Dynamical Systems

MA-375/M75

Lecture Notes

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Chapter I

One-dimensional dynamical systems

SECTION 1: Introduction

1.1 Definition

- DYNAMICAL SYSTEM describes how a point in a space evolves in time.
- The space is called the STATE SPACE.
  It may be e.g. a Euclidean space or more abstract space (metric, topological etc.).
  We will deal mostly with $\mathbb{R}$ (this chapter) and $\mathbb{R}^2$ (next chapter).

1.2 Discrete time *vs.* continuous time

Let $\mathbb{R}$ be the state space and let $x \in \mathbb{R}$ be a point. Let $f : \mathbb{R} \to \mathbb{R}$.

- If time changes discretely, e.g. $n = 0, 1, 2, \ldots$, the dynamical system is given through a DIFFERENCE EQUATION, e.g.

$$x(n+1) = f\big(x(n)\big). \tag{1.1}$$

- If time changes continuously, e.g. $t \in \mathbb{R}_+ := [0, \infty)$, the dynamical system is give through a DIFFERENTIAL EQUATION, e.g.

$$x'(t) = f\big(x(t)\big). \tag{1.2}$$

- For both difference and differential equation one can consider an **INITIAL CONDITION**: the value of x at a fixed 'initial' moment of time, e.g. at time 0:

$$x(0) = x_0 \in \mathbb{R}. \tag{1.3}$$

- The difference equation (1.1) can be then successively solved:

$$x(1) = f(x(0)) = f(x_0),$$
$$x(2) = f(x(1)) = f(f(x_0)),$$

  and so on.

  For some 'simple' difference equations one can even find an explicit formula for $x(n)$ as a function of $n \in \mathbb{N} \cup \{0\}$.
- The differential equation (1.2) also can be solved in very limited cases: this is an equation with separated variables, hence one needs that $\int \frac{1}{f(x)}\,dx$ can be integrated explicitly that is, typically, not the case.
- Actually, the course of the dynamical systems is about how to get properties of solutions when one can not find the solutions explicitly.

1.3 Remark

- It may look more natural if we write x_n instead of $x(n)$. We use here the latter notation to stress that x is a function of (discrete) time.
- We may be interested not only in the behaviour of solutions for 'future times', but also for the past times. In this case one can consider e.g. discrete time $n \in \mathbb{Z}$ or continuous time $t \in \mathbb{R}$.

1.4 Example: Compound interest

Let, for an $r \in (0, 1)$, $f(x) = (1+r)x$, and

$$x(n+1) = f\big(x(n)\big) = (1+r)x(n),$$

where $x(n)$ represents money on an account at the end of the year n. Therefore, after the year $n+1$ the existing money $x(n)$ will increase on $r\,x(n)$ i.e. r represents the interest rate. Then, if initially there was $x(0) = x_0 > 0$ on that account, we get that

$$x(n) = x_0(1+r)^n.$$

We normally expects a positive interest rate, i.e. $r \in (0,1)$. Then, naturally, $x(n)$ increases when n rises.

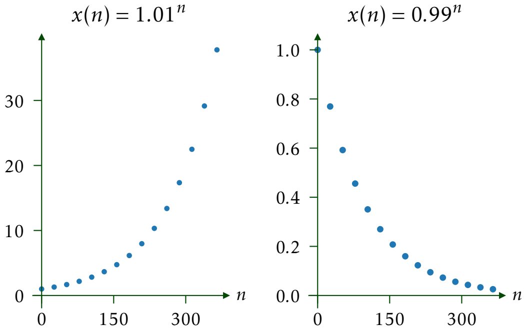
If, however, $r \in (-1,0)$, then $0 < 1 + r < 1$ and $x(n)$ decreases when n rises.

Let $x(0) = 1$ and let n represent days of year, and $x(n)$ represents the productivity. Then, for $r = 0.01$,

$$x(365) = (1 + 0.01)^{365} \cdot 1 \approx 37.783,$$

whereas, for $r = -0.01$,

$$x(365) = (1 - 0.01)^{365} \cdot 1 \approx 0.026.$$



1.5 Example: Malthus equation

Let, for $r \in \mathbb{R}$, $f(x) = rx$ and

$$x'(t) = f\big(x(t)\big) = rx(t). \tag{1.4}$$

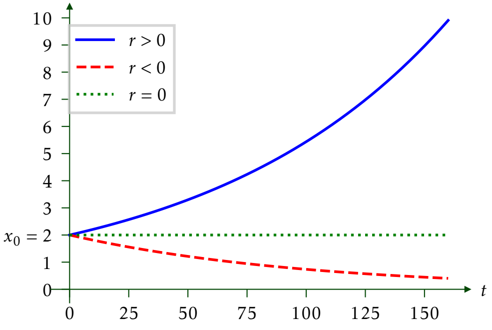
Separating variables one gets

$$\frac{dx}{dt} = rx, \qquad \frac{dx}{x} = r\,dt, \qquad \ln|x| = rt + C,$$

for a $C \in \mathbb{R}$, and hence $x(t) = C_1 e^{rt}$, where $C_1 = \pm e^C \in \mathbb{R}$.

Taking $t = 0$, we get $C_1 = x(0) =: x_0$. Then

$$x(t) = x_0 e^{rt}.$$



1.6 From discrete to continuous time

Comparing solutions for the *compound interest* and for the *Malthus equation* (1.4): one can rewrite, respectively,

$$x(t) = x_0(1+r)^t, \qquad t = n \in \mathbb{N},$$

and

$$x(t) = x_0(e^r)^t, \qquad t \in \mathbb{R}_+.$$

Surely, the answers are different, but there is a simularity. Note that, for small values of $|r|$, $e^r \approx 1+r$, that follows from the Taylor expansion:

$$e^r = 1 + r + \frac{r^2}{2} + \dots$$

To observe this simularity for any value of r, suppose now that the interest rate is paid more often, for example, each $\frac{1}{k}$ part of a year for some $k \in \mathbb{N}$.

Then each year will be k payments, and after n years there will be nk payments.

Naturally, the interest for each payment will be smaller, for example, let it be $\frac{r}{k}$.

Surely, the amount of money will be now different from the considered in Example 1.4, so we denote it by $\tilde{x}(n)$.

We have then

$$\begin{aligned}\tilde{x}(n) &= x_0\left(1+\frac{r}{k}\right)^{nk}\\ &= x_0\left(1+\frac{r}{k}\right)^{\frac{k}{r}\,r\,n}.\end{aligned}$$

Let now $k \to \infty$, then the equality

$$\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{\frac{k}{r}} = e$$

implies that

$$\tilde{x}(n) = x_0 e^{r\,n},$$

that coincides with the solution to the Malthus equation (at discrete moments of time).

1.7 Logistic-type model in discrete time

Rewriting the *compound interest model* in the equivalent form

$$x(n+1)-x(n)=rx(n),$$

one gets the *population model* discovered by Thomas Malthus in 1798: the population of the UK increased each 10 years by the number proportional to the previous one.

In reality, however, population growth has more complicated behaviour. For example, one can consider, for some $g:\mathbb{R}\to\mathbb{R}$,

$$x(n+1)-x(n)=g\big(x(n)\big)$$

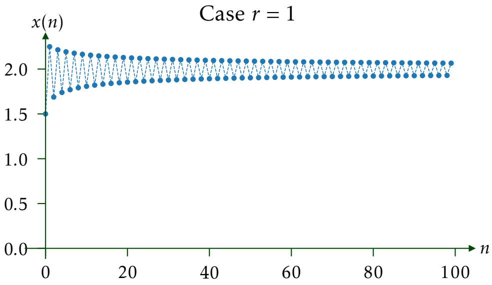
We consider a *logistic-type model* where $g(x)=rx(B-x)$ for some $r,B>0$:

$$x(n+1)-x(n)=rx(n)\big(B-x(n)\big).$$

Then

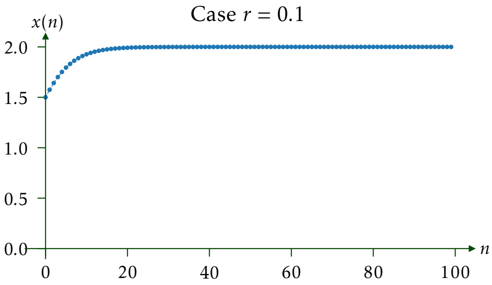
$$\begin{aligned}
x(n)<B &\Longrightarrow x(n+1)>x(n),\\
x(n)=B &\Longrightarrow x(n+1)=x(n),\\
x(n)>B &\Longrightarrow x(n+1)<x(n).
\end{aligned}$$

Then, e.g. for $x_0 = x(0) = 1.5 < B = 2$ and $r = 1$, one has



The solution oscillates around the value $x = B$ and the amplitude of the oscillation decays with (discrete) time.

However, discrete-time dynamical systems are very sensitive to the values of parameters, the same model for $r = 0.1$ behaves as follows:



The rigorous analysis of discrete-time dynamical systems can be done, but it requires good knowledge of difference equations.

Instead, in this course, we will deal with continuous-time dynamical systems and differential equations.

Again, to get the continuous time analogue for the logistic-type model, one should assume that the changes take places very often, say, after time $h > 0$ rather than time 1, i.e.

$$x(t+h) - x(t) = r_h x(t)\Big(B - x(t)\Big).$$

We wrote r_h as the rate should depend on h. For example, one can consider $r_h = rh$ for some constant $r > 0$. Then

$$\frac{x(t+h) - x(t)}{h} = rx(t)\Big(B - x(t)\Big).$$

Passing h to 0, one gets a differential equation, called the **LOGISTIC EQUATION**,

$$\boxed{x'(t) = rx(t)\Big(B - x(t)\Big).} \tag{1.5}$$

In the next sections, we discuss how to analyse behaviour of solutions to certain classes of differential equations, including (1.5).

Section 2: Differential Equations

2.1 Terminology

- Let $f : \mathbb{R} \to \mathbb{R}$. Consider the differential equation

$$x'(t) = f\big(x(t)\big), \qquad t \in \mathbb{R}. \tag{2.1}$$

- We will normally omit t by writing just

$$x' = f(x).$$

- Recall, that one can interpret t as time (if $t < 0$ it hence corresponds to the 'past').
- If we set the value of x at some t_0, i.e. $x(t_0) =: x_0 \in \mathbb{R}$, (2.1) is called the **initial value problem** (and, recall, x_0 is called *the initial condition*). Typically, $t_0 = 0$.
- Equation (2.1) defines a continuous **dynamical system**.

2.2 Remark

- Differential equation (2.1) defines a **first-order** dynamical system, because its right-hand side does not depend explicitly on t.
- For example, $x' = \sin x$ defines a first-order dynamical system, whereas $x' = t + \sin x$ defines a **SECOND-ORDER** dynamical system (because its right-hand side depends on two arguments: t and x).
- This (standard) terminology slightly contradicts with the (standard) terminology of the theory of differential equation, where (2.1) is called the *first-order* differential equation because its maximal order of derivatives is 1 (i.e. both $x' = \sin x$ and $x' = t + \sin x$ are first-order differential equations).
- Note that, in the theory of differential equations, the equation (2.1) is called *autonomous* since its right-hand side does not depend on t (whereas, e.g. $x' = t + \sin x$ is *non-autonomous*).
- To summarize: a first-order dynamical system is a first-order autonomous differential equation (with a clash of notions).

2.3 Theorem: Existence and uniqueness

Let $f = f(x)$ be a continuously differentiable (a.k.a. *smooth*) real-valued function on an interval $x \in (a, b) \subseteq \mathbb{R}$ (i.e. both f and f' are continuous on (a, b)).

Let $t_0 \in R$ and $x_0 \in (a, b)$. Then

- the initial value problem

$$x'(t) = f\big(x(t)\big), \quad x(t_0) = x_0 \tag{2.2}$$

  has a unique solution on the time interval $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$;
- there exists a maximum (possibly infinite) time interval $(T_-, T_+) \subseteq \mathbb{R}$ where (2.2) has a unique solution;
- the solution $x = x(t)$ is continuously differentiable in $t \in (T_-, T_+)$;
- if $T_\pm \neq \pm\infty$, then there are exactly two possibilities:
  - $|x(t)|$ becomes unbounded (arbitrary large) when $t \to T_\pm$ (it is called the *explosion in finite time*);
  - $\lim\limits_{t \to T_\pm} x(t) \in \{a, b\}$(the solution leaves the domain where f is smooth).

2.4 Example: Non-uniqueness

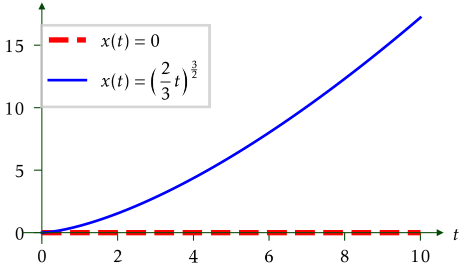
Consider the initial value problem

$$x' = x^{\frac{1}{3}}, \qquad x(0) = 0. \tag{2.3}$$

Then evidently $x(t) = 0$ for all $t \geq 0$ is a solution to (2.3). However, if $x \neq 0$, we get

$$\frac{dx}{dt} = x^{\frac{1}{3}}, \qquad \frac{dx}{x^{\frac{1}{3}}} = dt, \qquad \frac{3}{2}x^{\frac{2}{3}} = t + C.$$

Take $t = 0$, then $C = 0$, as as $x(0) = 0$. Hence, there is another solution to (2.3): $x = x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$. The reason for the non-uniqueness of solutions is that, the function $f(x) = x^{\frac{1}{3}}$ has unbounded (discontinuous) derivative: $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ is unbounded near $x_0 = 0$.



2.5 Example: Explosion in finite time

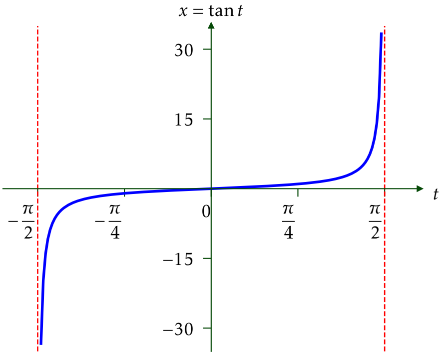
Consider the initial value problem

$$x' = x^2 + 1, \qquad x(0) = 0. \tag{2.4}$$

Then

$$\frac{dx}{x^2+1} = dt, \qquad \tan^{-1} x = t + C$$

and taking $t = 0$ one gets $C = 0$. Hence, $x = \tan t$ is the unique solution to the initial value problem (2.4). This solution can not be continued beyond the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ since $\lim\limits_{t \to \pm\frac{\pi}{2}} \tan(t) = \infty$.



2.6 Example: Non-expandable time interval

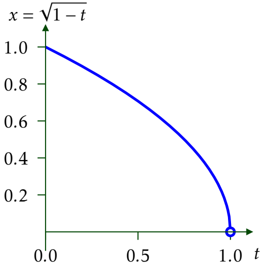
Consider the initial value problem

$$x' = -\frac{1}{2x}, \qquad x(0) = 1. \tag{2.5}$$

Note that here $f(x) = -\frac{1}{2x}$ is *continuously differentiable* on $(a,b) = (0,\infty)$ and $1 \in (a,b)$. We have

$$2x dx = -dt, \qquad x^2 = -t + C,$$

and taking $t = 0$ one gets $1 = x(0)^2 = C$, hence $x = \pm\sqrt{1-t}$ and since $x(0) = 1 > 0$, one gets the unique solution to (2.5): $x(t) = \sqrt{1-t}$. In contrast to Example 2.5, the solution is bounded, however, it cannot be extended beyond $t \in [0,1)$, see r.h.s. of (2.5); indeed, $\lim_{t\to 1} x(t) = x(1) = 0 = a \notin (a,b) = (0,\infty)$.



2.7 Remark

- If f is continuously differentiable on an infinite interval (e.g. on $\mathbb{R}$) then both possibilities presented in Theorem 2.3, actually, coincide.
- In this course, we will deal mainly with differential equations $x' = f(x)$ where f is continuously differentiable on $\mathbb{R}$. Then the cases considered in Examples 2.4 and 2.6 are impossible. How to recognise the case similar to that in Example 2.5, we will discuss below.

2.8 Definition: Trajectories

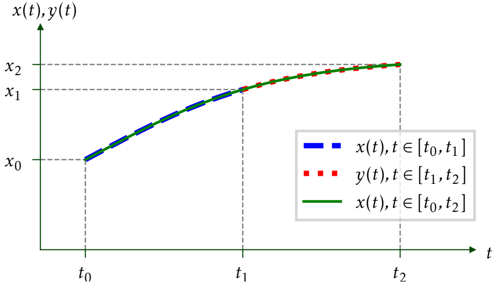
- Let $x(t)$ be a solution to the differential equation $x'(t) = f(x(t))$ on a time interval $t \in (T_-, T_+) \subseteq \mathbb{R}$. Then the set of points

$$\left\{ \Big(t, x(t)\Big) \in \mathbb{R}^2 \;\Big|\; t \in (T_-, T_+) \right\}$$

  is called a **TRAJECTORY**.
- The graph of a trajectory on the coordinate plane $\Big(t, x(t)\Big)$ is called the **SPACE-TIME DIAGRAM**.

2.9 Flow property

Let f be continuosly differentiable on an interval (a, b), so that the equation $x' = f(x)$ has the unique solution on a time interval (T_-, T_+). Let $x(t_0) = x_0 \in (a, b)$ for some $t_0 \in (T_-, T_+)$. Take any t_1 such that $t_0 < t_1 < T_+$. Denote $x_1 := x(t_1) \in (a, b)$. Consider now **the same** differential equation $y' = f(y)$ (written just for another *notation* for the solution) and consider the initial condition for it: $y(t_1) = x_1$. By the uniqueness, one gets that $x(t_2) = y(t_2) =: x_2$ for any $t_2 \in (t_1, T_+)$.



In other words, if a solution evolves on $[t_0, t_1]$, then stops at the moment of time t_1 and immediately starts (at the same value) evolving again on $[t_1, t_2]$, then the result is the same if the solution evolved, starting from the moment of time t_0, on $[t_0, t_2]$.

This is called the **FLOW PROPERTY**. If we denote by $U(t)$ the mapping which maps any $x_0 \in (a, b)$ into the solution of the initial value problem $x' = f(x)$, $x(0) = x_0$ (notice that the initial time is 0 here), i.e. $U(t)x(0) = x(t)$, then denoting $t = t_1 - t_0 \geq 0$ and $s = t_2 - t_1 \geq 0$ in the above, one gets

$$U(s)U(t)x_0 = U(t+s)x_0. \tag{2.6}$$

Equality (2.6) is called the **SEMIGROUP PROPERTY** of the flow.

2.10 Remark: Why 'flow'

It is often called that U passes the solution to the equation $x' = f(x)$ along its trajectory (like the water flow is passed along a stream).

