{\pi}{2} - \phi) = -\cos(\phi)$ and similarly that $\cos(\frac{\pi}{2} - \phi) = \sin(\phi)$. Therefore we can rewrite $ref_L(e_2) = \begin{bmatrix} \sin(2\mu) \\ -\cos(2\mu) \end{bmatrix}$. Let us say that $2\mu = \theta$ Thus ref_L matrix is $\begin{bmatrix} ref_L(e_1) & ref_L(e_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$.

(b) First recall that the standard matrix of a rotation counterclockwise by θ is given by $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin((\theta) & \cos(\theta) \end{bmatrix}$, and from part a that the standard matrix of a reflection is given by $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Suppose I have an orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Furthermore as we proofed on the worksheets, we then know that the columns of such a standard matrix for an orthogonal linear transformation must be orthonormal. Therefore $a^2 + c^2 = 1$, $b^2 + d^2 = 1$, and $ab + cd = 0$. As $\begin{bmatrix} a \\ c \end{bmatrix}$ is a unit vector we know it lies on the unit circle and thus we can write it as $\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$. Therefore we know as the columns are orthogonal that $b\cos(\theta) + d\sin(\theta) = 0$ Furthermore if $\cos(\theta) \neq 0$ then we know that we can rewrite this as $b = -d\frac{\sin(\theta)}{\cos(\theta)}$. Therefore it follows that $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -d\frac{\sin(\theta)}{\cos(\theta)} \\ d \end{bmatrix} = \begin{bmatrix} -d\frac{\sin(\theta)}{\cos(\theta)} & \sin(\theta) \\ d\frac{\sin(\theta)}{\cos(\theta)} & \cos(\theta) \end{bmatrix} = d\frac{\sin(\theta)}{\cos(\theta)} \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$. From above as these vectors are unit length we know that $(-d\frac{\sin(\theta)}{\cos(\theta)})^2 + d^2 = 1$

$\implies d^2\frac{\sin(\theta)^2}{\cos(\theta)^2} + d^2 = 1$

$\implies d^2(\frac{\sin(\theta)^2}{\cos(\theta)^2} + 1) = 1$

$\implies d^2\frac{\sin(\theta)^2 + \cos(\theta)^2}{\cos(theta)} = 1$

$\implies \frac{d^2}{\cos(\theta)^2} = 1$

$\implies \pm\frac{d}{\cos(theta)} = \pm 1$. Thus $\begin{bmatrix} b \\ d \end{bmatrix} = d\frac{\sin(\theta)}{\cos(\theta)} \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix} = \pm \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$

Therefore we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Recall from above that these are the standard matrices for either rotation clockwise by θ or reflection. Thus if a linear transformation is an orthogonal transformation then it is either a rotation or a reflection.

39. (a) Recall the least squares solution $A^TA\vec{x}^* = A^T\vec{b}$. Further recall that $A = QR$, where Q is a $m \times n$ matrix with orthonormal columns and $\mathbb{R}$ is an upper triangular invertible matrix with positive entries on its diagonal. Therefore it follows that $A^TA\vec{x}^* = A^T\vec{b}$
$\implies (QR)^T(QR)\vec{x}^* = (QR)^T\vec{b}$
$\implies R^TQ^TQR\vec{x}^* = (R^TQ^T)\vec{b}$. As R is invertible we know that R^T must also be invertible. Therefore there exists $(R^T)^{-1}$. Dotting both sides with $(R^T)^{-1}$ we find that $(R^T)^{-1}R^TQ^TQR\vec{x}^* = (R^T)^{-1}(R^TQ^T)\vec{b}$
$\implies Q^TQR\vec{x}^* = (Q^T)\vec{b}$. Furthermore as Q is an matrix with orthonormal columns then we know that $Q^TQ = \begin{bmatrix} q_1^Tq_1 & \cdots & q_1^Tq_n \\ \vdots & \ddots & \vdots \\ q_n^Tq_1 & \cdots & q_n^Tq_n \end{bmatrix} =$

$\begin{bmatrix} q_1 \cdot q_1 & \cdots & q_1 \cdot q_n \\ \vdots & \ddots & \vdots \\ q_n \cdot q_1 & \cdots & q_n \cdot q_n \end{bmatrix} = I_n$ as Q columns are orthonormal. Therefore we can conclude that $Q^T Q = I_n$ and thus we can rewrite our equality above as $R\vec{x}^* = (Q^T)\vec{b}$. Thus the least square solution is a solution to the system. We can simply show that this is the only solution by dotting both sides of $R\vec{x}^* = (Q^T)\vec{b}$ with RT^{-1} as R is an invertible matrix. Therefore we find that $\vec{x}^* = R^{-1}(Q^T)\vec{b}$ is the only solution for this system which is what we found above. Thus $\vec{x}^*$ is the unique solution to the system.

(b) Let $b = \begin{bmatrix} 10 \\ \sqrt{2} \\ 0 - \sqrt{2} \end{bmatrix}$ From part a problem 5 we found that Q is $\begin{bmatrix} \frac{1}{10} & \frac{-1}{2} & 0 \\ \frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$ and R is $\begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$. I propose then that $\mathbb{R}^{-1}$ is $\begin{bmatrix} \frac{1}{10} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ as we know that R is an invertible matrix and that $R^{-1}R = I_3$. Furthermore $Q^T = \begin{bmatrix} \frac{1}{10} & \frac{7}{10} & \frac{1}{10} & \frac{7}{10} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$ Using what we found in part B we know that $\bar{x}^* = R^{-1}Q^T\vec{b}$. Therefore plugging in our values above we first find that $Q^T\vec{b} = \begin{bmatrix} \frac{1}{10} & \frac{7}{10} & \frac{1}{10} & \frac{7}{10} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 10 \\ \sqrt{2} \\ 0 - \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 5\sqrt{2} \\ 2 \end{bmatrix}$. Therefore $\bar{x}^* = R^{-1}(Q^T\vec{b}) = \begin{bmatrix} \frac{1}{10} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 5\sqrt{2} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{51-10\sqrt{2}}{10} \\ -5 \\ \sqrt{2} \end{bmatrix}$. Thus the least square solution is $\begin{bmatrix} \frac{51-10\sqrt{2}}{10} \\ -5 \\ \sqrt{2} \end{bmatrix}$.

40. (a) Assume that $B = (v_0 \bar{v_0})$ is a basis for $\mathbb{C}$. Assume for contradiction that $a = 0$. Therefore it follows that the basis is $bi, -bi$. This is not linearly independent as $-bi$ is just the scalar product of the first element and -1. Therefore as it is not linearly independent it is not a basis, and thus our assumption must be wrong.
Assume for contradiction that $b = 0$ then it follows that we can write that basis as a, a which is clearly not linearly independent and thus again our initial assumption that $b = 0$ must be false. Therefore

given B is a basis for $\mathbb{C}$ we can conclude z_0 is neither real or purely imaginary.

Let $v \in \mathbb{C}$ where $v = c + di$ and $c, d \in \mathbb{R}$. Suppose that $c = (c_1 + c_2)a$ and $d = (c_1 - c_2)b$ where $c_1, c_2, b \in \mathbb{R}$. Therefore rewriting v we see that $v = (c_1 + c_2)a + (c_1 - c_2)bi = c_1a + c_2a + c_1bi - c_2bi = c_1(a + bi) + c_2(a - bi)$. Which is now clearly a linear combination of the basis B therefore B spans v and thus B spans $\mathbb{C}$. As we know that $\dim(\mathbb{C}) = 2$ and that B spans $\mathbb{C}$ we can then conclude that B must be linearly independent and thus B is a basis for $\mathbb{C}$.

(b) First recall that $S_{B\to E} = \begin{bmatrix}[b_1]_E & [b_2]_E\end{bmatrix}$. Where $[b_1]_E = \begin{bmatrix}a \\ b\end{bmatrix}$ as $a(1)+b(i) = a+bi$. Furthermore $[b_2]_E = \begin{bmatrix}a \\ -b\end{bmatrix}$ as $a(1)+-b(i) = a-bi$. Thus $\begin{bmatrix}a & a \\ b & -b\end{bmatrix}$ Recall that we know that $a^2+b^2 = 1$ when the columns of the matrix to form an orthonormal basis. Furthermore we know that $(a + b)(a - b) = 0 \implies a^2 - b^2 = 0 \implies a^2 = b^2$. Therefore replacing b^2 with a^2 we find that $2a^2 = 1$ and thus $a = \pm\frac{1}{\sqrt{2}}$. Similarly replacing a^2 with b^2 we find that $b = \pm\frac{1}{\sqrt{2}}$. Therefore as $z_0 = a + bi$, for the columns of the matrix to form an orthonormal basis we know that $z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $z_0 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, $z_0 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or $z_o = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. Further recall that when we know that the columns of the matrix from an orthonormal basis then we further know that its inverse in orthonormal and thus $S^{-1}_{B\to E}$ columns form an orthonormal basis.

(c) Recall from homework 7 part b problem $1a$ that $S_{C\to B} = S_{E\to B}S_{C\to E}$. Further recall that $S^{-1}_{B\to E} = (S_{B\to E})^T = S^T_{B\to E} = \begin{bmatrix}a & b \\ a & -b\end{bmatrix}$. Furthermore in a similar approach to part a, $S_{C\to E} = \begin{bmatrix}[c_1]_E & [c_2]_E\end{bmatrix}$ Where $[c_1]_E = \begin{bmatrix}c \\ d\end{bmatrix}$ as $c(1)+d(i) = c+di$. Furthermore $[c_2]_E = \begin{bmatrix}c \\ -d\end{bmatrix}$ as $c(1) + -d(i) = c - di$. Thus $\begin{bmatrix}c & c \\ d & -d\end{bmatrix}$. Therefore we have that $S_{C\to B} = \begin{bmatrix}a & b \\ a & -b\end{bmatrix}\begin{bmatrix}c & c \\ d & -d\end{bmatrix} = \begin{bmatrix}ac+bd & ac-bd \\ ac-bd & ac+bd\end{bmatrix}$. Lastly, similarly as above we know that $S_{B\to C} = S^{-1}_{C\to B} = (S_{C\to B})^T = S^T_{C\to B} = \begin{bmatrix}ac+bd & ac-bd \\ ac-bd & ac+bd\end{bmatrix}$.