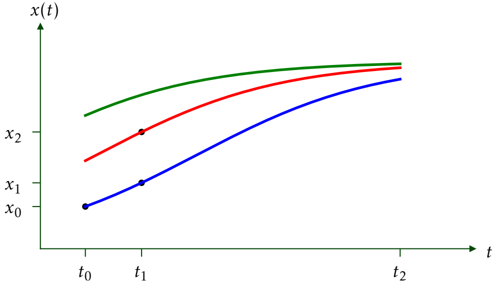
2.11 Remark

It was crucial, of course, that the initial value for the 'second' solution was the termination value for the 'first' solution, and because of this, the solution is, actually, the same.

2.12 Trajectories on the space-time diagram

Let f be continuously differentiable on an interval (a,b) (e.g. $(a,b) = \mathbb{R}$). Then, by Theorem 2.3 there exists time-interval (T_-, T_+) (finite or infinite), such that, for any $t_0 \in (T_-, T_+)$ and for any $x_0 \in (a,b)$ the initial value problem $x' = f(x)$, $x(t_0) = x_0$ has the unique solution on (T_-, T_+). The corresponding trajectory is a *continuous* (and even *smooth*) curve on the space-time diagram; it passes through the point (t_0, x_0). By the uniqueness property, if we take another point, then

- either this point, (t_1, x_1), belongs to the same trajectory, i.e. $x_1 = x(t_1)$,
- or this point, (t_1, x_2), is such that $x_2 \neq x(t_1)$, and one can consider **the same** differential equation $y' = f(y)$ with another initial condition $y(t_1) = x_2$. In the latter case, one gets **another** trajectory.



Therefore, through each point inside the open (possibly infinite) rectangle $(T_-, T_+) \times (a, b)$ on the space-time diagram there passes a trajectory. These trajectories either fully coincide or do not have common points at all (i.e. they neither intersect nor touch each other).

Section 3: Phase portrait

3.1 Definition

Let $f : (a, b) \to \mathbb{R}$ be a smooth function. A point $x_* \in (a, b)$ such that

$$f(x_*) = 0 \tag{3.1}$$

is called a **fixed point** (a.k.a. **stationary point**) of the dynamical system (or differential equation)

$$x' = f(x). \tag{3.2}$$

3.2 Remark

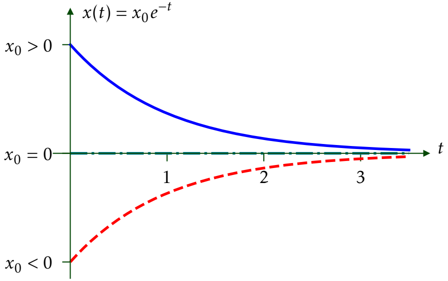
- We will always assume (without further notice) that the equation $f(x) = 0$ has finitely or countably many solutions $x \in (a, b)$.
- In particular, it may not have solutions at all; but we exclude e.g. the case when $f(x) = 0$ on an interval.

3.3 Properties of fixed points

- The constant function $x(t) = x_*$ for all $t \in \mathbb{R}$ (or another time interval) **is a solution** to the differential equation (3.2) (it is called the STATIONARY SOLUTION).
- The graph of this solution, that is the horizontal line $x = x_*$, is, hence, a *trajectory* of (3.2).
- As a result, any other trajectory of (3.2) does not cross nor touch that line.

3.4 Example

Consider the Malthus equation $x' = -x$. Here $f(x) = -x$, hence the only fixed point is $x_* = 0$. We consider now three trajectories for different signs of $x_0 := x(0)$, including $x_0 = x_* = 0$.



- Suppose, we don't know how to solve the equation $x' = -x$.
- One can still observe that $x_* = 0$ is the fixed point, hence $x(t) \equiv 0$ is a stationary solution.
- Next, we notice that, for $x > 0$, $x' = -x < 0$, i.e. the function $x = x(t)$ is **decreasing** everywhere in the upper halfplane.
- Similarly, for $x < 0$, $x' = -x > 0$, i.e. the function $x = x(t)$ is **increasing** everywhere in the lower halfplane.
- Finally, we know that the graphs of $x = x(t)$, i.e. the trajectories, do not intersect the line $x = 0$.
- Hence, the trajectories are like on the space-time diagram above.

3.5 Remark

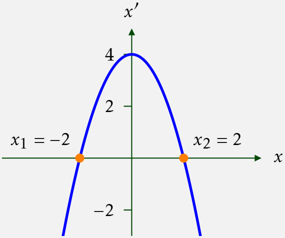
We will discuss below (see Theorem 3.9) why, in this example, $x = 0$ is the horizontal asymptote for all other trajectories.

3.6 Phase diagram

- Similarly, for a general equation $x' = f(x)$, we are interested where $f(x)$ is positive (hence, x increases), and where $f(x)$ is negative (hence, x decreases).
- To this end, one can consider the graph of $f(x)$. Since we are studying the equation $x' = f(x)$, one may think about x' as a function of x.
- Since x 'runs' over $\mathbb{R}$ which is, recall, called the **phase space**, the graph of $f(x)$ on the coordinate plane (x', x) is called the PHASE DIAGRAM.
- We find (all) fixed points (that are solutions to $f(x) = 0$).

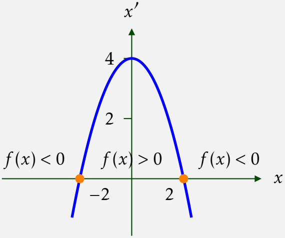
3.7 Example

Consider $x' = f(x) = 4 - x^2$. The fixed points are solutions to $4 - x^2 = 0$, i.e. $x_1 = -2$ and $x_2 = 2$.

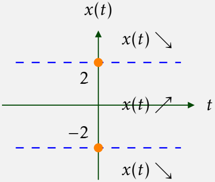


- The fixed points split the horizontal axis on the *phase diagram* on several parts (the first and the last are infinite).
- Naturally, the fixed points also split the vertical axis on the *space-time diagram* on the respectively equal parts (the lowest and the highest are infinite).
- Since f is continuous, it preserves the sign on each of the parts bordered by the fixed points.

In our example, one gets then three intervals on the phase diagram.

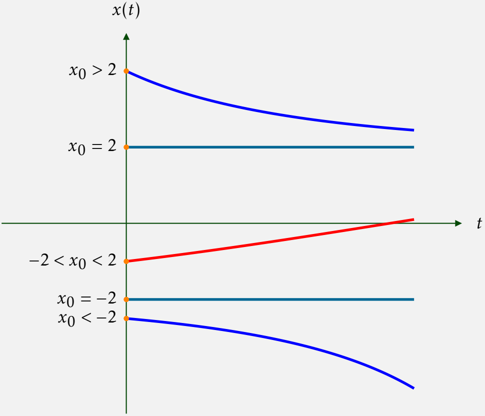


And the corresponding three intervals on the space-time diagram.



- One can get then sketches of the trajectories for the equation $x' = f(x)$ on the *space-time diagram*.
- Recall that, for each initial condition $x_0 = x(t_0)$, e.g. at time $t_0 = 0$, there will be **another** trajectory.
- Recall also that the horizontal lines which passes through the fixed points are trajectories, and that different trajectories do not have common points.

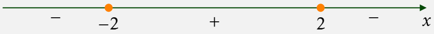
In our example, we have:



3.8 Phase portrait

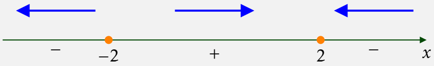
- To get the sketch of the space-time diagram, we actually need to know the fixed points and how $f(x)$ (and hence, x') alternates signs on the intervals bordered by the fixed points.

In the example above ($x' = 4 - x^2$), one actually needs this diagram only (a part of the phase diagram):



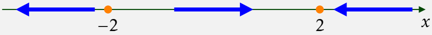
- Recall, that our main aim is understand how the position of the point x evolves with time, i.e. how $x(t)$ changes when t changes.
- Then $x' = x'(t)$ describes the velocity of this point. $x' > 0$ corresponds to the motion to the right direction, $x' < 0$ corresponds to the motion to the left direction.

Denoting the motion by arrows, one has (for $x' = 4 - x^2$):



- The arrows along the phase space (that is the real line) describe the point's motion qualitatively (i.e. we know where the point going, but we do not specify explicitly where the point is at each moment of time).
- The phase space with marked fixed points and arrows, representing the directions of the point's motion, form together the **PHASE PORTRAIT** of the dynamical system.

The phase portrait of the dynamical system $x' = 4 - x^2$ is hence



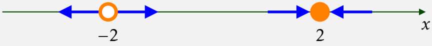
- The fixed points are like 'traps': the point can't leave them.
- If the arrows are directed **towards** a fixed point from both its sides, the fixed point is called a **STABLE FIXED POINT** or a **SINK**.
- If the arrows are directed **outwards** a fixed point (in both sides), the fixed point is called an **UNSTABLE FIXED POINT** or a **SOURCE**.

- Therefore, a point always moves away from an unstable fixed point, and it always move towards a stable fixed point.
- However, the point will never reach that stable fixed point (because of the uniqueness, as fixed points correspond to solutions), nor, evidently, ‘jump’ over it, as we consider the continuous motion.
- As a result, the point always remains in that interval of the phase bordered by the fixed points, where the point was at the initial time.
- We will denote **stable** and **unstable** fixed points by

   and ,

  respectively.

Therefore, the *phase portrait* for the dynamical system $x' = 4 - x^2$ is as follows:



Recall that, from the phase portrait, one can get the space-time diagram (as above).

3.9 Theorem: Convergence

Let $f : (a,b) \to \mathbb{R}$ be a smooth function. Let $x_* \in (a,b)$ be a *stable fixed point* of the dynamical system $x' = f(x)$. Let $x(t_0) =: x_0 \in (a,b)$ be such that there are no other fixed points of the dynamical system between x_0 and x_*. Then

$$\boxed{\lim_{t\to\infty} x(t) = x_*.} \tag{3.3}$$

3.10 Sketch of the proof

Suppose e.g. that $x_0 < x_*$. Since $f(x_*) = 0$ and there are no other fixed points on (x_0, x_*), function $f(x)$ preserves the sign on (x_0, x_*). Since x_* is stable, the arrow on the phase portrait goes *from* x_0 *towards* x_*, hence $f(x) > 0$ for $x \in (x_0, x_*)$, hence $x' > 0$, i.e. $x(t)$ increases starting from $t = t_0$.

The trajectory starts at (t_0, x_0), increases, and can't reach the line $x = x_*$. Therefore, $x(t) \le x_*$, hence, $x(t)$ cannot become unbounded. Also $(x_0, x_*) \subset (a,b)$. Hence both alternative in Theorem 2.3 are impossible, and, as a result, $x(t)$ exists and is unique for $t \in (t_0, \infty)$.

Denote $x_n = x(n)$, $n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is an increasing bounded sequence, therefore, there exists $x_\infty \in (x_0, x_*]$, such that

$$x_n \to x_\infty, \qquad n \to \infty.$$

By the mean-value theorem, for each $n \in \mathbb{N}$, there exists $\theta_n \in (0,1)$, such that

$$\begin{aligned} x(n+1) - x(n) \\ = x'(n+\theta_n)(n+1-n) \\ = x'(n+\theta_n) = f\big(x(n+\theta_n)\big). \end{aligned} \tag{3.4}$$

Since $n < n+\theta_n < n+1$ and x is increasing, $x_n \le x(n+\theta_n) \le x_{n+1}$. Then $x(n+\theta_n) \to x_\infty$, $n \to \infty$. Since f is continuous,

$$f\big(x(n+\theta_n)\big) \to f(x_\infty), \quad n \to \infty.$$

Therefore, by (3.4), $f(x_\infty) = 0$. Thus, x_∞ is a fixed point, but there are no fixed points on (x_0, x_*). Hence, $x_\infty = x_*$.

Therefore, the convergence in (3.3) takes place when t runs over N. Since x is increasing, we have

$$x([t]) \le x(t) \le x([t]+1)$$

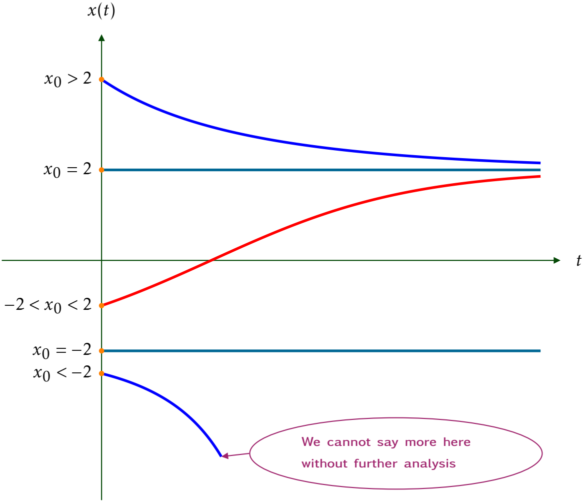
where $[t]$ is the entire part of t; that proves the convergence in (3.3) for real t.

3.11 Horizontal asymptotes of trajectories

As a result, on the space-time diagram, the horizontal trajectories corresponding to the stationary solutions at **stable fixed points** are asymptotes for other trajectories.

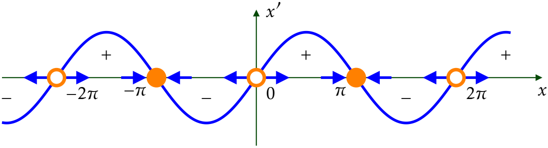
3.12 Example: Re-visited Example 3.7

We considered the dynamical system $x' = 4 - x^2$. It has two fixed points: $x = 2$ (stable) and $x = -2$ (unstable). Hence, the space-time diagram looks as follows:

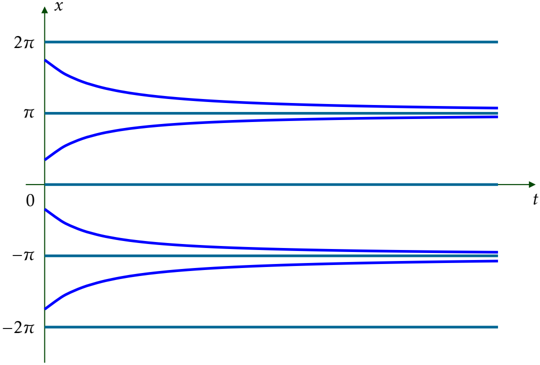


3.13 Example

Consider the dynamical system $x' = \sin x$. The fixed points are the solutions to $\sin x = 0$, i.e. $x = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$, or just $x = k\pi$, $k \in \mathbb{Z}$. Draw the graph of $f(x) = \sin x$ and consider the signs of f:



By drawing the corresponding arrows we conclude that $x = 0, 2\pi, -2\pi$ and, in general, $x = 2k\pi$, $k \in \mathbb{Z}$, are unstable fixed points, whereas $x = \pm\pi$ and, in general, $x = (2k+1)\pi$, $k \in \mathbb{Z}$, are stable fixed points. Then, one can sketch (a part of) the space-time diagram for different initial conditions $x_0 = x(0)$.



3.14 Unstable fixed points

- By the discussed above, if a trajectory starts between an unstable fixed point and a stable fixed point, it will converge asymptotically to the stationary solution (horizontal line) corresponding to the stable fixed point.
- However, if a trajectory starts (at a time t_0) between an unstable fixed point and infinity ($+\infty$ or $-\infty$), then two further scenarios are possible:
  - the solution $x = x(t)$ exists on a *finite* time interval (t_0, T_+) and then

$$\lim_{t \to T_+} x(t) = +\infty \text{ (or } -\infty);$$

  - the solution $x = x(t)$ exists on an *infinite* time interval (t_0, ∞) and then

$$\lim_{t \to \infty} x(t) = +\infty \text{ (or } -\infty).$$

- There is no general criteria which of scenarios takes place...

We start with an example demonstrating the first scenario.

3.15 Example: Once again Example 3.7

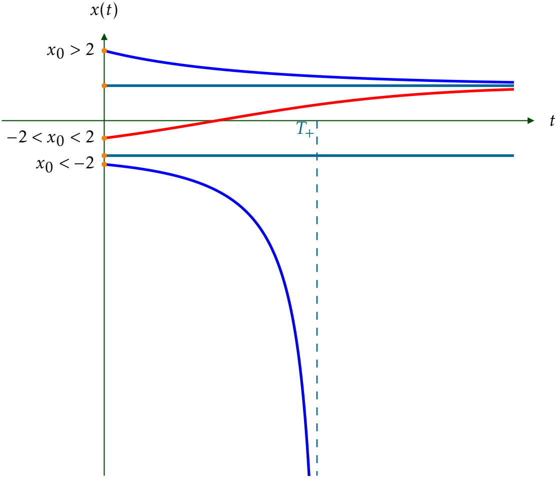
The differential equation $x' = 4 - x^2$ can be solved explicitly:

$$x(t) = 2\frac{x_0 - 2 + (x_0+2)e^{4t}}{2 - x_0 + (x_0+2)e^{4t}}, \tag{3.5}$$

where $x_0 = x(0)$ is the initial condition. When $x_0 < -2$, the denominator converges to 0 iff

$$e^{4t} \to \frac{x_0-2}{x_0+2} > 1, \quad t \to \frac{1}{4}\log\frac{x_0-2}{x_0+2} =: T_+ > 0.$$

One gets then the full space-time diagram:



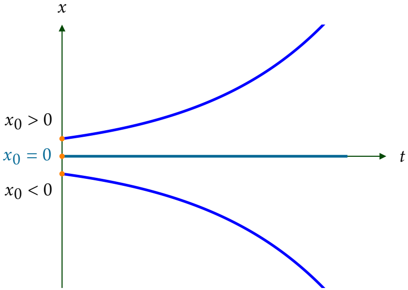
3.16 Remark

Note that, for $x_0 > 2$, the denominator in (3.5) also converges to 0 under the same conditions, however, in this case, $\frac{x_0-2}{x_0+2} \in (0,1)$, hence this takes place iff $t \to \frac{1}{4} \log \frac{x_0-2}{x_0+2} < 0$ that is impossible (stress that this is just confirm our previous analysis).

3.17 Example

The simplest example demonstrating the second scenario in 3.14 is the dynamical system $x' = x$. Here $f(x) = x$, hence the phase portrait is obvious:

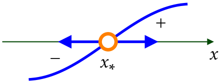
Thus, the only fixed point $x = 0$ is *unstable*. Recall that the solution is $x(t) = x_0 e^t$, where $x_0 = x(0)$. The space-time diagram is hence:



Hence the solution exists on $(0, \infty)$.

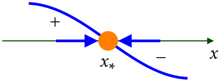
3.18 Linear stability analysis

- Let $x' = f(x)$ be a dynamical system with a smooth function f.
- So far, we obtained the phase portrait byeither solving the inequality $f(x) > 0$ (or $f(x) < 0$) or by sketching the graph of $f(x)$.
- Let x_* be a fixed point of $x' = f(x)$, i.e. $f(x_*) = 0$.
- If $f(x)$ is *negative* from the left of x_* and is *positive* from the right of x_*, then x_* is an **unstable** fixed point:



  Hence f is increasing around x_*, therefore, $f'(x*) \geq 0$. (To see why the equality is possible, think about $f(x) = x^3$.)