(d) i. Let $\gamma : C \to C$ be the conjugation mapping. Then $[\gamma]_B = \begin{bmatrix}[\gamma(b_1)]_B & [\gamma(b_2)]_B\end{bmatrix}$ It follows that $[\gamma(b_1)]_B = [\gamma(a+bi)]_B = \begin{bmatrix}0 \\ 1\end{bmatrix}$ as $0(a + bi) + 1(a - bi) = a - bi = \gamma(a + bi) = \gamma(b_1)$. Similarly $[\gamma(b_2)]_B = [\gamma(a - bi)]_B = \begin{bmatrix}1 \\ 0\end{bmatrix}$ as $1(a + bi) + 0(a - bi) = a + bi =$

$\gamma(a-bi)=\gamma(b_1)$. Therefore we have $[\gamma]_B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

ii. Let $\mu : C \to C$ be multiplication by i. Then $[\mu]_E=\begin{bmatrix}[\mu(e_1)]_E & [\mu(e_2)]_E\end{bmatrix}$. It follows that $[\mu(e_1)]_E=[\mu(1)]_E=\begin{bmatrix}0\\1\end{bmatrix}$ as $0(1)+1(i)=i=\mu(1)=\mu(e_1)$. Similarly $[\mu(e_2)]_E=[\mu(i)]_E=\begin{bmatrix}-1\\0\end{bmatrix}$ as $-1(1)+0(i)=-1=i^2=\mu(i)=\mu(e_1)$. Therefore we have $[\mu]_E=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$. Further more from change of basis theorem for coordinates on worksheet 15 we know that $[\mu]_B=S_{E\to B}[\mu]_E$. Where $S_{E\to B}=\begin{bmatrix}[e_1]_B & [e_2]_B\end{bmatrix}$. It follows that $[e_1]_B=[1]_B=\begin{bmatrix}\frac{1}{2a}\\ \frac{1}{2a}\end{bmatrix}$, as $\frac{1}{2a}(a+bi)+\frac{1}{2a}(a-bi)=\frac{2a}{2a}=1$. Similarly, $[e_2]_B=[i]_B=\begin{bmatrix}\frac{1}{2b}\\ -\frac{1}{2b}\end{bmatrix}$, as $\frac{1}{2b}(a+bi)+\frac{1}{2b}(a-bi)=\frac{2bi}{2b}=i$. Therefore we have that $S_{E\to B}$ is $\begin{bmatrix}\frac{1}{2a} & \frac{1}{2b}\\ \frac{1}{2a} & -\frac{1}{2b}\end{bmatrix}$. Therefore it follow then that $[\mu]_E=\begin{bmatrix}\frac{1}{2a} & \frac{1}{2b}\\ \frac{1}{2a} & -\frac{1}{2b}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{2b} & -\frac{1}{2a}\\ -\frac{1}{2b} & \frac{1}{2a}\end{bmatrix}$.

41. (a) Let $T:\mathbb{C}\to V$ where $z\mapsto T(z)$. Let $x=\begin{bmatrix}c & -d\\ d & c\end{bmatrix}$ where $x\in V$. Let $y=c+di$ where $y\in\mathbb{C}$ therefore $T(y)=x$, and therefore for an arbitrary element of V there exists an element in C that it maps to. Thus we can conclude that T is surjective. Furthermore notice that $\begin{bmatrix}c & -d\\ d & c\end{bmatrix}$ is only the zero matrix when $a=b=0$ and therefore when $z\in\mathbb{C}$ is $z=0+oi=0$. Therefore the $ker(T)=\{0\}$. Thus as T is a linear transformation that is both injective and surjective we can conclude that it is an isomorphism.

(b) Let $z=a+bi$, and let $T(z)=A=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$. Therefore $T(\frac{1}{z})=T(\frac{1}{a+bi})=T(\frac{a-bi}{a^2+B^2})=\begin{bmatrix}\frac{a}{a^2+b^2} & \frac{b}{a^2+b^2}\\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2}\end{bmatrix}$. Further note that $A^{-1}=\frac{1}{a^2+b^2}\begin{bmatrix}a & b\\ -b & b\end{bmatrix}=\begin{bmatrix}\frac{a}{a^2+b^2} & \frac{b}{a^2+b^2}\\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2}\end{bmatrix}$. Therefore $A^{-1}=\begin{bmatrix}\frac{a}{a^2+b^2} & \frac{b}{a^2+b^2}\\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2}\end{bmatrix}=T(\frac{1}{z})$. Furthermore we know that $A^T=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$. Further note that $\bar{z}=a-bi$ and thus $T(\bar{z})=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$. Therefore $A^T=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}=T(\bar{z})$.

(c) Let $z=a+bi$, and let $T(z)=A=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$. It was shown in part b

that $A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & b \end{bmatrix} = \frac{1}{a^2+b^2}A^T$. Furthermore from part b as we know that $A^{-1} = T(\frac{1}{z})$ and that $T(\bar{z}) = A^T$ Therefore we further know that $T(\frac{1}{z}) = \frac{1}{a^2+b^2}T(\bar{z})$. Therefore from part a as we know that T is an isomorphism, we know that it is therefore injective and thus if $T(x) = T(y)$ then $x = y$. Therefore we then know that in general $\frac{1}{z} = \frac{1}{a^2+b^2}\bar{z} \implies \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$.

(d) In general we know that for an orthogonal matrix that $A^T = A^{-1}$. If the columns of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have unit length such that $a^2 + b^2 = 1$ then we know it must be orthogonal. Therefore we can conclude that $A^T = AT{-1}$ and therefore $T(\bar{z}) = T(\bar{\frac{1}{z}}$, which as T is injective we know then that if $T(\bar{z}) = T(\frac{1}{z})$ then $\bar{z} = \frac{1}{z}$.

(e) Let $z_1 = a + bi$ and let $z_2 = c + di$ where $a, b, c, d \in \mathbb{R}$. Therefore it follows that $z_1 \cdot z_2 = (a + bi)(c + di) = ac + adi + bci - bd = ac - bd + (ad + bd)i$. Therefore it follows that $T(ac{-}bd{+}(ad{+}bd)i) = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}$. Furthermore $T(z_1) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $T(z_2) = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ and therefore $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -da - bc \\ bc + ad & -bd + ac \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad - bc) \\ ad + bc & ac - bd \end{bmatrix}$. Thus $T(z_1 \cdot z_2) = \begin{bmatrix} ac - bd & -(ad - bc) \\ ad + bc & ac - bd \end{bmatrix} = T(z_1)T(z_2)$ and therefore $T(z_1{\cdot}z_2) = T(z_1)T(z_2)$.

42. (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ have eigenvalues λ_1 and λ_2. Therefore it follows that A^2 have eigenvalues λ_1^2 and λ_2^2. Further recall Therorem 7.2.8 which states that $trace(A) = \lambda_1 + \lambda_2$. Therefore it follows that $(A^2) = \lambda_{1^2+\lambda_2^2}$. Mutliplying A by A we find that $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$, and thus $(A^2) = a^2{+}bc{+}bc{+}d^2 = a^2{+}d^2{+}2bc$. Thus as $a^2{+}d^2{+}2bc = trace(A^2) = \lambda_{1^2+\lambda_2^2}$, then $\lambda_{1^2+\lambda_2^2=a^2+d^2+2bc}$.

(b) Suppose $a, b, c, d \in \mathbb{R}$ and let $b = c$ then we know that $bb = bc \implies 2bc = 2bb \implies 2bc = b^2 + b^2 \implies 2bc = b^2 + c^2 \implies a^2 + b^2 + 2bc = a^2 + b^2 + c^2 + d^2$. Separately $a, b, c, d \in \mathbb{R}$ and let $b < c$ then we know $b - b < c - b \implies 0^2 < (c - b)^2 \implies 0 < c^2 - 2bc + b^2 \implies 2bc{+}b^2{+}c^2 \implies a^2{+}b^2{+}2bc < a^2{+}b^2{+}c^2{+}d^2$. Now note that from part a, $\lambda_1^2 + \lambda_2^2 = a^2 + d^2 + 2bc$. therefore $\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + c^2 + d^2$ or $\lambda_1^2 + \lambda_2^2 < a^2 + b^2 + c^2 + d^2$ thus $\lambda_1^2 + \lambda_2^2 \leq a^2 + b^2 + c^2 + d^2$.

(c) When A is a symmetric matrix, then we know that $A = A^T$. Thus calculating A^2 similar to in part a we notice that $A^2 = AA^T$ where λ_1^2 and λ_2^2 are eigenvalues for A^2 and thus $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} =$ $a^2 + b^2ac + bd$

$ca + bdc^2 + d^2$. Therefore $(A^2) = a^2 + b^2 + c^2 + d^2$. Furthermore like above, by Theorem 7.2.8 we know that $\lambda_1^2 + \lambda_2^2 = (A^2)$. Therefore $\lambda_1^2 + \lambda_2^2 = (A^2) = a^2 + b^2 + c^2 + d^2$ and therefore $\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + c^2 + d^2$.

43. (a) True, Suppose that T is diagonolizable, therefore there exists an eigenbasis $B = (v_1, v_2, ...v_n)$ for V. Therefore it follows that given the eigenbasis B where $Av_1 = \lambda_1 v_1, ...Av_n = \lambda_n v_n$ that $S = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ will diagonalize $[T]_B$, and thus $B = S^{-1}[T]_B S$. Note that this is directly from Theorem 7.1.3 in the textbook. Furthermore recall Homework 7 problem 4 a which states that if B, C are similar then B^k, C^k is also similar for all $k \in R$. Thus as we know that B and $[T]_B$ are similar then by this homework we further know that $B^k = S^{-1}[T]_B^k S$ for $k \in \mathbb{R}$, and thus we can conclude that T^k by definition is diagonalizable.

(b) False, Counterexample, let T have a matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, therefore $T(T(v)) = A^2 v$ and thus $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ where A^2 is diagnolizable, and thus must have an eigenbasis. This can be seen by computing the eigenvectors for this we set that $\begin{bmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}$ and thus A^2 has a characteristic polynoial $(-1-\lambda)^2$ and thus $gemu(-1) = 2$. To find such eigenvectors for this eigenbasis we can plug -1 back in to find that the eigenvectors are anything that span this matrix and thus we can assume the standard basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore as $(2) = \dim(source(T))$ then we can assume that $(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$ is an eigenbasis for V and thus diagonolizable, however note that $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ has eigenvalues $\pm i$ as $\begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix}$ and thus this has a charateristic polynomial $\lambda^2 + 1$ which implies that $\lambda = \pm i$. Therefore as this has no real eigenvalues there are no real eigenvectors that span R and thus this cannot be diagonolizable as an eigenbasis exist. Thus if T^2 is diagonalziable this does not imply that T is diagonalizable.