- Similarly, if $f(x)$ is *positive* from the left of x_* and is *negative* from the right of x_*, then x_* is an **stable** fixed point:



  Hence f is decreasing around x_*, therefore, $f'(x*) \leq 0$. (Again, the equality is possible, e.g. for $f(x) = -x^3$.)

3.19 Theorem: Sufficient conditions

Let $f : (a, b) \to \mathbb{R}$ be smooth on (a, b), and let $x_* \in (a, b)$ be a fixed point of the dynamical system

$$\boxed{x' = f(x),} \tag{3.6}$$

i.e. let $f(x_*) = 0$. Then

- if $\boxed{f'(x_*) > 0}$, then x_* is an UNSTABLE fixed point of (3.6);
- if $\boxed{f'(x_*) < 0}$, then x_* is a STABLE fixed point of (3.6).

3.20 Remark

- The mentioned above examples $f(x) = \pm x^3$ demonstrate why the conditions of Theorem 3.19 are sufficient, but not necessary.
- If $f'(x_*) = 0$, **one cannot conclude anything** about (un)stability of x_*; one needs an additional analysis (see also Remark 3.27 below).

3.21 Analytical explanation

Suppose that, for a smooth f,

$$f(x_*) = 0, \qquad f'(x_*) \neq 0.$$

Let $x(t)$ be close to x_*, i.e. $|x(t) - x_*|$ is small. Denote

$$\eta(t) := x(t) - x_*,$$

then $|\eta(t)|$ is small. By Taylor's formula,

$$\begin{aligned} f(x(t)) &= f(x_* + \eta(t)) \\ &= f(x_*) + f'(x_*)\eta(t) + o(\eta(t)), \end{aligned}$$

$$\text{where} \lim_{\eta(t)\to 0} \frac{o(\eta(t))}{\eta(t)} = 0.$$

But $f(x_*) = 0$, hence

$$f(x(t)) \approx f'(x_*)\eta(t).$$

On the other hand,

$$\eta'(t) = (x(t) - x_*)' = x'(t) = f(x(t)).$$

Therefore,

$$\eta'(t) \approx f'(x_*)\eta(t).$$

In other words, $x(t) = x_* + \eta(t)$, where $\eta(t)$ behaves (for its small values) like the solution to the Malthus equation

$$\eta' = f'(x_*)\eta.$$

The latter equation has the solution

$$\eta(t) = \eta(0)e^{f'(x_*)t}$$

Therefore, if $f'(x_*) < 0$, then $x(t) - x_* = \eta(t) \searrow 0$ as $t \to \infty$, i.e.

$$f'(x_*) < 0 \quad \Longrightarrow \quad x(t) \to x_*, \; t \to \infty.$$

On contrary, if $f'(x_*) > 0$, then $\eta(t)$ grows. Stress, that one **cannot say** that it will go to ∞, since as soon $|\eta(t)|$ becomes large, the previous considerations do not work, but, at least, we have that

$$f'(x_*) > 0 \implies x(t) \text{ 'runs away' from } x_*.$$

3.22 Remark

- What we actually did, we replaced *the study of* a nonlinear differential equation $x' = f(x)$ by *the study of* its **LINEARISATION** $\eta' = c\eta$, where $c = f'(x_*)$.
- Recall that $\eta(t) = x(t) - x_*$. One can hence 'shift' the phase space, to have the origin at x_*. For the shifted dynamical system, $x_* = 0$ is the fixed point, and we 'replaced' $x' = f(x)$ by $x' \approx f'(x_*)x$.

3.23 Example: Logistic equation

Consider the dynamical system

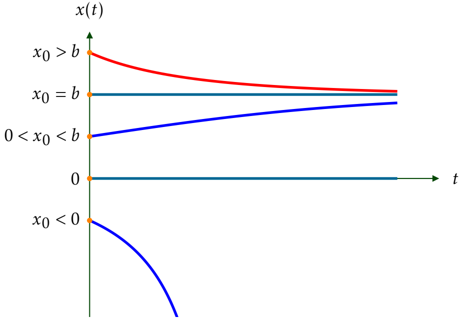
$$x' = ax(b - x), \qquad a, b > 0, \tag{3.7}$$

that is nothing but the logistic equation (1.5). For $f(x) = ax(b - x) = abx - ax^2$, we have two fixed points: $x_1 = 0$ and $x_2 = b > 0$. One can easily plot the graph of $f(x)$ as we did in Example 3.7, but we will use the linear stability analysis instead of. We have

$$\begin{aligned} f'(x) &= ab - 2ax, \\ f'(x_1) &= ab > 0, \\ f'(x_2) &= ab - 2ab = -ab < 0. \end{aligned}$$

Therefore, $x_1 = 0$ is an *unstable* fixed point, whereas $x_2 = b$ is a *stable* fixed point. We immediately have then the phase portrait

Therefore, taking also into account Theorem 3.19, one gets the space-time diagram:



3.24 Remark

The previous problem is, actually, almost equivalent to that in Example 3.7. Namely, consider $y = \frac{4}{b}x - 2$, then $x = \frac{b}{4}(y+2)$ and

$$\begin{aligned} y' &= \frac{4}{b}x' = \frac{4}{b}ax(b-x) \\ &= \frac{4}{b}a\frac{b}{4}(y+2)\left(b - \frac{b}{4}(y+2)\right) = \frac{ab}{4}(y+2)(2-y). \end{aligned}$$

Next, one can **change time**, by writing

$$t = t(\tau) := \frac{4}{ab}\tau.$$

Then, considering $z(\tau) = y(t(\tau))$, one gets

$$\begin{aligned}\frac{d}{d\tau}z(\tau) &= \frac{d}{d\tau}y(t(\tau)) = \frac{d}{dt}y(t)\cdot\frac{dt}{d\tau}\\ &= \frac{ab}{4}(y+2)(2-y)\cdot\frac{4}{ab}\\ &= 4-y^2 = 4-z^2,\end{aligned}$$

that is nothing but the equation from Example 3.7. In the same way, one can show that *all* equations of the form $x' = p(x-q)(r-x)$, $p,q,r>0$, have, actually, the same phase portraits. The 'simplest' equation is, of course $x' = x(1-x)$. The procedure described above is called **nondimensionalisation**: we exclude all parameters (a,b,p,q,r) from the equation, hence the space of parameters has not a dimension.

3.25 Remark

The logistic equation (3.7) has important applications, in particular, in biology. Namely, we interpret $x(t) \geq 0$ as the density of a population at time $t \geq 0$.

Recall that in the Malthus equation $x' = rx$ the **non-negative** solutions, as $t \to \infty$, may *either* converge to $+\infty$ (when $r > 0$, demonstrating that the population growth without any bounds) *or* converge to 0 (when $r < 0$ demonstrating that the population extincts). In the logistic equation (3.7) (called also the Verhulst equation), the population stabilises, converging to b.

3.26 Definition: Half-stable point

If x_* is a fixed point to a dynamical system $x' = f(x)$, so that the phase portrait (the direction of motion along the phase space) is not changing around x_*, then x_* is called a HALF-STABLE fixed point.

3.27 Remark

Since f is smooth, we have that **if** x_* is a *half-stable* fixed point, **then** $f'(x_*) = 0$.

However, the opposite is not true: **if** $f'(x_*) = 0$ **then** x_* may be: *stable* (e.g. for $f(x) = -x^3$), *unstable* (e.g. for $f(x) = x^3$), *half-stable* (e.g. for $f(x) = x^2$ or, differently, for $f(x) = -x^2$).

3.28 Graphical analysis

- When we sketched the graph of $f(x)$ and got from this the phase portrait for the dynamical system $x' = f(x)$, we, actually, performed a simple *graphical analysis*.
- The *linear stability analysis* is often a faster and a more straightforward way to get the phase portrait.
- However, the graphical analysis becomes especially efficient, when e.g. cannot find values of the fixed points explicitly, thus, one cannot justify the sign of $f'(x_*)$.
- Surely, the graphical analysis itself cannot find the values of the fixed points in this situation, however, it *may* show how many fixed points are and what are their types (stable/unstable).

3.29 Example

Consider the dynamical system

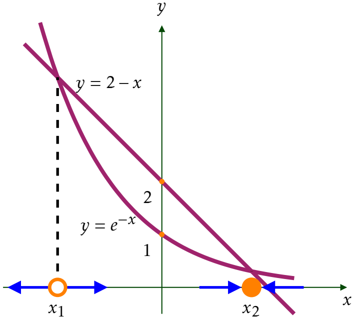
$$x' = 2 - x - e^{-x}.$$

The fixed points satisfy the equation

$$2 - x = e^{-x};$$

they are not given in elementary functions.

One can sketch the graph of $f(x) = 2-x-e^{-x}$, but it would be efficient if we proceed differently. One needs to know where e.g. $f(x) > 0$, or, equivalently, where $2-x > e^{-x}$. We plot the graphs of both functions $y = 2-x$ and $y = e^{-x}$ on the same coordinate plane:



Comparing the points of intersections with the y-axis, one concludes that there are two points where the graphs intersect each other, let $x_1 < x_2$ be their x-coordinates.

By the sketch, for $x_1 < x < x_2$, one has that $2-x > e^{-x}$, therefore, $f(x) = 2-x-e^{-x} > 0$, whereas, for $x < x_1$ or $x > x_2$, one gets that $f(x) < 0$. This yields the phase portrait.

Therefore, x_1 is an *unstable* fixed point, and x_2 is a *stable* fixed point. (In particular, for each $x(0) > x_1$, $x(t) \to x_2$ as $t \to \infty$.)

3.30 Convexity of trajectories

- Recall that, a function $x(t)$ is CONVEX (respectively, CONCAVE) on an interval $[t_1, t_2]$, if its graphs lies below (respectively, above) the segment connecting the border points: $(t_1, x(t_1))$ and $(t_2, x(t_2))$.
- Recall, also that a smooth $x(t)$ is convex (respectively, concave) on an interval if and only if $x''(t) > 0$ (respectively, $x''(t) < 0$) on that interval.
- For a dynamical system $x' = f(x)$, we have

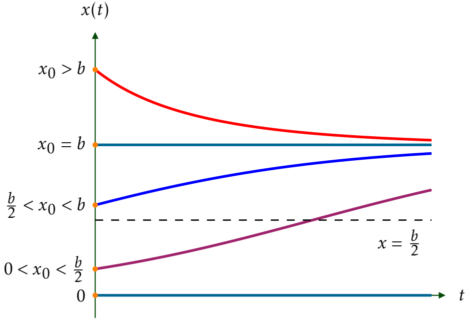
$$\begin{aligned} x''(t) &= \frac{d}{dt}x'(t) = \frac{d}{dt}f(x(t)) \\ &= \frac{d}{dx}f(x(t))\frac{d}{dt}x(t) = f'(x(t))x'(t) \\ &= f'(x(t))f(x(t)), \end{aligned}$$

  or, in brief, $x'' = f'(x)f(x)$.
- As a result, one can gets a simple rule:
  - $x(t)$ is convex when x takes values on the intervals, where f and f' have equal signs;
  - $x(t)$ is convex when x takes values on the intervals, where f and f' have opposite signs.

3.31 Example: Back to the logistic equation

Consider again the logistic equation $x' = f(x) := ax(b-x)$. We will be interested in non-negative solutions only: $x \geq 0$ (having in mind biological applications). Clearly, $f(x) > 0$ for $x \in (0,b)$ and $f(x) < 0$ for $x > b$. Next, $f'(x) = a(b-2x)$, then $f'(x) > 0$ for $x \in (0, \frac{b}{2})$ and $f'(x) < 0$ for $x > \frac{b}{2}$.

As a result, $x(t)$ is convex when $x \in (0, \frac{b}{2})$ or $x > b$; and $x(t)$ is concave when $x \in (\frac{b}{2}, b)$. In particular, if initially $0 < x(0) < \frac{b}{2}$, then $x(t)$ will increase being convex until it reaches the level $\frac{b}{2}$, and then it will become concave. If $b > x(0) \geq \frac{b}{2}$, then it will be always concave.



3.32 Remark: Oscillations

As we could see, the motion of x over the phase space, for the solution $x = x(t)$ of a one-dimensional dynamical system $x' = f(x)$ with a smooth f, is very straightforward: x always moves in a certain direction over some interval of the phase space bordered by fixed point(s) of the dynamical system.

If one of the interval borders is a stable fixed point (or a half-stable fixed point and x started from its stable part), x will converge to that point, otherwise x will go to infinity (the latter may be within a finite or infinite interval of time). Actually, arrows on the phase portrait gives the full description of the motion and limits when time growth. On the space-time diagram, we see that solution is always monotone (increasing or decreasing).

Therefore, if x goes in one direction over the phase space, it can't stop and start moving in other direction. In particular, oscillations around fixed points are impossible. From the course of Classical Mechanics, however, we know that oscillations are typical for some differential equations (and hence dynamical systems) on the real line, e.g. describing the angular displacement of a simple pendulum from the equilibrium vertical position.

The corresponding solution is periodic as a function of time, not monotone. There is not any contradiction, as the corresponding equation includes $mx''(t)$ (where $x = x(t)$ denotes that angular displacement) because of the Second Newton's Law. More generally, differential equations from Mechanics are typically of the form

$$mx''(t) = F(x(t), x'(t), t)$$

(here m is a mass), and hence are second-order differential equations, i.e. all the previous theory does not work for them. If F above does not depend on t, i.e. if we have

$$mx''(t) = F(x(t), x'(t)),$$

then one can introduce new function $y(t) = x'(t)$ and then (since $x'' = y'$) we obtain a **system** of two **coupled** first-order autoinomous differential equations:

$$x' = y, \qquad y' = \frac{1}{m}F(x, y).$$

The latter is an example of a dynamical system on plane $(x, y) \in \mathbb{R}^2$, and it will be considered in the Chapter II of this course. We will see that then, indeed, periodic solutions are possible.

3.33 Remark: Half-stable points and stability

In the Analytical explanation 3.21, we considered the solution to a dynamical system $x' = f(x)$ near its fixed point:if $f(x_*) = 0$, then we considered $x(t) = x_* + \eta(t)$ with a small $|\eta(t)|$ (on some small time interval), and then the Taylor expansion yields:

$$\begin{aligned}\eta'(t) = x'(t) = f(x(t))\\ &= f(x_*) + f'(x_*)\eta(t) + \frac{1}{2}f''(x_*)\eta(t)^2\\ &\quad + \frac{1}{6}f'''(x_*)\eta(t)^3 + \dots.\end{aligned}$$

Since $f(x_*) = 0$, the evolution of η is approximately given by $\eta' = f'(x_*)\eta$ for small η. This works if $f'(x_*) \neq 0$, and then x_* is stable or unstable. If $f'(x_*) = 0$, then we get

$$\eta'(t) = \frac{1}{2}f''(x_*)\eta(t)^2 + \frac{1}{6}f'''(x_*)\eta(t)^3 + \dots.$$

If now $f''(x_*) \neq 0$, then, for small η, η^3 is much smaller than η^2, and hence

$$\eta'(t) \approx \frac{1}{2}f''(x_*)\eta(t)^2$$

Therefore, in a small neihghbourhood of x_*, the dynamical system behaves(recall, that $x = x_* + \eta$ where x_* is just a constant)like $\eta' = c\eta^2$ and thus x_* is half-stable.

Otherwise, if $f''(x_*) = 0$, but $f'''(x_*) \neq 0$, then

$$\eta'(t) \approx \frac{1}{4} f'''(x_*)\eta(t)^3$$

and hence x_* is either stable, if $f'''(x_*) < 0$, or unstable, if $f'''(x_*) > 0$.

If, however, $f'''(x_*) = 0$, then the forth derivative has to be considered, and so on.

Section 4: Bifurcations

4.1 Meaning

- Recall that solutions of one-dimensional dynamical systems may either converge (as time tends to infinity) to stable fixed points (a.k.a. EQUILIBRIUM or ATTRACTOR) or converge to ∞ (as time tends to a finite or infinite limit).
- They also depend on various parameters that makes the analysis interesting and non-trivial.
- Important are the critical values of parameters, so that the dynamical system demonstrates different qualitative behaviours for parameters smaller and larger than the values.

4.2 Definition

Consider a dynamical system $x' = f(x)$, where $f(x)$ is a **smooth** function of a parameter $r \in \mathbb{R}$, i.e. we assume that $f(x) = f(x, r)$. BIFURCATION is the phenomenon when the dynamical system demonstrates **different phase portraits** when r changes its values from $r < r_{\text{cr}}$ to $r > r_{\text{cr}}$ for some critical value r_{cr} called the BIFURCATION POINT.

4.3 Definition: Saddle-node bifurcation

For a dynamical system $x' = f(x, r)$, a (**local**) SADDLE-NODE BIFURCATION takes place for a critical value of the parameter $r = r_{\text{cr}}$ if, for values of r from one side of r_{cr} there are not fixed points, whereas as soon as r takes values from the other side of r_{cr} **there appear** two fixed points. Thus one could said that two fixed points collide (when $r \to r_{\text{cr}}$, from one side) and disappear (annihilate).

4.4 Example

Consider $x' = x^2 + r$. Then

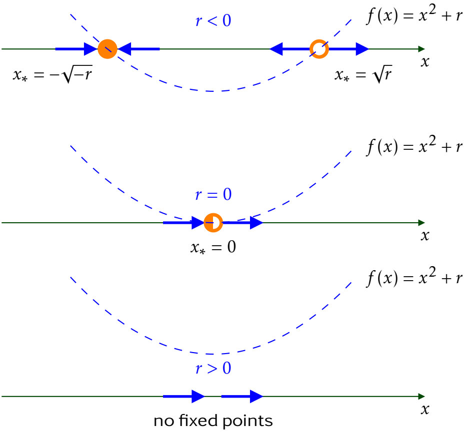
- for $r > 0$, the equation $x^2 + r = 0$ has not (real) roots, i.e. there are no fixed points. Note also that then $x' > 0$ i.e. $x(t)$ increases for all t (it will be on a finite time interval, similarly to Example 3.7);
- for $r < 0$, the equation $x^2 + r = 0$ has two fixed points: $x = \pm\sqrt{-r}$. As it was in Example 3.7, one can show that $x = -\sqrt{-r}$ is stable and $x = \sqrt{-r}$ is unstable. The same can be also shown using the linear stability analysis: $f(x) = x^2 + r$, then

$$f'(x) = 2x, \quad f'(\pm\sqrt{-r}) = \pm 2\sqrt{-r}$$

  and hence we get the same statement;

- also we note that for $r = 0$, there is a unique fixed point, $x^2 = 0$ yields $x = 0$, but since $x^2 > 0$ for all $x \neq 0$ we conclude that $f(x)$ does not change its sign, and hence this fixed point is neither stable nor unstable, it is a half-stable fixed point.