(c) True, Suppose that T is diagonalizable with eigenvalues 0 or 1 then there exists an eigenbasis $B = (v_1, v_2, ...v_n)$ for V consisting of eigenvectors. Therefore we know that $[T]_B = \begin{bmatrix} [T(v_1)]_B & \cdots & [T(v_n)]_B \end{bmatrix} =$

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$ where $\lambda_1, ..., \lambda_n = 1$ or 0. Further note that

$$[T]_B[T]_B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \underset{\dagger}{=} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix},$$

where $\dagger$ follows as if the eigenvalues are 0 or 1 then multiplying any by itself we get the same thing as $0 * 0 = 0$ and $1 * 1 = 1$. Therefore we have that $[T]_B = [T]_B^2$. Now recall Homework 8 where we showed that $[T]_B = [T]_B^2 = [T^2]_B$. Thus $[T]_B[T^2]_B$ are equal they have the same columns and thus are the same and thus we can imply that $T = T^2$.

(d) True, Suppose $T^2 = T$ for T we know $T(v) = \lambda v$ and similarly $T(T(v)) = \lambda^2 v$ by definition. As $T = T^2$ it follows that $\lambda^2 v = T(T(v)) = T(v) = \lambda(v)$. Therefore $\lambda^2(v) = \lambda(v) \implies \lambda^2 = \lambda$. Suppose first that $\lambda = 0$ therefore $\lambda^2 = \lambda \implies 0^2 = 0$ and thus $\lambda = 0$. Suppose now that $\lambda \neq 0$ therefore we can rewrite this as $\frac{\lambda^2}{\lambda} = \frac{\lambda}{\lambda} \implies \lambda = 1$. Thus when $T = T^2$, it follows then that $\lambda = 1$ or $\lambda = 0$.

44. (a) Let $f(a, b) \in C^\infty(\mathbb{R}^2)$ and let $f(c, d) \in C^\infty(\mathbb{R}^2)$. Furthermore let $k \in \mathbb{R}$. Therefore $\Delta(f(a, b)) = \frac{\partial^2 f}{\partial a^2}(a, b) + \frac{\partial^2 f}{\partial b^2}(a, b) = \frac{\partial^2}{\partial a^2}(a) + \frac{\partial^2}{\partial b^2}(b)$ Thus $c(\Delta(f(a, b)) = c(\frac{\partial^2}{\partial a^2}(a) + \frac{\partial^2}{\partial b^2}(b))$. Furthermore, $\Delta(ca, cb) = \frac{\partial^2 f}{\partial a^2}(ca, cb) + \frac{\partial^2 f}{\partial b^2}(ca, cb) = \frac{\partial^2}{\partial a^2}(ca) + \frac{\partial^2}{\partial b^2}(cb)$ Therefore $\Delta(f(ca, cb)) = c(\Delta(f(a, b))$. Therefore T preserves scalar multiplication. Further note that $\Delta(f(c, d)) = \frac{\partial^2}{\partial c^2}(c) + \frac{\partial^2}{\partial d^2}(d)$ Therefore it follows that $\Delta(f(c, d)) + \Delta(f(a, b)) = \frac{\partial^2}{\partial(a+c)^2}(a + c) + \frac{\partial^2}{\partial(b+d)^2}(b + d)$. Computing $\Delta(f(a + c, b + d)) = \frac{\partial^2 f}{\partial(a+c)^2}(a + c, b + d) + \frac{\partial^2 f}{\partial(b+d)^2}(a + c, b + d) = \frac{\partial^2}{\partial(a+c)^2}(a + c) + \frac{\partial^2}{\partial(b+d)^2}(b + d)$Thus it is clear that $\Delta(f(c, d)) + \Delta(f(a, b)) = \Delta(f(a + c, b + d))$ and thus T is closed under addition. As T is then closed under addition and scalar multiplication it follows then that T is a linear transformation.

(b) Let $c \in \mathbb{R}$ and let $f_c(x, y) = \sin(cx)\cos(cy)$. Therefore $T(f_c(x, y)) = \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y)$ To compute this let us first find the first partial derivative of x, and then its second partial derivative note that $\frac{\partial f}{\partial x}\sin(cx)\cos(cy) = c{*}\cos(cx)cos(cy)$ and thus then $\frac{\partial^2 f}{\partial x^2}c{*}\cos(cx)cos(cy) = -c^2\sin(cx)\cos(cy)$ Similarly $\frac{\partial f}{\partial y}\sin(cx)\cos(cy) = -c\sin(cx)\sin(cy)$ and therefore it follows that $\frac{\partial f}{\partial x} - c\sin(cx)\sin(cy) = -c^2\sin(cx)\cos(cy)$. Therefore we have that $\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = -c^2\sin(cx)\cos(cy) +$

$-c^2\sin(cx)\cos(cy) = -2c^2(\sin(cx)\cos(cy))$ Now note the definition of an eigenvalue is when $T(v) = \lambda v$ where v is an eigenvector. Thus given $T(f_c(x,y)) = -2c^2(\sin(cx)\cos(cy)$. Similiarly note given $\lambda = -2c^2$ we find that this equals $-2c^2(\sin(cx)\cos(cy))$. Therefore we can conclude that $f_c(x,y)$ is an eigenvector with eigenvalue of $-2c^2$ for Δ.