The phase portraits changes accordingly when r changes its values from $r < r_{\text{cr}} = 0$ to $r > r_{\text{cr}}$:



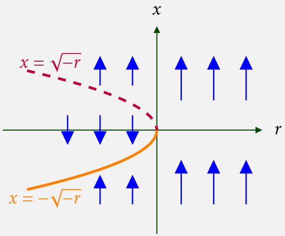
4.5 Remark

The same saddle-node biffurcation takes place for $x' = x^2 - r$, just the 'order' of the phase portraits will be opposite.

4.6 Bifurcation diagram

Recall that a phase portrate shows how the solution $x = x(t)$ behaves with time (increasing or decreasing) in certain intervals of values of that x. In the case of bifurcation for a dynamical system $x' = f(x, r)$, the behaviour of x depends on r. The diagram which shows the dependence of x on r is called the **BIFURCATION DIAGRAM**.

In Example 4.4, we have shown that, for $r \geq 0$, $x = x(t)$ increases; whereas for $r < 0$, we got that $x = x(t)$ increases when $x > \sqrt{-r}$ and also when $x < -\sqrt{-r}$, and we got that $x = x(t)$ decreases when $-\sqrt{-r} < x < \sqrt{-r}$. We have then the bifurcation diagram:



The dashed curve consists of unstable fixed points and the solid curve consists of stable fixed points.

4.7 Remark

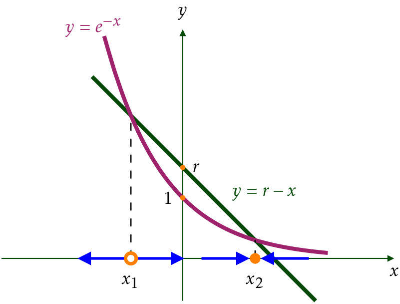
If one fixes some $r_\circ \in \mathbb{R}$ and consider the corresponding vertical line $r = r_\circ$ on the bifurcation diagram, we will get the phase portrait for $x' = f(x, r_\circ)$ placed vertically.

4.8 Example

Consider the dynamical system

$$\boxed{x' = r - x - e^{-x}}, \qquad r \in \mathbb{R}. \tag{4.1}$$

This example generalises Example 3.29 (where $r = 2$). We will use again graphical arguments by exploring the graphs of $y = e^{-x}$ and $y = r - x$. All (straight) lines $y = r - x$ for $r \in \mathbb{R}$ are parallel. For r big enough, we will have two points of intersection, the smaller is unstable, the larger is stable (by the same arguments as it was done in Example 3.29).

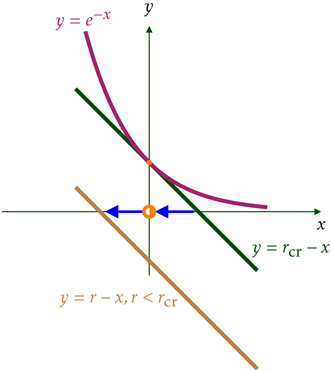


Next, for some critical value $r = r_{\mathrm{cr}}$, the line $y = r - x$ is the tangent line to the curve $y = e^{-x}$ and hence there is only one fixed point to (4.1). The tangent line to $y = e^{-x}$ at a point x_* has the gradient

$$y'(x_*) = -e^{-x_*},$$

whereas the gradient of the line $y = r - x$ is -1. Hence, to have it tangent, one needs $-e^{-x_*} = -1$, i.e. $x_* = 0$. Therefore, the (tangent) line passess through $(0, e^{-0} = 1)$. Since the line is $y = r_{\mathrm{cr}} - x$ and $y(0) = 1$, we conclude that $r_{\mathrm{cr}} = 1$.

Finally, for $r < r_{\mathrm{cr}}$, the line $y = r - x$ will not intersect the curve $y = e^{-x}$, and then $e^{-x} > r - x$, and hence $x' = r - x - e^{-x} < 0$ for all x.



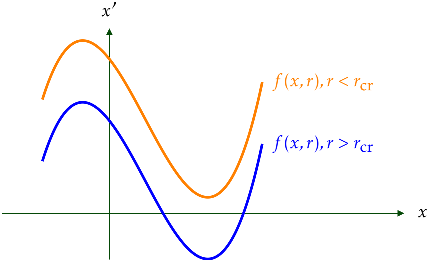
4.9 Conditions for the saddle-node bifurcation

One can ask ourselves, why does the dynamical systems $x' = x^2 + r$ and $x' = r - x - e^{-x}$ have the same variety of phase portraits. The answer is thew following: consider the second dynamical system in the neighbourhood of $x = 0$, by Taylor's expansion:

$$x' = r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2} + \dots\right)$$
$$= (r-1) - \frac{x^2}{2} + \dots.$$

Hence, for x near 0, the solutions to $x' = r - x - e^{-x}$ behave like solutions to $y' = r_1 - y^2$ up to certain rescaling (see e.g. Remark 3.24).

More generally, to have the saddle-node bifurcation for $x' = f(x,r)$ at some $r = r_{\mathrm{cr}}$ one needs e.g. the following phase diagram



(Note: the diagram might be flipped and the signs between r and r_{cr} might be reverted.)

By the continuity arguments, for $r = r_{\mathrm{cr}}$, the graph of $y = f(x, r)$ should have $y = 0$ as the tangent line. Let $x = x_*$ be the tangent point Consider Taylor's expansion for $f(x, r)$ at the neighbourhood of $x = x_*$ and $r = r_{\mathrm{cr}}$:

$$\begin{aligned} x' = f(x, r) = f(x_*, r_{\mathrm{cr}}) \\ &+ (x - x_*)\frac{\partial f}{\partial x}(x_*, r_{\mathrm{cr}}) + (r - r_{\mathrm{cr}})\frac{\partial f}{\partial r}(x_*, r_{\mathrm{cr}}) \\ &+ \frac{1}{2}(x - x_*)^2\frac{\partial^2 f}{\partial x^2}(x_*, r_{\mathrm{cr}}) \\ &+ (x - x_*)(r - r_{\mathrm{cr}})\frac{\partial^2 f}{\partial x \partial r}(x_*, r_{\mathrm{cr}}) \\ &+ \frac{1}{2}(r - r_{\mathrm{cr}})^2\frac{\partial^2 f}{\partial r^2}(x_*, r_{\mathrm{cr}}) + \dots \end{aligned} \tag{$*$}$$

If the graph of $y = f(x, r_{\mathrm{cr}})$ touches $y = 0$ at $x = x_*$, then both the value and the gradient of $f(x, r_{\mathrm{cr}})$ are equal to 0 at $x = x_*$, i.e.

$$\begin{aligned} f(x_*, r_{\mathrm{cr}}) &= 0, \\ \frac{\partial f}{\partial x}(x_*, r_{\mathrm{cr}}) &= 0. \end{aligned} \tag{4.2}$$

Therefore, for the saddle-node bifurcation the system behaves near the bifurcation point $r = r_{\mathrm{cr}}$ approximately as follows:

$$
\begin{aligned}
x' &\approx (r - r_{\text{cr}})\frac{\partial f}{\partial r}(x_*, r_{\text{cr}}) + \frac{1}{2}(x - x_*)^2 \frac{\partial^2 f}{\partial x^2}(x_*, r_{\text{cr}}) \\
&= a(r - r_{\text{cr}}) + b(x - x_*)^2,
\end{aligned}
$$

that corresponds to the model equaiton we considered in Example 4.4, provided that

$$
\begin{aligned}
\frac{\partial f}{\partial r}(x_*, r_{\text{cr}}) &\neq 0, \\
\frac{\partial^2 f}{\partial x^2}(x_*, r_{\text{cr}}) &\neq 0.
\end{aligned} \tag{4.3}
$$

Together, (4.2) and (4.3) provide, respectively, necessary and sufficient conditions to have a saddle-node bifurcation.

4.10 Definition

Let $x' = f(x, r)$ where f is a smooth function of two variables. Let (4.2) and (4.3) hold. The dynamical system

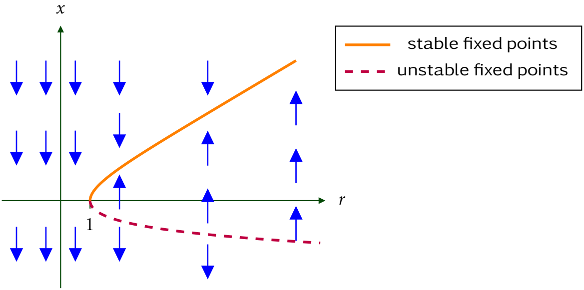
$$
x' = a(r - r_{\text{cr}}) + b(x - x_*)^2, \tag{4.4}
$$

where $a := \frac{\partial f}{\partial r}(x_*, r_{\text{cr}}) \neq 0$, $b := \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_*, r_{\text{cr}}) \neq 0$ is called the **NORMAL FORM** of $x' = f(x, r)$ in the case of the saddle-node bifurcation.

4.11 Remark

Bifurcation diagrams for the case of a saddle-node bifurcation look similarly, because of the normal form (4.4), they differ only by the forms of the curves $x = x(r)$ of fixed points which solves $f(x, r) = 0$ and also because they may flipped vertically or horizontally depending on signs of a and b.

For example, the bifurcation diagram for the considered above dynamical system $x' = r - x - e^{-x}$ is as follows (compare it with the bifurcation diagram for $x' = r + x^2$ above):



Note that here the curves $x = x(r)$ here are determined by the equality $r - x = e^{-x}$ and can not be expressed in elementary functions, but the fact that they are uniquely determined and also are monotone follow from out previous graphical analysis for Example 4.8.

4.12 Remark

By a bifurcation diagram, one can reconstruct space-time diagrams, at least qualitatively. For example, the bifurcation diagram described in Remark 4.11 tells us that, for $r < 1$, $x = x(t)$ decreases regardless of $x(0)$, whereas, for $r > 0$, $x(t)$ will, for example, monotonically converge to certain finite value (a point on the orange curve) if only $x(0)$ is larger then certain *negative* value (a point on the purple curve). In particular, $r > 1$ and $x(0) \geq 0$ imply together that

$$\lim_{t\to\infty} x(t) < \infty.$$

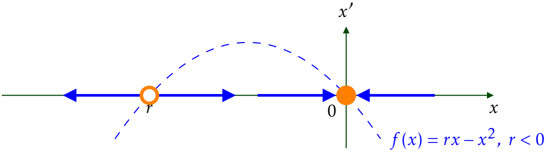
(Sketch the corresponding space-time diagrams by yourself!)

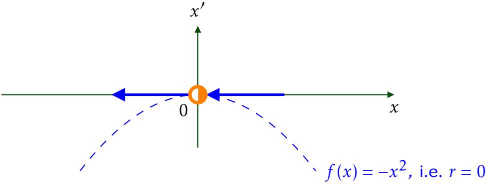
4.13 Definition: Transcritical bifurcation

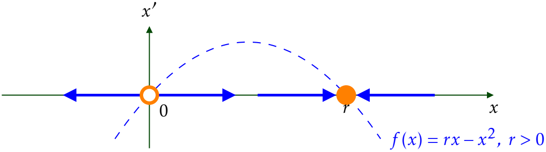
For a dynamical system $x' = f(x, r)$, a (**local**) TRANSCRITICAL BIFURCATION takes place for a critical value of the parameter $r = r_{\mathrm{cr}}$ if the dynamical system has two fixed points for all of values of r around r_{cr}, but they swap their stability types when r passes through r_{cr}. (The distance between the fixed point converge to 0 a $r \to r_{\mathrm{cr}}$.)

4.14 Example

Consider $\boxed{x' = rx - x^2}$. Since $rx - x^2 = 0$ yields $x(r - x) = 0$, the system has two fixed points $x = 0$ and $x = r$. The phase portraits however are different for $r < 0$, $r = 0$, and $r > 0$:



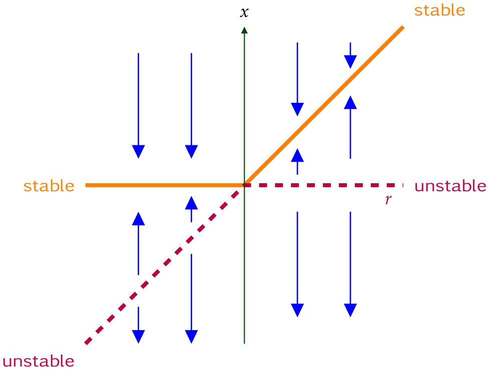




Namely,

- for $r > 0$, $x = 0$ is an unstable fixed point and $x = r$ is a stable fixed point;
- for $r = 0$, $x = 0$ is a half-stable fixed point;
- for $r < 0$, $x = 0$ becomes a stable fixed point and $x = r$ is an unstable fixed point.

We can hence draw the bifurcation diagram on the plane (x, r): the line $x = r$ is stable for $r > 0$ and unstable for $r < 0$, whereas the line $x = 0$ is stable for $r < 0$ and unstable for $r > 0$:



4.15 Example

Consider the dynamical system

$$x' = x(1 - x^2) - a(1 - e^{-bx}), \qquad a, b \in \mathbb{R}.$$

Denote the right hand side by $f_{a,b}(x)$. We see that $x = 0$ is a fixed point, as $f_{a,b}(0) = 0$ for all a, b. Consider the Taylor expansion of $f_{a,b}(x)$ in x for small values of x.

We have

$$\begin{aligned} x' &= x(1-x^2) - a(1-e^{-bx}) \\ &= x + o(x^2) - a\left(bx - \frac{1}{2}b^2x^2 + o(x^2)\right), \end{aligned}$$

where, recall, $\lim\limits_{x\to 0} \frac{o(x^2)}{x^2} = 0$; in particular, $x^3 = o(x^2)$ when $x \to 0$. Therefore,

$$x' = (1-ab)x + \frac{ab^2}{2}x^2 + o(x^2).$$

As a result, for x near the fixed point $x_* = 0$, $x = x(t)$ behaves like the solution to

$$\boxed{x' = (1-ab)x + \frac{ab^2}{2}x^2} \tag{4.5}$$

and hence the transcritical bifurcation occurs when $ab = 1$. Stress that the bifurcation occurs not for a particular values of parameters a, b, but for all pairs (a, b) which lie on the **BIFURCATION CURVE** $ab = 1$ on the plane (a, b) (here there actually two curves, hyperbolas, $b = \frac{1}{a}$).

The second (non-zero) fixed point for (4.5) exists hence for $ab \neq 1$ as, otherwise, two fixed points coincide and we have the half-stable fixed point at 0 only).

Hence, for $ab \neq 1$, the second (non-zero) fixed point of (4.5) satisfies the equation $1 - ab + \frac{ab^2}{2}x = 0$, i.e.

$$x_* = \boxed{\frac{2(ab-1)}{ab^2}}. \tag{4.6}$$

To determine the stability depending on a, b, it is easier to use the linear stability analysis (to do not consider various signs of a and b). Stress that the result should be the same for the original function $f_{a,b}(x)$ and for 'approximate' function $\tilde{f}_{a,b}(x) := (1-ab)x + \frac{ab^2}{2}x^2$, namely,

$$\begin{aligned} f'_{a,b}(x) &= 1 - 3x^2 - abe^{-bx}; \quad & f'_{a,b}(0) &= 1 - ab; \\ \tilde{f}'_{a,b}(x) &= 1 - ab + ab^2x^2; \quad & \tilde{f}'_{a,b}(0) &= 1 - ab. \end{aligned}$$

Therefore, for $ab > 1$, $x = 0$ is a stable fixed point and hence (4.6) is an **unstable** fixed point, whereas for $ab < 1$, $x = 0$ is an unstable fixed point and (4.6) is a **stable** fixed point.
Since our previous considerations worked for small values of x only, we require that (4.6) is a small number, i.e. that ab is close to 1. More detailed graphical analysis for the function $f_{a,b}(x)$ would allow to receive the same result without that restriction.

4.16 Definition

Let $x' = f(x,r)$ where f is a smooth function of two variables. Let (4.2) hold, i.e. $\boxed{f(x_*, r_{cr}) = \frac{\partial f}{\partial x}(x_*, r_{cr}) = 0}$. Let also (see formula $(*)$ in item 4.9 above)

$$\boxed{\begin{aligned} \frac{\partial f}{\partial r}(x_*, r_{cr}) &= 0, \\ a := \frac{\partial^2 f}{\partial x \partial r}(x_*, r_{cr}) &\neq 0, \\ b := \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_*, r_{cr}) &\neq 0. \end{aligned}}$$

Then the dynamical system

$$\boxed{x' = a(r - r_{cr})(x - x_*) + b(x - x_*)^2,}$$

is called the **NORMAL FORM** of $x' = f(x,r)$ in the case of the transcritical bifurcation.

4.17 Definition

For a dynamical system $x' = f(x,r)$, a (**local**) **PITCHFORK BIFURCATION** takes place for a critical value of the parameter $r = r_{cr}$ if the number of fixed points changes from 1 to 3 when r passes through r_{cr}. (The distances between fixed points converge to 0 as $r \to r_{cr}$.)

- More precisely, if, for $r < r_{\mathrm{cr}}$ there are three fixed points: two unstable and one stable, and, if, for $r > r_{\mathrm{cr}}$ two unstable ones disappear and the stable one becomes unstable (and the unique), it is called a **SUBCRITICAL PITCHFORK BIFURCATION** (as three fixed points exist *before* the critical value r_{cr}).
- On contrary, if, for $r < r_{\mathrm{cr}}$ there is a unique stable fixed point and it becomes an unstable one when $r > r_{\mathrm{cr}}$ and also two new stable fixed points appear, then it is called a **SUPERCRITICAL PITCHFORK BIFURCATION** (as it takes place *after* the critical value r_{cr}).

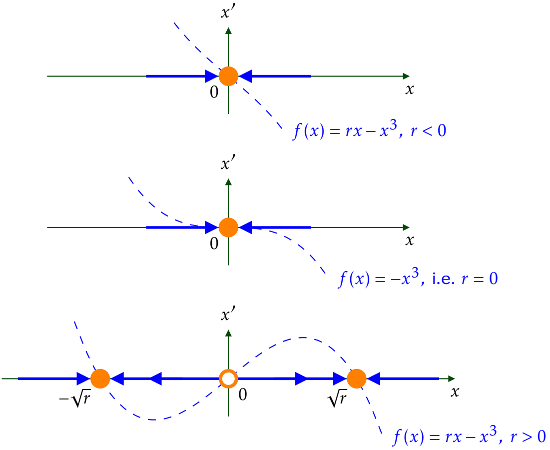
4.18 Example

Consider $x' = rx - x^3 := f(x)$. Since $f(x) = x(r - x^2)$, we conclude that, for $r < 0$, there is a unique fixed point: $x = 0$, and since $f'(x) = r - 3x^2$, one has $f'(0) = r < 0$, i.e. $x = 0$ is a stable fixed point. However, for $r > 0$, there are three fixed points: $x = 0$ and $x = \pm\sqrt{r}$, and

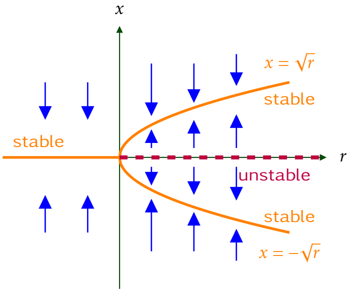
$$f'(0) = r > 0, \quad f'(\pm\sqrt{r}) = r - 3r = -2r < 0,$$

thus $x = 0$ is an unstable fixed point and $x = \pm\sqrt{r}$ are stable fixed points.