(c) Suppose the eigenvalues c_i are distinct and non-negative for the set $(f_{c1},...f_{ci})$. Recall Theorem 3 on Workbook 24 which states that if we have a set of eigenvectors, in this case, $(f_{c1},...f_{ci}$ have distinct eigenvalues, a fact we know from the hypothesis, therefore by Workbook 5, Theorem 3 $\{v_1,...v_n\}$.

(d) Let $f(x,y) = 3$ Therefore $\frac{\partial^2 f}{\partial x^2}(x,y) = 0$ and $\frac{\partial^2 f}{\partial y^2}(x,y) = 0$ Therefore $\Delta(f(x,y)) = 0$ and thus this a harmonic function.

Secondly, let $g(x,y) = x+y$. Therefore $\frac{\partial^2 g}{\partial x^2}(x,y) = 0$ and $\frac{\partial^2 g}{\partial y^2}(x,y) = 0$ as the second derivative of both x and y are both 0 Therefore $\Delta(g(x,y)) = 0$ and thus this a harmonic function.

Thirdly, let $h(x,y) = x+3$. Therefore $\frac{\partial^2 g}{x^2}(x,y) = 0$ and $\frac{\partial^2 h}{\partial y^2}(x,y) = 0$ as the second derivative of both x and 3 are 0 Therefore $\Delta(h(x,y)) = 0$ and thus this is a harmonic function.

45. (a) Given $n = 2$ then $A = \begin{bmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}$. Therefore $\det(A) = (1)\alpha_2 - 1(\alpha_1) = \alpha_2 - \alpha_1$. Furthermore when $n = 2$, $\prod_{1\le i<j\le 2}(a_j - a_i) = \alpha_2 - \alpha_1$ Thus $\det(A) = \prod_{1\le i<j\le 2}(a_j - a_i)$.

Given $n = 3$ then $A = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{bmatrix}$. Therefore by Laplace Expansion we can compute $\det(A)$. Note then that

$\det(A) = 1\begin{bmatrix} 2 & \alpha_2^2 \\ \alpha_3 & \alpha_3^2 \end{bmatrix} - 1\begin{bmatrix} a_1 & \alpha_1^2 \\ \alpha_3 & \alpha_3^2 \end{bmatrix} + 1\begin{bmatrix} \alpha_1 & \alpha_1^2 \\ \alpha_2 & \alpha_2^2 \end{bmatrix}$
$= \alpha_2\alpha_3^2 - \alpha_3\alpha_2^2 - (\alpha_1\alpha_3^2 - \alpha_3\alpha_1^2) + \alpha_1\alpha_2^2 - \alpha_2\alpha_1^2$
$= \alpha_1^2(\alpha_3 - \alpha_2) + \alpha_2^2(\alpha_1 - \alpha_3) + \alpha_3^2(\alpha_2 - \alpha_1)$

Furthermore when $n = 3$,
$\prod_{1\le i<j\le 3}(\alpha_j - \alpha_i)$
$= (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)$
$= \alpha_3^2\alpha_2 - \alpha_1\alpha_3^2 + \alpha_3\alpha_1^2 - \alpha_2^2\alpha_3 + \alpha_2^2\alpha_1 - \alpha_2\alpha_1^2$
$= \alpha_1^2(\alpha_3 - \alpha_2) + \alpha_2^2(\alpha_1 - \alpha_3) + \alpha_3^2(\alpha_2 - \alpha_1)$.
Thus $\prod_{1\le i<j\le 3}(\alpha_j - \alpha_i) = \alpha_1^2(\alpha_3 - \alpha_2) + \alpha_2^2(\alpha_1 - \alpha_3) + \alpha_3^2(\alpha_2 - \alpha_1) = \det(A)$ and therefore $\det(A) = \prod_{1\le i<j\le 3}(\alpha_j - \alpha_i)$ when $n = 3$.

(b) From Homework 7, part 5b, we found that $[T]_{(B,C)}[u]_b = [T(u)]_C$. Therefore in the terms of this problem we know that $[T]_{(B,E)}[u]_B =$

$[T(u)]_E$ for $u \in \mathcal{P}_{n-1}$. Therefore using the key theorem we know that $[T]_{(B,E)} = \begin{bmatrix} \vdots & & \vdots \\ [T(1)]_E & \cdots & [T(x^{n-1})]_E \\ \vdots & & \vdots \end{bmatrix}$. Therefore it follows that $T(1) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and thus $[T(1)]_C = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Therefore by the general key theorem we know that the first column of the matrix that represents $[T]_{(B,E)}$ is $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Similarly for $T(x)$ we find that $T(x) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$, and therefore the second column of this matrix is $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$. Therefore for a general case for x^k where $1 < k \leq n$ we find that $T(x^k) = \begin{bmatrix} \alpha_1^k \\ \alpha_2^k \\ \vdots \\ \alpha_n^k \end{bmatrix}$.