Therefore, a supercritical pitchfork bifurcation occurs for this dynamical system with the bifurcation point at $r = 0$. Consider also how the phase portraits change.



The bifurcation diagram is hence as follows:

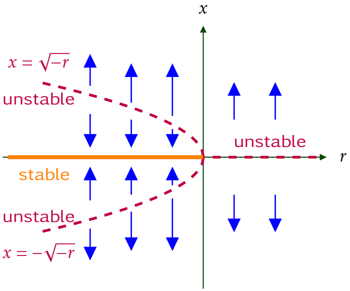


4.19 Remark

The phase portraits for $r < 0$ and for $r = 0$ are the same. However, for small x, we have that $x' \approx rx$ (as $x^3 = o(x)$) and $x(t) \approx x(0)e^{rt} \to 0$ for $r < 0$ exponentially fast as $t \to \infty$. In contrast, $x' = -x^3$ implies $x(t) = \frac{x(0)}{\sqrt{1+2x(0)^2 t}} \to 0$ as $t \to \infty$, but much slower.

4.20 Remark

The model example for the subcritical pitchfork bifurcation is $\boxed{x' = rx + x^3}$. Sketch the different phase portraits for $r < 0$, $r = 0$, $r > 0$ by yourself! The 'pitchfork' on the corresponding bifurcation diagram is then 'flipped':



4.21 Example

Consider the dynamical system

$$x' = rx - \frac{x^3}{1+x^4} =: f(x), \quad r \in \mathbb{R}. \tag{4.7}$$

Clearly, $x = 0$ is a fixed point. Expanding the right-hand side in Taylor's series for small x, one gets

$$x' = rx - x^3(1 - x^4 + x^8 + \dots) \approx rx - x^3,$$

hence the the supercritical pitchfork bifurcation occurs with the bifurcation point at $r_{\text{cr}} = 0$ and the fixed point at $x_* = 0$.

However, the right-hand side of (4.7) is also equal to zero if

$$r = \frac{x^2}{1+x^4} \tag{4.8}$$

This implies that $r \geq 0$. The case $r = 0$ leads to $x = 0$, and it was considered before (but we will return to it below). For $r > 0$, we have

$$rx^4 - x^2 + r = 0, \qquad x^2 = \frac{1 \pm \sqrt{1-4r^2}}{2r}.$$

Since $r > 0$ and $1-4r^2 \leq 1$, one gets that non-zero fixed points exist if and only if

$$\boxed{0 < r \leq \frac{1}{2}}.$$

Each such r provides **four** fixed points except $r = \frac{1}{2}$ that implies $x^2 = 1$, i.e. $x = \pm 1$. However, to get the stability type, one needs some long computations.

It may be easier done graphically directly on the bifurcation diagram. Since (4.8) provides dependence r on x, we revert the order of axes on the bifurcation diagram. We have

$$f(x) = x\Big(r - g(x)\Big), \qquad g(x) := \frac{x^2}{1+x^4} \geq 0.$$

To sketch the graph of $r = g(x)$, we note that

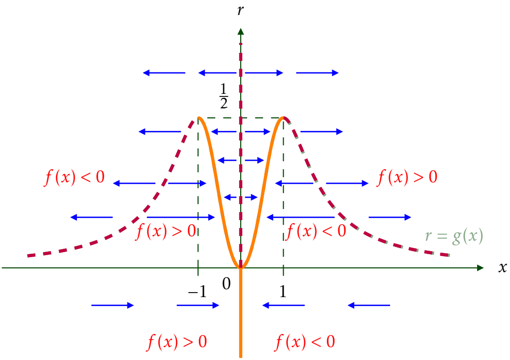
$$g(x) = 0 \Leftrightarrow x = 0; \qquad \lim_{x\to\pm\infty} g(x) = 0;$$

$$g'(x) = \frac{2(1-x)x(x+1)(x^2+1)}{(1+x^4)^2}.$$

Therefore, $x = 0$ is the point of local minimum for $g(x)$ and $x = \pm 1$ are points of local maximum with $g(\pm 1) = \frac{1}{2}$.

Sketch now the graph of $r = g(x)$ on the bifurcation diagram (note that the scales are different for x and r axes). Then, one can get the signs of $f(x) = x(r - g(x))$ (and hence the directions of arrows, note that they are horizontal now):

- for $x > 0$, $f(x) > 0$ if $r > g(x)$ and $f(x) < 0$ if $r < g(x)$;
- for $x < 0$, $f(x) > 0$ if $r < g(x)$ and $f(x) < 0$ if $r > g(x)$.



Recall that $x = 0$ is always a fixed point, as $f(x) = x(r - g(x))$. We see, now, graphically, that for $r > \frac{1}{2}$ or $r < 0$ there no other fixed points. The line $r = \frac{1}{2}$ is tangent to the curve $r = g(x)$, hence at $r = \frac{1}{2}$ saddle-node bifurcations occur for both appearing (half-stable) fixed points $x = 1$ and $x = -1$.

They are saddle-node, since as soon as $r < \frac{1}{2}$, each fixed point is 'split' by two: stable and unstable. Stress that the line $r = 0$ is also tangent to the graph of $r = g(x)$ at $x = 0$, however, here the bifurcation is not saddle-node, as $x = 0$ has additional 'multiplicity' as a fixed point for $f(x) = x(r - g(x))$, and it has a pitchfork bifurcation around $x = 0$, that we have shown before.

4.22 Remark

The value $r = 0$ in the previous example shows the difference between the so-called *local* and *global* bifurcations. Namely, the analysis we provided for small x which uses the Taylor expansion, shows the local bifurcation (supercritical pitchfork) around $x = 0$. It stated that for $r > 0$ two additional fixed points appeared around 0. However, the global phase portrait visible from the bifurcation diagram shows that for $r < 0$ there are indeed no fixed points for (4.7), but as soon as $r > 0$, appeared **five** fixed points: $x = 0$ and four solutions of $g(x) = r$. Note that, for small $r > 0$, two solutions have very large absolute values (and hence remained invisible for the local analysis).

Chapter II

Two-dimensional dynamical systems

Section 5: General ideas

5.1 Definition

We will define a TWO-DIMENSIONAL DYNAMICAL SYSTEM as a pair of (usually coupled) one-dimensional dynamical systems:

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y). \end{cases} \tag{5.1}$$

Here $x = x(t) \in \mathbb{R}$, $y = y(t) \in \mathbb{R}$, $t \in I$ for some $I \subseteq \mathbb{R}$, and $f, g : \mathbb{R}^2 \to \mathbb{R}$.

5.2 Basic notions

- The PHASE SPACE for the dynamical system (5.1) is $\mathbb{R}^2$; i.e. (5.1) describes the motion of a point $(x,y) = (x(t), y(t))$ on the plane $\mathbb{R}^2$; the motion depends on (time) $t \in I \subseteq \mathbb{R}$.
- For any $t_0 \in I$, a pair $x_0 = x(t_0) \in \mathbb{R}$, $y_0 = y(t_0) \in \mathbb{R}$ (or, equivalently, a point $(x(t_0), y(t_0)) \in \mathbb{R}^2$) is called the INITIAL CONDITION for (5.1).
- The dynamical system (5.1) together with an initial condition (x_0, y_0) is called the INITIAL VALUE PROBLEM.

5.3 Definition: Smooth function in $\mathbb{R}^2$

We will say that functions f and g are smooth on some region $D \subseteq \mathbb{R}^2$, e.g. $D = (a,b) \times (c,d) \subseteq \mathbb{R}^2$, if all functions

$$f,\ g,\ \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial g}{\partial x},\ \frac{\partial g}{\partial y}$$

are continuous on D.

5.4 Theorem: Existence and uniqueness

- Let f and g in (5.1) be **smooth** on some $D = (a,b) \times (c,d) \subseteq \mathbb{R}^2$. Consider the initial value problem for (5.1) given by

  $$x(t_0) =: x_0 \in (a,b), \quad y(t_0) =: y_0 \in (c,d)$$

  (or, in short, $(x_0, y_0) \in D$). Then there exists a unique solution to the initial value problem on some time-interval $I := (t_0 - \delta, t_0 + \delta)$ for a $\delta > 0$, i.e. there exists a unique pair of functions $x(t), y(t)$ which solve (5.1) and satisfy the initial conditions.
- The functions $x(t), y(t)$ are also smooth in $t \in I$.
- The time-interval may extended to a maximal time-interval $(T_-, T_+) \subseteq \mathbb{R}$.

5.5 Trajectories

- Let $x(t), y(t)$ be solutions to the dynamical system (5.1) on a time interval $t \in I \subseteq \mathbb{R}$. Then the set of points

$$\left\{\Big(x(t), y(t)\Big) \in \mathbb{R}^2 \,\middle|\, t \in I\right\}$$

is called a **TRAJECTORY** of (5.1).

5.6 Remark

- Notice the difference with Definition 2.8. The time itself is not a coordinate of the trajectory for a two-dimensional dynamical system, in contrast to the one-dimensional dynamical system.
- In other words, the trajectory of a one-dimensional dynamical system $x'(t) = f(x(t))$ lies on the plane (t, x), whereas the trajectory of the two-dimensional dynamical system (5.1) lies on the plane (x, y).
- In particular, a trajectory of (5.1) is not the graph of solutions $x = x(t)$, $y = y(t)$ to the dynamical system (5.1).

Consider the corresponding examples.

5.7 Example

Functions $x(t) = \cos t$, $y(t) = \sin t$, $t \geq 0$, solve the initial value problem

$$\begin{cases} x' = -y, & x(0) = 1, \\ y' = x. & y(0) = 0. \end{cases} \tag{5.2}$$

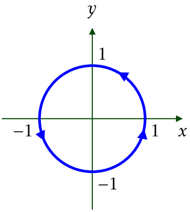
Since

$$x(t)^2 + y(t)^2 = \cos^2 t + \sin^2 t = 1, \qquad t \geq 0,$$

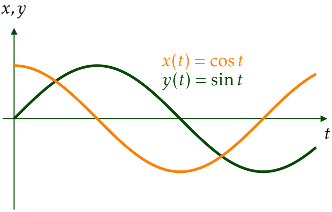
the trajectory of (5.2) is just the circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

whereas the **space-time diagram**, that shows the graphs of $x = x(t)$ and $y = y(t)$, is different (note that the graphs may be shown also on different diagrams):



trajectory



space-time diagram

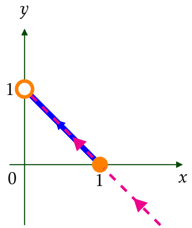
Note that the arrows on the trajectory show the **direction of motion** of the point $(x(t), y(t)) \in \mathbb{R}^2$ when $t \in [0, \infty)$ grows.

5.8 Example

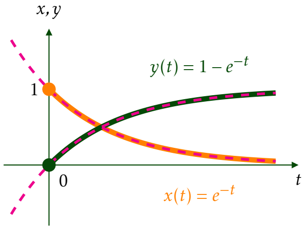
Functions $x(t) = e^{-t}$, $y(t) = 1 - e^{-t}$, $t \in \mathbb{R}$, solve the initial value problem

$$\begin{cases} x' = y - 1, & x(0) = 1, \\ y' = x. & y(0) = 0. \end{cases} \tag{5.3}$$

Since $x(t) + y(t) = e^{-t} + (1 - e^{-t}) = 1$, the trajectory of (5.3) **lies** on the line $\{(x, y) \in \mathbb{R}^2 \mid x + y = 1\}$, whereas the space-time diagram is, of course, different. Moreover, one can consider (5.3) for $t \in [0, \infty)$, then the whole trajectory is just a segment closed from one side, as then $0 < x = e^{-t} \leq 1$ and $0 \leq y = 1 - e^{-t} < 1$.



trajectories: $t \geq 0$, $t \in \mathbb{R}$

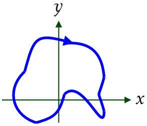


space-time diagram

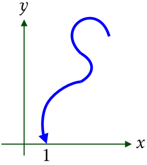
However, if we consider (5.3) for $t \in \mathbb{R}$, the trajectory becomes an infinite ray. In both cases, the point $(0, 1) = \lim\limits_{t \to +\infty} (x(t), y(t))$ **does not belong** to the trajectory.

5.9 Remark

Below, we will be mainly working with finding/describing trajectories; the space-time diagram may be much more challenging, especially for less trivial dynamical system. On the other hand, having a trajectory, one can say something about the behaviour of the solutions. For example, if one gets a **CLOSED TRAJECTORY**



then it means that each point (x_1, y_1) of this trajectory is 'visited' during the motion along the trajectory again and again, i.e. $(x(t), y(t)) = (x_1, y_1)$ at infinitely many moments of time t, thus both $x(t)$ and $y(t)$ are periodic-like functions. Or, if the trajectory is directed towards a point which does not belong to it:



would mean the an both $x(t)$ and $y(t)$ have finite limits as $t \to \infty$ (in the latter picture, say, the limits are 1 and 0, respectively).

5.10 Theorem

Let f, g be smooth functions on $D \subseteq \mathbb{R}^2$ and $I \subseteq \mathbb{R}$ be a time-interval so that for each $t_0 \in I$ and $(x_0, y_0) \in D$ the initial value problem to (5.1) with $x(t_0) = x_0$, $y(t_0) = y_0$ has unique solutions $x(t), y(t)$ for $t \in I$. Then any two trajectories either coincide or do not have common points (i.e. do not cross nor touch each other).

5.11 Example

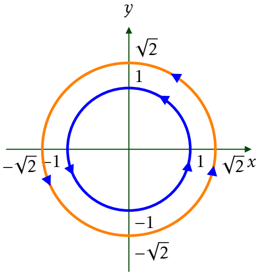
Consider again (see Example 5.7) the dynamical system (5.2). We have shown that the trajectory which passes through the point $(x_1, y_1) = (1, 0) \in \mathbb{R}^2$ and hence which is corresponding to the initial condition $x(0) = 1$, $y(0) = 0$ is given by $x(t) = \cos t$ and $y(t) = \sin t$; and is just a unit circle centred at the origin. Consider now another initial condition: $x(0) = 1 = y(0)$. The corresponding solution to (5.2) is then

$$x(t) = \cos t - \sin t, \qquad y = \cos t + \sin t,$$

(check by yourself!) and one has

$$x(t)^2 + y(t)^2 = 2\cos^2 t + 2\sin^2 t = 2,$$

hence, the trajectory is the circle of the radius $\sqrt{2}$ centred at the origin.



5.12 Remark

The crucial is that Theorem 5.10 states that **trajectories** do not have common points, whereas graphs of e.g. $x(t)$, corresponding to different initial conditions, may intersect.

In Example 5.11, $x(0) = 1, y(0) = 0$ lead to $x(t) = \cos t$ whereas $x(0) = 1 = y(0)$ led to $x(t) = \cos t - \sin t$. The trajectories, that are circles of radius 1 and $\sqrt{2}$ do not intersect whereas graphs of $x = \cos t$ and $x = \cos t - \sin t$ have (infinitely many) common points (namely, when $t = k\pi$, $k \in \mathbb{Z}$).

5.13 Definition

A point $(x_*, y_*) \in \mathbb{R}^2$ such that

$$\begin{cases} f(x_*, y_*) = 0, \\ g(x_*, y_*) = 0 \end{cases} \tag{5.4}$$

is said to be a **FIXED POINT** of the dynamical system (5.1).

5.14 Properties of fixed points

- If $(x_*, y_*) \in \mathbb{R}^2$ is a fixed point of (5.1), then $x(t) \equiv x_*$ and $y(t) \equiv y_*$ is a pair of **STATIONARY SOLUTIONS** to (5.1).
- The corresponding trajectory hence consists of **one point only**: (x_*, y_*).
- By Theorem 5.10, any other trajectory does not path through (x_*, y_*).
- The corresponding graphs on the space-time diagram are straight lines.

5.15 Corollary

Let f, g in (5.1) be smooth on a $D \subseteq \mathbb{R}^2$. Then each point from D belongs to a unique trajectory of (5.1).

5.16 Direction field

- Let $(x(t), y(t))$ be a solution to (5.1) for $t \in I \subseteq \mathbb{R}$. Then, for each $t \in I$, the vector

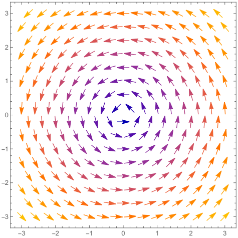
$$\big(x'(t), y'(t)\big) = \big(f(x(t), y(t)), g(x(t), y(t))\big)$$

  is the **tangent vector** to the corresponding trajectory of (5.1) at the point $(x(t), y(t)) \in \mathbb{R}^2$ of that trajectory.

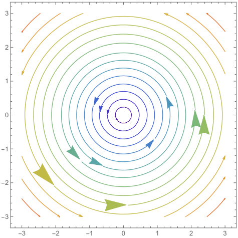
- The **DIRECTION FIELD** (a.k.a. **VECTOR FIELD**) of (5.1) with smooth functions $f, g : D \to \mathbb{R}$ is the set of vectors with directions $(f(x,y), g(x,y))$ for all $(x,y) \in D$.

5.17 Example

Consider again the dynamical system from Example 5.7, i.e. with $f(x,y) = -y$, $g(x,y) = x$. The corresponding direction gives a hint how the trajectories may look like:



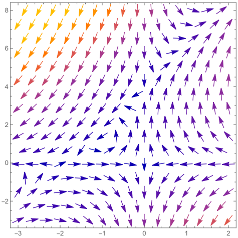
direction field



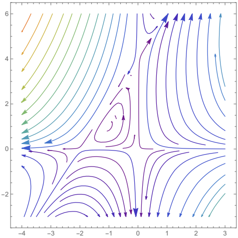
trajectories

5.18 Example

Usually, the recovering of trajectories from the direction field is less trivial. Consider, for example, the dynamical system (5.1) with $f(x,y) = -2x - x^2 + xy$, $g(x,y) = 4y + 3xy - y^2$. Then



direction field



trajectories

5.19 Phase portrait

- Our aim will be to get the **PHASE PORTRAIT** of the dynamical system (5.1) for certain smooth f and g, that is the sketch of 'typical' trajectories around each of the fixed point of the dynamical system.
- What does 'typical' mean we will discuss for each particular type of problems considered below.

Section 6: Linear systems

6.1 Preliminaries

- Consider (5.1) with linear f and g:

$$\begin{cases} x' = ax + by, \\ y' = cx + dy. \end{cases} \tag{6.1}$$

  Here $a, b, c, d \in \mathbb{R}$ are some real numbers.

- Consider the matrix A associated with the system (6.1):

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.2}$$

- Then, using the column vector notation $(x, y)^T := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ and coordinate-wise differentiation:

$$\begin{pmatrix} x \\ y \end{pmatrix}' := \begin{pmatrix} x' \\ y' \end{pmatrix},$$

  one can rewrite (6.1) as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6.3}$$

- A **fixed point** of (6.1) is, hence, a solution to the linear (algebraic) system

$$\begin{cases} ax + by = 0, \\ cx + dy = 0. \end{cases} \tag{6.4}$$

- Therefore, the **origin** $(0, 0)$ is **always** a fixed point of the linear system (6.1).
- The origin is the **unique** fixed point of the linear system (6.1) if and only if

$$\det A := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc \neq 0.$$

- If, however, $\det A = 0$, then both equations in (6.4) determine **the same** line on the phase plane. **All points** of this line are then fixed points of (6.1). We will discuss this case later on. Until that, we will always assume below that

$$\det A \neq 0. \tag{6.5}$$

- Recall that a number $\lambda \in \mathbb{C}$ is said to be an EIGENVALUE of a matrix (6.2), if there exists an EIGENVECTOR $v = (v_1, v_2)^T \in \mathbb{C}^2$, $v \neq (0, 0)^T$, such that $Av = \lambda v$.

- Recall that if $v \in \mathbb{C}^2$ is an eigenvector of a matrix A corresponding to a fixed eigenvalue $\lambda \in \mathbb{C}$, then, **for any** $p \in \mathbb{C}$, $pv \in \mathbb{C}^2$ is also an eigenvector of A corresponding to the eigenvalue λ.
- Eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are solutions to the following CHARACTERISTIC EQUATION of (6.1):

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0, \quad \text{i.e.}$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{6.6}$$

- Using the notation $\operatorname{tr} A := a + d$ for the TRACE of A, one can rewrite (6.6)

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0.$$

- By Vieta's formulas, $\lambda_1 \cdot \lambda_2 = \det A$. Therefore, the condition (6.5) means that

$$\lambda_1 \neq 0, \qquad \lambda_2 \neq 0. \tag{6.7}$$

- The characteristic equation (6.6) has real or complex roots depending on the value of its discriminant $\Delta := (\operatorname{tr} A)^2 - 4 \det A$.

6.2 Case 1: Real DIFFERENT eigenvalues ($\Delta > 0$)

- In the case $\Delta > 0$, the characteristic equation (6.6) has two **different real** eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2$.
- Then the corresponding eigenvectors are also real: $v_1, v_2 \in \mathbb{R}^2$.
- The general solution to the linear system (6.1) is then given by

$$\big(x(t), y(t)\big)^T = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constant. Or, component-wise, if $v_1 = (v_{11}, v_{12})^T$ and $v_2 = (v_{21}, v_{22})^T$, then

$$\begin{aligned} x(t) &= C_1 v_{11} e^{\lambda_1 t} + C_2 v_{12} e^{\lambda_2 t}, \\ y(t) &= C_1 v_{21} e^{\lambda_1 t} + C_2 v_{22} e^{\lambda_2 t}, \end{aligned} \tag{6.8}$$

(stress that C_1, C_2 are the same in the expressions for $x(t)$ and $y(t)$).

- Each initial point

$$(x_0, y_0) = (x(t_0), y(t_0)) \in \mathbb{R}^2$$

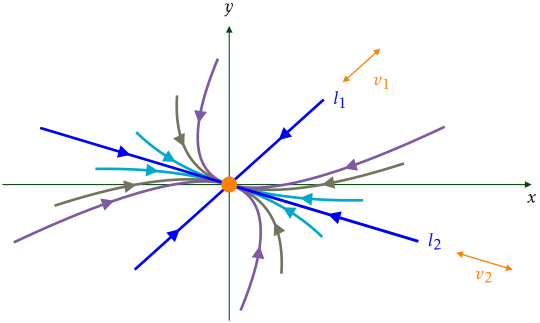
determines only one pair $C_1, C_2 \in \mathbb{R}$ if we take $t = t_0$ in system (6.8), since eigenvectors v_1 and v_2 are **linearly independent**.

6.3 Case 1.1: Negative DIFFERENT eigenvalues

- If $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_1 \neq \lambda_2$, then

$$\boxed{x(t) \to 0, \quad y(t) \to 0, \quad \text{as } t \to +\infty.} \tag{6.9}$$

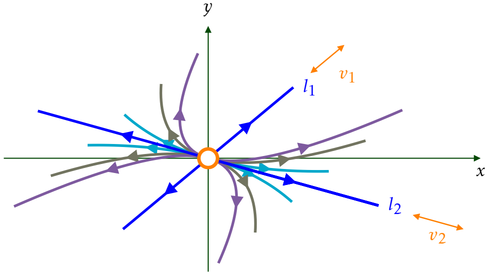
- In this case, the fixed point $(x_*, y_*) = (0, 0)$ is called a **STABLE NODE** (a.k.a. **SINK** or **ATTRACTOR**).
- It is also said, that $(0, 0)$ is then a **globally asymptotically stable** fixed point or a **global attractor**, where 'global(ly)' is to stress that (6.9) holds regardless how far was initially $(x(t_0), y(t_0))$ from $(0, 0)$.
- Hence the trajectories of (6.1) tend then to the origin, but **do not** contain it (as it is a fixed point).
- Let e.g. $\boxed{|\lambda_1| > |\lambda_2|}$, i.e. $\boxed{0 > \lambda_2 > \lambda_1}$, then the **phase portrait** of (6.1) is as follows:



- The phase portrait should show:
  - The fixed point (the origin) is a trajectory (orange dot).
  - The arrows on the trajectories are directed **towards** the origin.
  - The only straight trajectories lie on lines l_1, l_2 parallel to eigenvectors v_1, v_2, respectively, which pass through the origin. (Stress that there are **four** such trajectories, not two!)
  - All other trajectories are branches of skewed parabola-like curves. (Imagine that the 'parabolas' are sketched in the skewed coordinate plane with axes l_1 and l_2.)
  - The line l_2 ("**slow line**"), which corresponds to the eigenvalue λ_2 with **smaller absolute value**, is the **tangent line** to the all 'parabolas' (with the tangent point at the origin).
  - The line l_1 ("**fast line**"), which corresponds to the eigenvalue λ_1 with **larger absolute value** is the 'axis' for the all 'parabolas'. (Note that, however, the 'parabolas' are **not** symmetric with respect to l_1.)
  - Stress that each such 'parabola' consists of **two** disjoint trajectories.

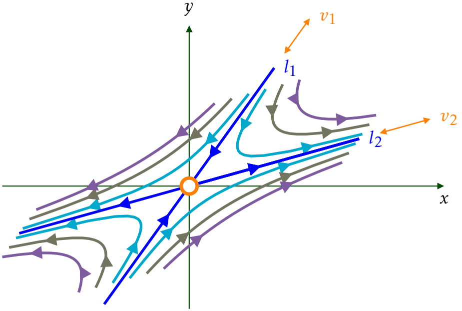
6.4 Case 1.2: Positive DIFFERENT eigenvalues

- If $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, then both $x(t)$ and $y(t)$ converge to ∞ ($+\infty$ or $-\infty$) as $t \to +\infty$.
- In this case, the fixed point $(x_*, y_*) = (0, 0)$ is called an **UNSTABLE NODE** (a.k.a. **SOURCE** or **REPELLER**).
- The trajectories tend then outwards the origin and towards the infinity. The trajectories may start at some (x_0, y_0) quite far from the origin, but we will always draw them starting close to the origin.
- The scheme of how to make the phase portrait is the same as for the stable node (**stress** that now $|\lambda_1| > |\lambda_2|$ implies $\lambda_1 > \lambda_2 > 0$), with the only **crucial** difference that the arrows on the trajectories are directed **outwards** the origin.



6.5 Case 1.3: Eigenvalues with OPPOSITE signs

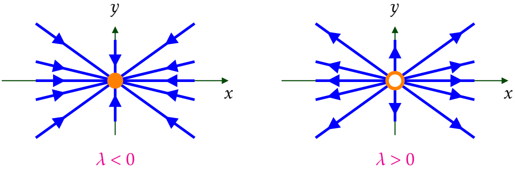
- Suppose that $\lambda_2 > 0 > \lambda_1$.
- Then $(x_*, y_*) = (0,0)$ is an **unstable** fixed point which is called the SADDLE POINT.
- The phase portrait looks as follows:



- The only straight trajectories lie on lines l_1, corresponding to $\lambda_1 < 0$ (STABLE MANIFOLD), and l_2, corresponding to $\lambda_2 > 0$ (UNSTABLE MANIFOLD).
- All other trajectories are branches of skewed hyperbola-like curves.
- The arrows on the trajectories are towards the origin while the motion is 'along' the line l_1, whereas the arrows are outwards the origin, when the motion is 'along' the line l_2.
- Unless the initial condition lies on l_1, both $x(t), y(t)$ converge to $\pm\infty$ as $t \to \infty$.

6.6 Case 2: Real EQUAL eigenvalues ($\Delta = 0$)

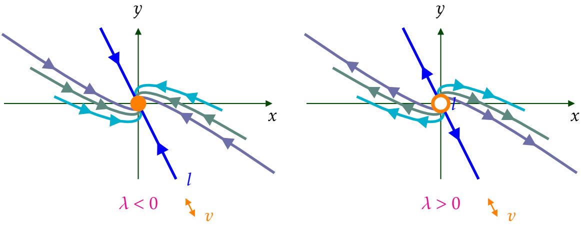
- Suppose that $\lambda_1 = \lambda_2 := \lambda \neq 0$. Then there possible two sub-cases:
- Matrix A is diagonal: $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.
  - Then (6.1) is split by two independent identical equations: $x' = \lambda x$, $y' = \lambda y$.
  - **All trajectories** then lie on straight lines which pass through the origin.
  - The arrows are directed towards the origin if $\lambda < 0$, and outwards the origin if $\lambda > 0$.
  - The fixed point (at the origin) is called then a (**stable** or **unstable**) STAR or STAR NODE.



- Matrix A is not diagonal. Then
  - There is a unique eigenvector $v \in \mathbb{R}^2$ corresponding to λ.
  - The general solution to (6.1) is

$$\left(x(t), y(t)\right)^T = (C_1 + C_2 t)e^{\lambda t} v.$$

- All the trajectories lie on ‘cubic parabola’-like curves, with the tangent line parallel to v.
- The arrows are directed towards the origin if $\lambda < 0$, and outwards the origin if $\lambda > 0$.
- The fixed point (at the origin) is called then a (**stable** or **unstable**) DEGENERATE NODE.



6.7 Remark

To choose, whether the ‘cubic parabola’ is like $y = x^3$ or $y = -x^3$, one can use tangent vectors from the direction field.

6.8 Case 3: Complex eigenvalues ($\Delta < 0$)

- Let $\Delta < 0$, i.e. the characteristic equation (6.6) has two complex roots

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta,$$

where $\alpha, \beta \in \mathbb{R}$, and $\beta \neq 0$.

- The corresponding eigenvectors are also complex (from $\mathbb{C}^2$) and conjugate:

$$v_1 = u + iw, \quad v_2 = u - iw,$$

where $u, w \in \mathbb{R}^2$ are real vectors.

- The general solution to (6.1) is then

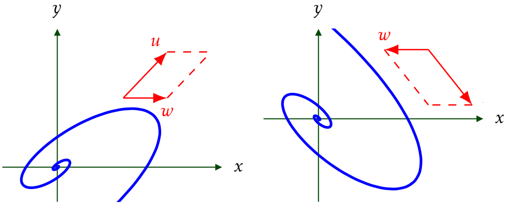
$$\begin{aligned}(x(t),y(t))^T &= C_1 \mathrm{Re}\left(e^{\lambda_1 t} v_1\right) + C_2 \mathrm{Im}\left(e^{\lambda_1 t} v_1\right) \\ &= C_1 e^{\alpha t}\left(\cos(\beta t)u - \sin(\beta t)w\right) \\ &\quad + C_2 e^{\alpha t}\left(\sin(\beta t)u + \cos(\beta t)w\right) \\ &= e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)u \\ &\quad + e^{\alpha t}\left(C_2 \cos(\beta t) - C_1 \sin(\beta t)\right)w.\end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary (real!) constants.

- Again, each initial point $(x(t_0), y(t_0)) \in \mathbb{R}^2$ determines only one pair $C_1, C_2 \in \mathbb{R}$ if we set $t = t_0$ above.

6.9 Case 3.1: Non-zero real part ($\alpha \neq 0$)

- **Each** trajectory (except the fixed point at the origin itself) is a **spiral** around the origin.
- The arrows on the spiral are directed **towards** the origin if $\alpha < 0$ and **outwards** the origin if $\alpha > 0$. The fixed point $(x_*, y_*) = (0, 0)$ is called then STABLE SPIRAL (a.k.a. SPIRAL SINK) or UNSTABLE SPIRAL (a.k.a. SPIRAL SOURCE), respectively.
- The phase portrait should contain only one trajectory, i.e. one spiral (to do not create a mess).
- To sketch the spiral properly, we need to know its slope (how it is 'slanting') and also to determine the rotation of the spiral: whether it is clockwise or counter-clockwise.
- To determine the **slope** of the spiral , one can consider the parallelogram formed by vectors u and w.



- Stress, however, that u and w are **not** determined uniquely (even up to a factor).
- To determine whether the rotation of the spiral is clockwise or counter-clockwise, it is enough to find even one vector of the direction field, and 'agreed' its direction with the direction of arrows on the spiral. Consider an example.

6.10 Example

Consider the dynamical system

$$\begin{cases} x' = -2x + 6y \\ y' = -3x + 4y. \end{cases}$$

The corresponding matrix is $A = \begin{pmatrix} -2 & 6 \\ -3 & 4 \end{pmatrix}$, hence the characteristic equation is

$$\begin{vmatrix} -2-\lambda & 6 \\ -3 & 4-\lambda \end{vmatrix} = 0,$$

that yields

$$\lambda^2 - 2\lambda + 10 = 0,$$
$$\lambda_1 = 1 + 3i, \quad \lambda_2 = 1 - 3i,$$
$$\alpha = 1 > 0, \qquad \beta = 3.$$

Since $\alpha = 1 > 0$, the origin is an unstable spiral; and the arrows along the trajectory should be directed outwards the origin.
Find e.g. eigenvector $v_1 = (p,q)^T \in \mathbb{C}^2$

$$Av_1 = \lambda_1 v_1,$$
$$(-2p+6q, -3p+4q)^T = \lambda_1 (p,q)^T,$$

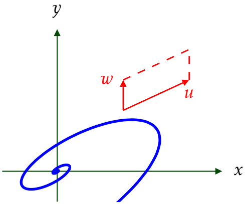
Stress that since the eigenvector can be found up to a factor before it only, it is enough to compare first coordinates here:

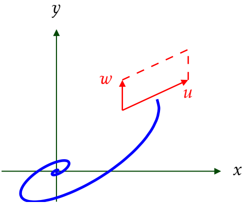
$$-2p+6q = (1+3i)p \Longrightarrow 2q = (1+i)p.$$

We choose now a 'convenient' value of p, e.g. $p=2$. Then $q = 1+i$. Hence $v_1 = (2, 1+i)^T$ is an eigenvector, i.e.

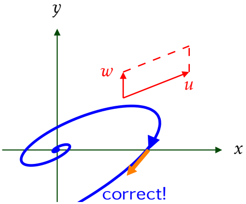
$$v_1 = (2,1)^T + i(0,1)^T,$$
$$u = (2,1)^T, \quad w = (0,1)^T.$$

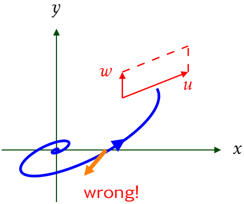
Despite non-uniqueness of the choice for u, w, one gets the slope of the spiral. But, which rotation is correct?





Consider a direction vector: take, for example, $x = 1$, $y = 0$, then $x' = -2$, $y' = -3$. The vector $n = (-2, -3)^T$ is, of course, just the first column of the matrix A.
Recall that each initial condition, e.g. each point on x-axis, will have own trajectory, i.e. an own spiral. All spirals have **the same** rotation rule (clockwise or counter-clockwise). Assume that we have the sketch of the spiral which passes through the considered point $(1, 0)$. Recall that since we got $\alpha = 1 > 0$, the arrows on the spiral are directed outwards the origin. And the arrows at $(1, 0)$ on the spiral and of the direction vector must be **the same**. Hence, only the first graph is correct.





6.11 Case 3.2: Zero real part ($\alpha = 0$)

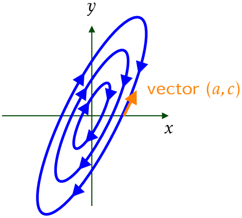
- Consider now the case when $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$.
- Note that the characteristic equation (6.6) takes then the form $\lambda^2 + \beta^2 = 0$.

- This is possible hence if and only if $\operatorname{tr} A = 0$, i.e. $a = -d$.
- The general solution to (6.1) then is:

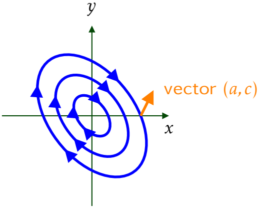
$$(x(t), y(t))^T = \Big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\Big)u \\ + \Big(C_2 \cos(\beta t) - C_1 \sin(\beta t)\Big)w,$$

  hence both $x(t)$ and $y(t)$ are **periodic** functions. In particular, they **do not** have any limits (finite nor infinite) when $t \to \infty$.
- The trajectories of (6.1) are then **ellipses**.
- The fixed point $(x_*, y_*) = (0, 0)$ is called then the **CENTRE**.
- The arrows on the trajectories can be determined by a direction vector. Recall that the direction vector should be also tangent to the trajectories.

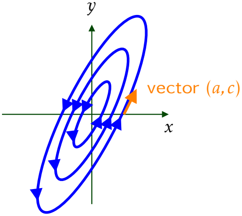
In the following sketch, $a > 0, c > 0$ (the first column of matrix A).



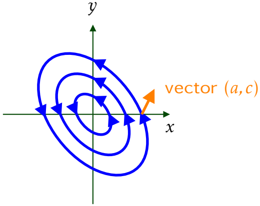
The directions of blue and orange arrows are opposite



The vector (a, c) is not tangent to the ellipse



Correct phase portrait



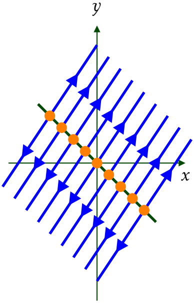
The vector (a,c) is not tangent to the ellipse

6.12 Remark

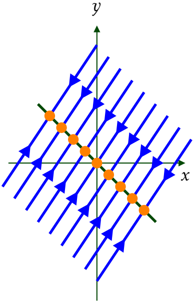
- Note that a centre is actually not stable nor unstable fixed point.
- More precisely, it is **Lyapunov stable** (stable in the sense of Lyapunov): namely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if the (Euclidean) distance between $(x(0), y(0))$ and $(x_*, y_*) = (0,0)$ is less than δ, then the distance between $(x(t), y(t))$ and $(x_*, y_*) = (0,0)$ is less than ε for all $t > 0$.
- However, a centre is asymptotically unstable as $(x(t), y(t))$ does not converge to $(x_*, y_*) = (0,0)$ as $t \to \infty$.