Furthermore note that in A the kth column is also $\begin{bmatrix} \alpha_1^k \\ \alpha_2^k \\ \vdots \\ \alpha_n^k \end{bmatrix}$. Therefore as this follows in a general case for $[T]_{(B,E)}$ then we know that this must be equal to A.

(c) Let $C = (1, g_1, g_2, ..., g_k, ..., g_{n-1})$. Where $g_k = \prod_{i=1}^{k}(x - \alpha_i)$. Therefore suppose that for $c_1, c_2, ..., c_n \in \mathbb{R}$ such that $c_1(1) + c_2(g_2) + ...c_n(g_{n-1}) = 0$. Furthermore let us begin by setting $x = \alpha_1$, note then that $g_1 = (x - \alpha_i)$, $g_2 = (x - \alpha_1)(x - \alpha_2)$, this pattern continues to g_{n-1}. Therefore note that $g_1, ...g_{n-}$ all contain $(x - \alpha_1)$ therefore when $x = \alpha_1$ then this will be $(\alpha_1 - \alpha_1) = 0$ and thus $g_1, ...g_{n-1} = 0$. Therefore we have $c_1(1) + 0 + ... + 0 = 0$ and thus we can conclude that $c_1 = 0$. Furthermore suppose that now $x = \alpha_2$ therefore we see similarly that all elements besides 1, and g_1 have an element $(x - \alpha_2)$ which is 0 when $x = \alpha_2$ and thus $g_2, ...g_{n-1} = 0$. Therefore we see that $c_1(1) + c_2(\alpha_2 - \alpha_1) + c_3(0) + ...c_n(0) = 0$. Therefore as we know the value of c_1 is 0 we can disregard this element and have $c_2(\alpha_2 - \alpha_1) + c_3(0) + ...c_n(0) = 0$, therefore we can conclude that $c_2 = 0$. Now suppose for a general case that $x = \alpha_k$ as $g_k, ..g_{n-1}$ all contain the term $(x-\alpha_k)$ and thus when $x=\alpha_k$ all these will go to zero. Therefore we are left with $c_1(1)+...+c_{k-1}+c_k(g_{k-1})+(0)+...+0=0$. Therefore depending on the k as above we can disregard the previous elements besides $c_k(g_{k-1})$ and find that $c_k=0$. This can be found where $1<k\leq n-1$ and thus we find that $c_1=c_2=...=c_n=0$ and thus the only relation is the trivial one and thus C is linearly independent. Furthermore note that $\dim(C)=n$ as it contains n elements. Similarly $\dim(P_{n-1})=n$ Thus $\dim(C)=\dim(P_{n-1})$. Therefore as this set is the same dimension, and C is linearly independent, we can further include that C must span P_{n-1}. Therefore as C is linearly independent, and spans P_{n-1} then we can conclude that C is a basis for P_{n-1}.

(d) Let $B=(1,x,...,x^{n-1})$ and $C=(1,g_1,g_2,...,g_{n-1})$ where $g_k=\prod_{i=1}^{k}(x-\alpha_i)$. Thus for example $g_1=(x-\alpha_1)$ Recall that $S_{C\to B}$ is defined as $\begin{bmatrix} \vdots & & & & \vdots \\ [1]_B & \cdots & [g_k]_B & \cdots & [g_{n-1}]_B \\ \vdots & & & & \vdots \end{bmatrix}$. Therefore $1=1(1)+0(x)+...+(0)x^{n-1}$. Therefore $[1]_C=\begin{bmatrix}1\\0\\\vdots\\0\\0\\0\end{bmatrix}$. Similarly for g_k where $1<k\leq n-1$ we first note that $g_k=(x-\alpha_1)(x-\alpha_2)...(x-\alpha_k)$. Note that this will result in a polynomial $x^k+c_{k-1}x^{k-1}+...+c_1$ where $c_1,...,c_{k-1}\in\mathbb{R}$. Therefore $[g_k]_B=c_1(1)+c_2(x)+...c_{k-1}(x^{k-1})+1(x^k)$. Thus $[g_k]_B=\begin{bmatrix}c_1\\\vdots\\1\\0\\\vdots\\0\end{bmatrix}$ where this is then the kth column of $S_{B\to C}$. Thus it follows in the general case that $S_{B\to C}$ is upper triangular with 1s on the diagonal.

(e) Similar to part b, note that $[T]_{(C,E)}[u]_C=[T(u)]_E$. Thus $[T(u)]_E=\begin{bmatrix} \vdots & & & & \vdots \\ [T(1)]_E & \cdots & [T(g_k)]_E & \cdots & [T(g_{n-1})]_E \\ \vdots & & & & \vdots \end{bmatrix}$. $T(1)=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$ and thus

$[T(1)]_E = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Recall that $g_k = (x-\alpha_1)(x-\alpha_2)...(x-\alpha_k)$. Therefore it follows that $T(g_k) = \begin{bmatrix} 0_1 \\ \vdots \\ 0_k \\ (\alpha_{k+1}-\alpha_1)(\alpha_{k+1}-\alpha_2)...(\alpha_{k+1}-\alpha_k) \\ \vdots \\ (\alpha_n-\alpha_1)(\alpha_n-\alpha_2)...(\alpha_n-\alpha_k) \end{bmatrix}$. Thus $[T(g_k)]_E = \begin{bmatrix} 0_1 \\ \vdots \\ 0_k \\ (\alpha_{k+1}-\alpha_1)(\alpha_{k+1}-\alpha_2)...(\alpha_{k+1}-\alpha_k) \\ \vdots \\ (\alpha_n-\alpha_1)(\alpha_n-\alpha_2)...(\alpha_n-\alpha_k) \end{bmatrix}$. Therefore we have shown that for a general case that $[T]_{(C,E)}$ is in fact equal to the matrix we are verifying in the problem.