6.13 Case 4: Zero eigenvalue ($\det A = 0$)

- Consider finally the degenerate case when (6.5) fails, i.e. $\det A = ad - bc = 0$.
- Then the characteristic equation (6.6) takes the form $\lambda^2 - (\operatorname{tr} A)\lambda = 0$, i.e. $\lambda_1 = 0$ or $\lambda_2 = \operatorname{tr} A$. Then there are two sub-cases.
- *either* A is the zero matrix. Then:
- *or* A is not the zero matrix. Then both equations $ax + by = 0$ and $cx + dy = 0$ determine the same line, all points of which are fixed points. They are stable if $\lambda_2 = \operatorname{tr} A < 0$ and unstable if $\lambda_2 = \operatorname{tr} A > 0$. The trajectories are parallel to the eigenvector v_2 corresponding to λ_2. Finally, if $\operatorname{tr} A = 0$, the trajectories are parallel to the line of the fixed points.



Line of unstable fixed points



Line of stable fixed points

Section 7: Local phase portrait

7.1 Linearisation method

- Let $f, g : D \to \mathbb{R}$ be smooth functions on an open set $D \subset \mathbb{R}^2$ and $(x_*, y_*) \in D$ be a fixed point of the dynamical system (5.1):

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases} \tag{7.1}$$

  i.e. $f(x_*, y_*) = g(x_*, y_*) = 0$.

- Expanding $f(x,y)$ into the Taylor series in the neighbourhood of (x_*, y_*), one gets

$$f(x,y) = \underbrace{f(x_*, y_*)}_{=0} + \frac{\partial f}{\partial x}(x_*, y_*)(x - x_*) + \frac{\partial f}{\partial y}(x_*, y_*)(y - y_*) + \dots,$$

  where, for $|x - x_*| < \varepsilon$, $|y - y_*| < \varepsilon$, the omited terms are less than const $\cdot\, \varepsilon^2$. The same can be done for $g(x,y)$, hence:

$$g(x,y) \approx \frac{\partial g}{\partial x}(x_*, y_*)(x - x_*) + \frac{\partial g}{\partial y}(x_*, y_*)(y - y_*)$$

  for (x,y) close to (x_*, y_*).

- Denoting $u(t) := x(t) - x_*$, $v(t) = y(t) - y*$, and omitting the terms of the order less than ε, we get that

$$\begin{pmatrix} u \\ v \end{pmatrix}' \approx J(x_*, y_*) \begin{pmatrix} u \\ v \end{pmatrix},$$

where $J(x_*, y_*)$ is the value at (x_*, y_*) of the matrix-function

$$J(x,y) := \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial x}(x,y) \end{pmatrix}, \tag{7.2}$$

if only $|u|$ and $|v|$ are small enough, i.e. in a neigbourhood of $(u,v)^T = (0,0)^T \in \mathbb{R}^2$.

7.2 Definition

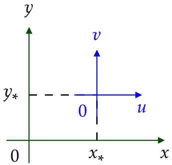
- The matrix (7.2) is called the JACOBIAN MATRIX of the system (7.1).
- The linear system

$$\begin{pmatrix} u \\ v \end{pmatrix}' = J(x_*, y_*) \begin{pmatrix} u \\ v \end{pmatrix}, \tag{7.3}$$

is called the LINEARISATION of (7.1) at (x_*, y_*), or just the **linearized system**.

7.3 Remark

The linearisation (7.3) is a linear system with on (u, v)-phase space with a fixed point at $(u = 0, v = 0)$. Note that $u = 0$, $v = 0$ implies $x = x_*$, $y = y_*$. Hence the phase portrait of the linear system (7.3) will appear 'around' the fixed point (x_*, y_*) on (x, y)-phase space.



7.4 Remark

With an abuse of notations, the linearized system (7.3) is typically written in terms of (x, y) rather than (u, v). (Keeping in mind that the result will be about the fixed point at (x_*, y_*), not at $(0, 0)$.)

7.5 Definition

Directions of eigenvectors of the Jacobian matrix $J'(x_*, y_*)$ corresponding to the linearisation (7.3) are called the **MAIN DIRECTIONS** of (7.1) at (x_*, y_*) (will be used for nodes and saddles only).

7.6 Definition

A fixed point (x_*, y_*) is called **HYPERBOLIC** if both eigenvalues λ_1 and λ_2 of $J(x_*, y_*)$ (i.e. of the Jacobian matrix (7.2) at (x_*, y_*)), have **non-zero real parts** $(\operatorname{Re}\lambda_1 \neq 0,\ \operatorname{Re}\lambda_2 \neq 0)$. (In particular, a fixed point is hyperbolic, if $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.) Therefore,

HYPERBOLIC	NON-HYPERBOLIC
node	centre
saddle	line of fixed points
spiral	plane of fixed points
star	
degenerate node	

7.7 Theorem: Hartman–Grobman

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions and (x_*, y_*) be a hyperbolic fixed point of (7.1). Then there exists a neighbourhood $U \subset \mathbb{R}^2$ of (x_*, y_*) and a homeomorphism $H : U \to \mathbb{R}^2$ (i.e. $H : U \to H(U)$ is a bijection and both H and H^{-1} are continuous) such that H maps trajectories of (7.1) around (x_*, y_*) to trajectories of the linearisation (7.3) around $(0, 0)$.

Moreover, the direction of motion, in particular, the stability will be preserved.

7.8 Certainty about the phase portraits

- Even if a fixed point (x_*, y_*) of the nonlinear system (7.1) is hyperbolic, the trajectories around (x_*, y_*) are usually be more 'screwed' comparing with the trajectories of its linearisation (7.3).
- In particular, straight line trajectories of the linearisation (7.3) **are not**, in general, trajectories of the nonlinear system (7.1).
- Nevertheless, the phase portrait remains quite similar if the fixed point (x_*, y_*) is either of node, saddle, spiral.
- For shortness, we will specify the type of the fixed point of the linearised system (7.3) as 'linear', and for the main system (7.1) as 'nonlinear'.
- Therefore, to show that a fixed point (x_*, y_*) of (7.1) is either of nonlinear node, nonlinear saddle, or nonlinear spiral, it is enough to show that it is linear node, saddle, or spiral, respectively.

7.9 Separatrices

A SEPARATRIX is a trajectory which divides the phase space into regions with distinctly different types of qualitative behaviour.

7.10 Trajectories around nodes

- The trajectories **locally** around a (nonlinear, stable or unstable) node of (7.1) at a fixed point look like branches of 'screwed' parabolas.
- As a draft sketch, they can be drawn with respect to the axes that are the main directions.
- In reality, however, the parabolas are drawn in 'screwed' axes formed by the separatrices, which are tangent to the main directions at the fixed point.
- Note that, in general, the main directions themselves (that are straight lines) do not contain trajectories.
- The line l_2 (a main direction) which corresponds to the eigenvalue with **smaller** absolute value remain **tangent to all** trajectories at the fixed point (including the corresponding separatrix).

7.11 Example

Consider the dynamical system

$$\begin{cases} x' = x, \\ y' = 2y + x^2. \end{cases} \tag{7.4}$$

A fixed point of (7.5) should satisfy both $x = 0$ and $2y + x^2 = 0$, hence the only fixed point is the origin $(0, 0)$. The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 1 & 0 \\ 2x & 2 \end{pmatrix}, \qquad J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and hence $(0, 0)$ is a (linear, and hence a nonlinear) unstable node. Since $J(0, 0)$ is a diagonal matrix, the eigenvectors are the standard basis: $v_1 = (1, 0)^T$, $v_2 = (0, 1)^T$; hence the main directions are the coordinate axis.

The linearised system here can be solved explicitly: $x(t) = x(0)e^t$, $y(t) = y(0)e^{2t}$, and hence the trajectories are just branches of parabolas $y = cx^2$, $c \in \mathbb{R}$ and of $x = 0$.

Then, the trajectories of (7.4) should behave similarly **at the neighbourhood of the origin**, however, the role of coordinate axis will be replaced by separatrices, which can be found for this (simple) system explicitly.

Note that the constant zero-function $x \equiv 0$ solves the first equation in (7.4), and being substituted to the second equation provides $y' = 2y$ (the same as in the linearisation), hence both rays $\{(0, y) \in \mathbb{R}^2 \mid y > 0\}$ and $\{(0, y) \in \mathbb{R}^2 \mid y < 0\}$ are trajectories of the nonlinear system (7.4) as well.

Note that the arrows are directed outwards (e.g. because $y = 0$ is an unstable fixed point for $y' = 2y$). Therefore, $x = 0$ is a separatrix itself.

For $x \not\equiv 0$, one can divide the second equation of (7.5) by the first one:

$$\frac{y'}{x'} = \frac{2y + x^2}{x}.$$

Considering $y = y(x)$, so that $y(t) = y(x(t))$, one gets

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Longrightarrow \frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt} = \frac{y'}{x'}.$$

Therefore,

$$\frac{dy}{dx} = \frac{2y}{x} + x;$$

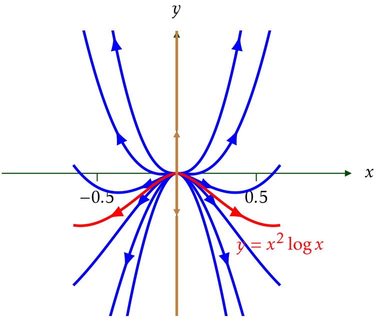
this is a linear non-homogeneous differential equation which yields

$$\boxed{y(x) = Cx^2 + x^2 \log x,}$$

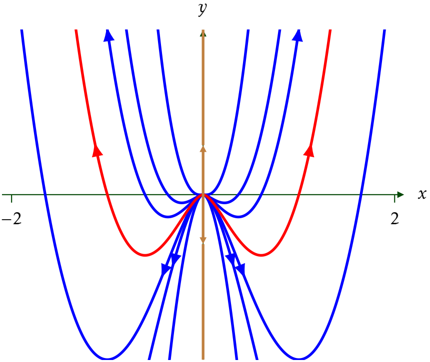
where $C \in \mathbb{R}$ is an arbitrary constant.

The curve $y_1(x) := x^2 \log x$ is hence the second separatrix at the origin: it corresponds to $C = 0$ and separates the curves with $C > 0$ from the curves with $C < 0$. It is also tangent to $y = 0$ at the origin.

The trajectories in the neighbourhood of the origin confirm the general rules:



Note that, however, the phase portrait far enough from fixed point may look quite different. Actually, trajectories which are more 'close' to the separatrix look more 'similar' to it.



7.12 Trajectories around saddles

- The trajectories around a saddle fixed point look like branches of 'screwed' hyperbolas.
- As a draft sketch, they can be drawn with respect to the axes made by the main directions.
- In reality, however, the 'hyperbolas' are drawn in 'screwed' axes formed by the separatrices, which are tangent to the main directions at the fixed point.
- Let, e.g. $\lambda_2 > 0 > \lambda_1$, then, similarly to the linear case, arrows on the 'hyperbolas' are directed **towards** the fixed point along the main direction l_1 corresponding to the **negative** eigenvalue $\lambda_1 < 0$, and **outwards** the fixed point along the main direction l_2 corresponding to the **positive** eigenvalue $\lambda_2 > 0$.
- The separatrix tangent to l_1 is called the **stable manifold** and the separatrix tangent to l_2 is called the **unstable manifold**. Their properties are summarised in the next theorem.

7.13 Theorem: Stable and unstable manifolds

Let (x_*, y_*) be a fixed point to (7.1) such that the linearisation (7.3) has eigenvalues $\lambda_2 > 0 > \lambda_1$ and hence a saddle fixed point. Let l_1, l_2 be straight lines which pass through (x_*, y_*) and which are parallel to the corresponding eigenvectors v_1, v_2.

Then, in a neighbourhood of (x_*, y_*), there exist two curves, s_1 and s_2, which intersect at (x_*, y_*) only, such that l_1 is tangent to s_1 and s_2 is tangent to l_2 at (x_*, y_*), and such that

- any $(x(0), y(0)) \in s_1$ implies, as $t \to \infty$, $(x(t), y(t)) \to (x_*, y_*)$ **moving along** s_1;
- any $(x(0), y(0)) \in s_2$, $(x(0), y(0)) \neq (x_*, y_*)$ implies, as t increases, $(x(t), y(t))$ 'runs away' from (x_*, y_*), **moving along** s_2.

The curves s_1 and s_2 are called STABLE and UNSTABLE MANIFOLDS of (7.1) at (x_*, y_*), respectively. They are separatrices: they separate zones where other trajectories lie.

7.14 Example

Consider the dynamical system

$$\begin{cases} x' = x, \\ y' = -y + x^2. \end{cases} \tag{7.5}$$

Proceeding similarly to Example 7.11, we get that the origin is the only fixed point, the Jacobian is

$$J(x,y) = \begin{pmatrix} 1 & 0 \\ 2x & -1 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and hence the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, i.e. the origin is (linear, and hence nonlinear) saddle point. The main directions again are given by the standard basis: $v_1 = (1,0)^T$, $v_2 = (0,1)^T$.

The trajectories of the linearised system are just hyperbolas $y = \frac{c}{x}$, $c \in \mathbb{R}$, and also the axis $x = 0$.

Therefore, the trajectories of the nonlinear system (7.5) should behave similarly at the neighbourhood of the origin, however, the role of coordinate axis will be replaced by stable and unstable manifolds, which can be also found here explicitly.

Again, the zero-function $x \equiv 0$ solves the first equation in (7.5), and yields $y' = -y$. Therefore, both rays $\{(0,y) \in \mathbb{R}^2 \mid y > 0\}$ and $\{(0,y) \in \mathbb{R}^2 \mid y < 0\}$ are trajectories of (7.5), directed towards the origin. Hence $x = 0$ is the **stable manifold** of (7.5).

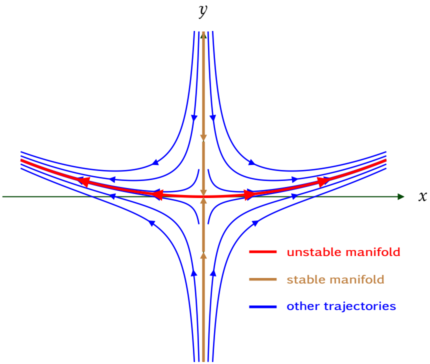
For $x \neq 0$, one can delete the second equation of (7.5) by the first oneand, similarly to Example 7.11, one gets

$$\frac{dy}{dx} = -\frac{y}{x} + x.$$

Therefore,

$$y(x) = \frac{C}{x} + \frac{x^2}{3},$$

where $C \in \mathbb{R}$ is an arbitrary constant. All the graphs but for $C = 0$ are indeed hyperbola-like. $C = 0$ corresponds to the parabola $y = \frac{x^2}{3}$, which is tangent at the origin to the main direction $y = 0$ (that was the unstable manifold for the linearisation). and hence this parabola is the **unstable manifold**.



7.15 Less certainty about the phase portraits

- If the linearisation (7.3) has, as the fixed point, a star or a degenerate node (which correspond to $\Delta = 0$ for the linear systems) the nonlinear system (7.1) may have quite different phase portrait even in the small neighbourhood of (x_*, y_*).
- Stress, however, that, since stability is kept, an e.g. stable star for linearisation may correspond to a stable node or stable spiral for the nonlinear system, but not to an unstable one.

7.16 Example

Consider the dynamical system

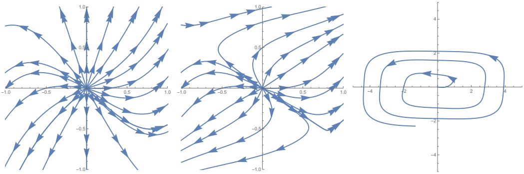
$$\begin{cases} x' = x + ry^3, \\ y' = y + x^3. \end{cases} \tag{7.6}$$

Clearly, $(0,0)$ is a fixed point. The Jacobian is

$$J(x,y) = \begin{pmatrix} 1 & 3ry^2 \\ 3x^2 & 1 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $J(0,0)$ is a diagonal matrix with $\lambda_1 = \lambda_2 = 1 > 0$, the linearised system has an unstable star fixed point.

Consider, however, the phase portraits (obtained numerically) for different values of $r \in \mathbb{R}$:



$r = 0$ 'star-like' trajectories | $r = 5$ 'node-like' trajectories | $r = -14$ a spiral trajectory

Note that the trajectories remain unstable in all cases.

7.17 Theorem: Linear centre

Let (x_*, y_*) is a linear centre of (7.1) (i.e. the linearisation (7.3) has a centre at the origin). Then (x_*, y_*) is (**locally**) either centre or spiral (stable or unstable).

7.18 Example

This example show that the linear center indeed does not imply the type of the nonlinear fixed point nor its stability.
Consider the dynamical system

$$\begin{cases} x' = -y + ax(x^2 + y^2), \\ y' = x + ay(x^2 + y^2), \end{cases} \qquad a \in \mathbb{R}. \qquad (7.7)$$

Clearly, $(0,0)$ is a fixed point, and this is the only fixed point. Indeed, if $x \neq 0$, then

$$\begin{cases} -y + ax(x^2+y^2) = 0, \\ x + ay(x^2+y^2) = 0 \end{cases}$$

implies $a(x^2+y^2) = \frac{y}{x}$, and then $x + \frac{y}{x}y = 0$, $x^2+y^2 = 0$, thus $x = y = 0$, a contradiction. The same holds if $y \neq 0$.
The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 3ax^2 + ay^2 & -1 \\ 1 + 2axy & ax^2 + 3ay^2 \end{pmatrix}.$$

Then

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, the eigenvalues are solution to

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0, \qquad \lambda^2 + 1 = 0, \qquad \lambda = \pm i,$$

i.e. $(0,0)$ is a linear centre.
Consider now the behaviour of the whole nonlinear system. We multiply the first equation in (7.7) by x, the second—by y, and add:

$$xx' + yy' = a(x^2+y^2)^2. \tag{7.8}$$

We pass now to the polar coordinates, (r,θ) in (7.7).