(f) Using the incredibly useful lemme we know that $[T]_{(B,E)}e_j$ is equal to the jth column of $[T]_{(B,E)}$. It follows then that $[T]_{(B,E)}e_j = [T(x^{j-1}]_E = T(x^{j-1})$. We can use a similiar method to find the jth column of $[T]_{C,E)}S_{B\to c}e_j$. Therefore it follows that $[T]_{C,E)}S_{B\to c}e_j$
$= [T]_{C,E)}[x^{j-1}]_C$
$= [T(x^{j-1}]_E = T(x^{j-1})$. Therefore as for an arbitrary column these two matrices all the same we can conclude that all their columns are the same and thus they are equal to each other. Therefore $[T]_{(B,E)} = [T]_{C,E)}S_{B\to c}$.

46. (a) If $p(x)$ passes through these points then we know that $p(\alpha_1) = \beta_1$, $p(\alpha_2) = \beta_2$, $p(\alpha_3) = \beta_3$, $p(\alpha_4) = \beta_4$. Thus for a polynomial of degree less than or equal to $n-1$ where $c_0, ...c_{n-1} \in \mathbb{R}$ we know that for $p(\alpha_1)$ that $c_0 + c_1(\alpha_1) + c_2(\alpha_1)^2 + ... + c_{n-1}(\alpha_1)^{n-1} = \beta_1$, and $p(\alpha_{2)=c_0+c_1(\alpha_2)+c_2(\alpha_2)^2+...+c_{n-1}(\alpha_2)^{n-1}=\beta_2}$, $p(\alpha_3) = c_0 + c_1(\alpha_3) + c_2(\alpha_3)^2 + ... + c_{n-1}(\alpha_3)^{n-1} = \beta_3$, and so on and so forth through $p(\alpha_n) = c_0 + c_1(\alpha_n) + c_2(\alpha_n)^2 + ... + c_{n-1}(\alpha_n)^{n-1} =_{\beta\ n}$. Now note that this is clearly a system of linear equations and then we can write this as a system of equations, justly $A = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{bmatrix}$,

as well as $v = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$ and $z = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$. Thus we have that $A\vec{v} = \vec{z}$ Notice at this point that A is matrix we are shown in the hypothesis in problem 4 therefore we know we can use Vandermonde Determinant to compute $det(A)$. Thus from part 4 we know that $\prod_{1 \leq i < j \leq n}(a_j - a_i)$ therefore this is equal to $(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)....(\alpha_n - \alpha_1)(\alpha_n - \alpha_2)...(\alpha_n - \alpha_{n-1})$ as it is given that $\alpha_1, ...\alpha_n$ are distinct then we know that $\alpha_1 \neq \alpha_2 \neq ... \neq \alpha_n$ and thus $det(A) \neq 0$. Therefore we can conclude that A must be invertible. As A is invertible we know there must exist A^{-1}, thus multiplying $A\vec{v} = \vec{z}$ by A^{-1} on both sides we find that $I\vec{v} = A^{-1}\vec{z}$, note that as $A^{-1}\vec{z}$ is equal to a $1 \times n$ column vector we find unique values for $\vec{v}$ which is in fact $c_0, ...c_{n-1}$ the coefficients for a polynomial that bases through these points.

(b) If $p(x)$ passes through these points then we know that $p(1) = 8$, $p(2) = -1$, $p(3) = -4$, $p(4) = 5$. Thus for a polynomial of degree less than or equal to 3 where $c_0, ...c_3 \in \mathbb{R}$ we know that for $p(1)$ that $c_0+c_1(1)+c_2(1)^2+c_3(1)^3 = 8$, and $p(2) = c_0+c_1(2)+c_2(2)^2+c_3(2)^3 = -1$, $p(3) = c_0 + c_1(3) + c_2(3)^2 + c_3(3)^3 = -4$, and $p(4) = c_0 + c_1(4) + c_2(4)^2 + c_3(4)^3 = 5$. Notice at this point that there are all linear equations and thus this is a linear system, we can then write this as a matrix, where we have $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 8 \\ 1 & 2 & 4 & 8 & -1 \\ 1 & 3 & 9 & 27 & -4 \\ 1 & 4 & 16 & 64 & 5 \end{bmatrix}$. Note we can also use the method found above in part a to compute this, though with the low degree of such a polynomial is is also quite simple to row reduce this to find values for $c_0, ...c_3$. Computing Gaussian-Jordan elimination of A we find that $rref(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 17 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ and thus $c_0 = 17$, $c_1 = -7$, $c_2 = -3$, $c_3 = 1$ Therefore we can plug these bag into the polynomial form above to construct the polynomial that passes through all these points, thus $p(x) = 17 + -7(x) + -3(x)^2 + x^3$ is the polynomial that intersects the four points given in the hypothesis.