Note that, actually, $r = r(t) \geq 0$ and $\theta = \theta(t) \in \mathbb{R}$. We have

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{7.9}$$
$$r^2 = x^2 + y^2 \tag{7.10}$$

and differentiating (7.10) we get

$$2rr' = 2xx' + 2yy', \qquad rr' = xx' + yy'.$$

Then (7.8) reads as follows: $rr' = ar^4$, i.e.

$$r' = ar^3. \tag{7.11}$$

- If $a > 0$ then $r' > 0$ for $r > 0$ (and $r = 0$ is an unstable fixed point of (7.11)), hence $r = r(t)$ increases in time.
- If $a < 0$ then $r' < 0$ for $r > 0$ (and $r = 0$ is a stable fixed point of (7.11)), hence $r = r(t)$ decreases, moreover, $r(t) \to 0$ as $t \to \infty$.
- If $a = 0$, then $r' = 0$, i.e. $r(t) = \text{const} = r(0)$ for all t.

Next, $x = x\cos\theta$ implies

$$x' = r'\cos\theta - r\sin\theta\cdot\theta' = ar^3\cos\theta - r\sin\theta\cdot\theta'.$$

On the other hand, by (7.7) and (7.9):

$$x' = axr^2 - y = ar^3\cos\theta - r\sin\theta.$$

Equating the right hand sides of the latter equalities, we obtain

$$\theta' = 1, \qquad \theta(t) = t + \theta(0),$$

i.e. the angle changes linearly with time.

- As a result, for $a > 0$, the trajectories will be unstable spirals, as the distance r from the origin will increase with increasing of the angle θ.
- For $a < 0$, the trajectories will will be stable spirals, as the distance r from the origin will decrease with increasing of the angle θ.
- For $a = 0$, the trajectories will be circles, as the distance r from the origin will remain constant while angle θ increases.

In other words, only for $a = 0$ the linear centre at the origin is indeed a nonlinear centre for (7.7). Moreover, both stability and unstability may take place for the same such linearisation.

Note that the direction of the rotation of spirals and the arrows for circles can be obtained using a direction vector for (7.7): e.g. at $(1,0)$, one has $(x',y') = (a,1)$, or just using that $\theta(t)$ is an increasing function and the angle in polar coordiantes is measured counter-clockwise.

Section 8: Global phase portrait

8.1 General scheme

- Let $f, g : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be smooth, and
$$\begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases} \tag{8.1}$$
for $t \in I \subseteq \mathbb{R}$, be such that, for each $t_0 \in I$ and $(x(t_0), y(t_0)) \in D$, (8.1) has a unique solution $(x(t), y(t)) \in D$ for $t \in I$.
- The global phase portrait of (8.1) shows the behaviour of all 'typical' trajectories of (8.1) over the whole domain D.
- To get the global phase portrait, one should, firstly, describe (if possible) the local phase portraits around each of fixed points of (8.1).
- Recall that it is always possible for fixed points which are nodes, saddles, or spirals. For linear centres, however, a further analysis is needed (see below).
- Then one should sketch the trajectories connecting the neihgbourhoods of the fixed points, so that the 'global' trajectories would reflect the revealed local phase portraits. Several tricks discussed below may help to do this in a non-contradictory way.

8.2 Trajectories on straight lines

- A vertical straight line $x = a$, $a \in \mathbb{R}$ (on the phase space $\mathbb{R}^2$) contains trajectories of (8.1) iff

$$f(a,y) = 0, \quad \text{for all } y \in \mathbb{R} \text{ s.t. } (a,y) \in D.$$

  In other words, the constant function $x(t) = a$ satisfies the first eqution in (8.1).

- The phase portrait on the line $x = a$ is determined then from the *one-dimensional* dynamical system

$$y' = g(a,y).$$

- Similarly, a horizontal straight line $y = b$, $b \in \mathbb{R}$, contains trajectories of (8.1) iff

$$g(x,b) = 0, \quad \text{for all } x \in \mathbb{R} \text{ s.t. } (x,b) \in D.$$

  In other words, the constant function $y(t) = b$ satisfies the second eqution in (8.1).

- The phase portrait on the line $y = b$ is determined then from the *one-dimensional* dynamical system

$$x' = f(x,b).$$

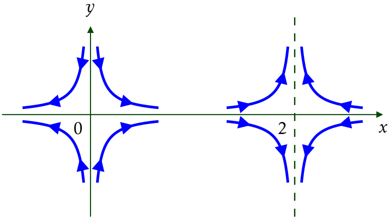
- Intersections of such vertical and horizontal lines are (some of) the fixed points of (8.1).

8.3 Example

Consider the dynamical system

$$\begin{cases} x' = 2x - x^2, \\ y' = -y + xy. \end{cases}$$

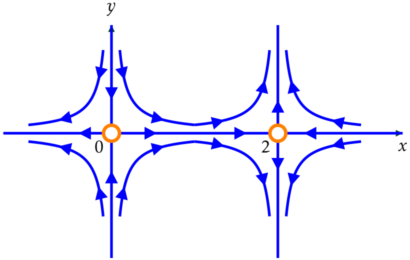
The fixed points are $O(0,0)$ and $A(2,0)$ (**check!**). The linearisation at O has eigenvalues $\lambda_1 = 2 > 0$ and $\lambda_2 = -1 < 0$ (**check!**), hence O is a **saddle** fixed point; and the corresponding eigenvectors are (**check!**) $v_1 = (1,0)^T$ and $v_2 = (0,1)^T$. Therefore, the horizontal line $y = 0$ is tangent to the unstable manifold and the vertical line $x = 0$ is tangent to the stable manifold **around** O.
The linearisation at A is also a **saddle**, since (**check!**) the eigenvalues $\lambda_1 = -2 < 0$ and $\lambda_2 = 1 > 0$; and the corresponding eigenvectors are $v_1 = (1,0)^T$ and $v_2 = (0,1)^T$. Therefore, the same horizontal line $y = 0$ is tangent now to the stable manifold, and the vertical line $x = 2$ is tangent to the unstable manifold **around** A.



To extend the local phase portraits around O and A to the global one, we have to 'join' disconnected trajectories. In general, it can eb done in different ways, however, here one can notice that $x(t) \equiv 0$ and $x(t) \equiv 2$ solve the first eqution, and $y(t) \equiv 0$ solves the second equation. Therefore, the vertical lines $x = 0$ and $x = 2$ and the horizontal line $y = 0$ contain trajectories of the given system. In particular, the mentioned stable and unstable manifolds (locally around the fixed points) lie on these lines.

On $y = 0$, the phase portrait is given by the first equation, there are three trajectories there, and $x = 0$ is unstable and $x = 2$ is stable there. Stress that the correspondin arrows **should be agreed** with the local phase portraits around O and A. On $x = 0$, the phase portrait is given by $y' = -y$, and on $x = 2$, the phase portrait is given by $y' = y$.

SInce the trajectories do not intersect, we get the global phase portrait



8.4 Remark

For smooth f, g, the solutions to (8.1) are continuous functions, which **conrinuously** depend on the initial conditions. Since each point inside D may be an initial condition, two trajectories in a small region which does not contain fixed points should behave similarly. In particular, the hyperbola-like trajectories in the previous example will asymptotically convrge to the corresponding horizontal and vertical trajectories.

8.5 Trajectories on straight lines (*cont.*)

- A **slanted** straight line $y = px + q$, $p \neq 0$, containes trajectories of (8.1) if

$$\frac{y'}{x'} = p \quad \text{along this line}$$

(the slope of the direction vector remains constant there), i.e. if

$$\frac{g(x, px+q)}{f(x, px+q)} = p$$

for all $x \in \mathbb{R}$ such that $(x, px+q) \in D$. (**Remark:** we will mostly have $D = \mathbb{R}^2$.)

- Such line often exists if two fixed points (nodes or saddles) have equal (or proportional) one of the main directions.

8.6 Example

Consider the dynamical system

$$\begin{cases} x' = x(6 - x + 2y), \\ y' = y(x - y - 4). \end{cases} \tag{8.2}$$

This system has four fixed points (**check the following by yourself!**):

- $O(0,0)$ is a **saddle** point: $\lambda_1 = 6$, $\lambda_2 = -4$, $v_1 = (1,0)^T$, $v_2 = (0,1)^T$;
- $A(6,0)$ is a **saddle** point: $\lambda_1 = -6$, $\lambda_2 = 2$, $v_1 = (1,0)^T$, $v_2 = (3,2)^T$;
- $B(0,-4)$ is a **saddle** point: $\lambda_1 = 4$, $\lambda_2 = -2$, $v_1 = (0,1)^T$, $v_2 = (3,2)^T$;
- $C(2,-2)$ is a **linear centre**: $\lambda_1 = 2i$, $\lambda_2 = -2i$; hence, the phase portrait at C **requires further analysis** (which we will consider later on).

Next, $x = 0$ and $y = 0$ solve the first and second equations, respectively. Therefore, the coordinate axes contain trajectories: $x = 0$ implies $y' = -y(y+4)$, where $y = 0$ is stable and $y = -4$ is unstable fixed points; similarly, $y = 0$ implies $x' = x(6-x)$, where $x = 0$ is unstable and $x = 6$ is stable fixed points. In particular, the (local) unstable manifold for O and the stable manifold for A lie on $y = 0$ and the (local) stable manifold for O and the unstable manifold for B lie on $x = 0$.

Also, we see that the other main directions for A and for B coinside. It gives a guess to try line AB; its equation is

$$\frac{x-6}{0-6} = \frac{y-0}{-4-0}, \quad y = \frac{2}{3}x - 4.$$

The slope $\frac{2}{3}$ of AB **is the slope of the common main direction** $(3,2)^T$ of saddles at A and at B. Let $(x(t), y(t))$ lie on AB, i.e. $y = \frac{2}{3}x - 4$. Then

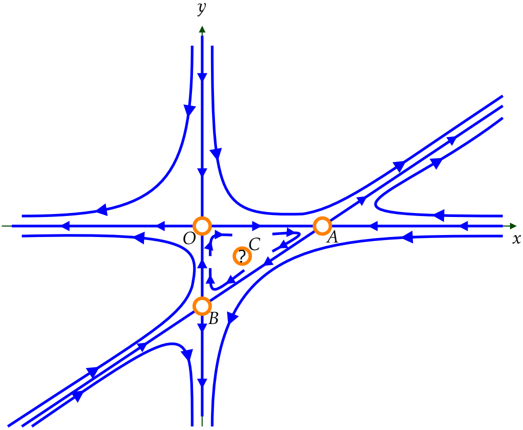
$$x' = x\Big(6 - x + \frac{4}{3}x - 8\Big) = \frac{1}{3}x(x-6),$$
$$y' = \Big(\frac{2}{3}x - 4\Big)\frac{1}{3}x = \frac{2}{9}x(x-6),$$

and hence

$$\frac{y'}{x'} = \frac{2}{3}.$$

Therefore, the line $y = \frac{2}{3}x - 4$ contain trajectories. Next, on this line, by the above,$x' > 0$ and $y' > 0$ for $x < 0$ and for $x > 6$, and $x' < 0$ and $y' < 0$ for $x \in (0,6)$. This gives the phase portrain on the line. (Note that the arrows on the line **should be agreed** with the local phase portraits around A and B.) Since the trajectories do not intersecr, the trangle OAB is a '**trap**': if $(x(t_0), y(t_0))$ lies inside this triangle, then $(x(t), y(t))$ will remain inside this triangle forever.

The phase picture on lines containing the sides of OAB and the phase picture outside OAB is now clear. Whether trajectories inside OAB are spirals or cycles, we will discuss later on.



8.7 Definition

- **Isocline** is a curve/line in the phase space at each point of which the direction vector (x', y') has the same gradient (slope).
- In other words, the tangent lines to trajectories, at the points where they intersect an isocline, are parallel.

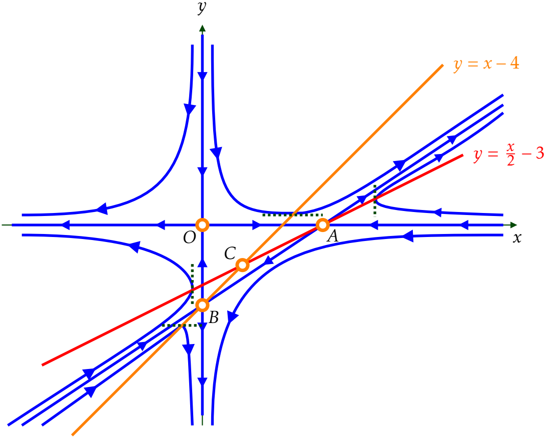
- An isocline is not, in general, a trajectory, unless the case of a straight line trajectory (which is trivially 'tangent' to itself at each point).
- The important particular cases of isoclines are **NULLCLINES**:
  - x-nullclines are curves/lines along which $x' = 0$, i.e. they are determined from the equation $f(x, y) = 0$;
  - y-nullclines are curves/lines along which $y' = 0$, i.e. they are determined from the equation $g(x, y) = 0$.
- x-nullclines intersect y-nullclines at the fixed points.

8.8 Example: continuation of Example 8.6

Find the nullclines of (8.2):

- x-nullclines are $x = 0$ (trivial as it is a trajectory) and $6 - x + 2y = 0$,
- y-nullclines are $y = 0$ (trivial) and $x - y - 4 = 0$.

These two lines hence contain turning points for the trajectories they intersect: on x-nullcline $y = \frac{x}{2} - 3$, the tangent lines to trajectories are vertical, and on y-nullcline $y = x - 4$, the tangent lines to trajectories are horizontal.



In particular, this provides more detailed description of the behaviour of solutions. E.g. if we know that

$$x(0) - 4 < y(0) < \frac{2}{3}x(0) - 4,$$

i.e. if $(x(0), y(0))$ lies above the (orange) y-nullcline and below the (blue) slanted straight line of trajectories, then $x(t)$ and $y(t)$ will both increase until a unique moment of time $t_1 > 0$ when $y(t_1) = x(t_1) - 4$, and after this $x(t)$ will continue increase, converging to 0 as $t \to \infty$, whereas $y(t)$ will decrease to $-\infty$.

8.9 Integral of motion

- Let f, g in (8.1) be smooth on $D \subseteq \mathbb{R}^2$ and let $D_1 \subseteq D$ be an open set. A smooth function $h : D_1 \to \mathbb{R}$ is said to be an **INTEGRAL OF MOTION** (a.k.a. a **FIRST INTEGRAL**) of (8.1) if $h(x(t), y(t))$ is constant for any solution $(x(t), y(t)) \in D_1$, $t \in I$, of (8.1).
- The characteristic property of an integral of motion is the identity $\dfrac{d}{dt} h(x(t), y(t)) = 0$, which can be rewritten as follows:

$$\frac{\partial}{\partial x} h(x(t), y(t)) \cdot f(x(t), y(t)) + \frac{\partial}{\partial y} h(x(t), y(t)) \cdot g(x(t), y(t)) = 0$$

  for all $(x(t), y(t)) \in D_1$, $t \in I$.
- Since $D_1 \subset D$ and, for any $(x, y) \in D$, there is a trajectory which passes through it, to find a first integral one needs to find $h : D_1 \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x} h(x, y) \cdot f(x, y) + \frac{\partial}{\partial y} h(x, y) \cdot g(x, y) = 0,$$

for all $(x, y) \in D_1$.

- The trajectories $(x(t), y(t)) \in D_1$ then lie on the curves, determine dby the **levels** of function h:

$$\boxed{h(x,y) = C}, \qquad (x,y) \in D_1,\ C \in J \subseteq \mathbb{R}.$$

  By the existence and uniquencess theorem, each such curve is a disjoint union of trajectories.
- If $h = h(x,y)$ is an integral of motion of (8.1) in D_1, then $\tilde{h}(x,y) := k(h(x,y))$ is also an integral of motion of (8.1) in D_1, for any smooth $k : \mathbb{R} \to \mathbb{R}$. In particular, $ah(x,y) + b$ is an integral of motion, for any $a, b \in \mathbb{R}$.
- The dynamical system (8.1) is called CONSERVATIVE if it has an integral of motion on the whole phase plane $\mathbb{R}^2$ (i.e. if $D_1 = D = \mathbb{R}^2$).

8.10 Remark

- Despite the name, integrals of motion do not reveal solutions to (8.1), they rather provide the shape of the trajectories.
- Integrals of motion are one of the main ways to determine nonlinear centres (see below).

8.11 Example: Predator-prey model

Consider the dynamical system

$$\begin{cases} x' = x(\mathrm{a} - \mathrm{b}y), \\ y' = y(\mathrm{d}x - \mathrm{c}). \end{cases}$$

Here $x \geq 0$ represents the density of a population of preys and y represents the density of a population of predators.
The coefficients are positive:

$$\mathrm{a} > 0, \mathrm{b} > 0, \mathrm{c} > 0, \mathrm{d} > 0.$$

Note that if, however, $\mathrm{b} = 0$, then $x' = \mathrm{a}x$ implies $x(t) = x(0)e^{\mathrm{a}t} \nearrow \infty$ for $x(0) > 0$ and $\mathrm{a} > 0$. In other words, the population of preys will exponentially grow if the predators will not influence them (we suppose that preys do not compete between themselves and have infinitely many food). In contrast, if $\mathrm{d} = 0$, then $y' = -\mathrm{c}y$ implies $y(t) = y(0)e^{-\mathrm{c}t} \searrow 0$ for $\mathrm{c} > 0$, i.e. the population of predators without preys (i.e. without food) will extinct.

For positive $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$, there are two fixed points (**check the following!**):

- the origin $O(0,0)$ which is a saddle point with the main directions along the coordinate axis which also contain trajectories; as a result, $x = 0$ is the stable manifold and $y = 0$ is an unstable manifold;
- the point $A\left(\frac{\mathrm{c}}{\mathrm{d}}, \frac{\mathrm{a}}{\mathrm{b}}\right)$ which is a linear centre.

To find an integral of motion, we divide equations (similar to Example 7.11):

$$\frac{dx}{dy} = \frac{x(\mathrm{a} - \mathrm{b}y)}{y(\mathrm{d}x - \mathrm{c})},$$

and one can separate variables:

$$\mathrm{d}x - \mathrm{c}\log x = \mathrm{a}\log y - \mathrm{b}y + C$$

Therefore,

$$h(x,y) = \mathrm{d}x - \mathrm{c}\log x - \mathrm{a}\log y + \mathrm{b}y$$

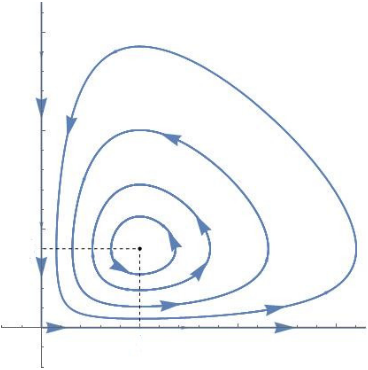
and the trajectories lie on the curves $h(x,y) = C \in \mathbb{R}$.

By Theorem 7.17, the fixed point at A may be either nonlinear centre or spiral. To distinguish these two cases note that, for a centre, any trajectory has a finite number of intersections with a line $x = x_1$ if x_1 is close enough to $\dfrac{\mathrm{c}}{\mathrm{d}}$.

Whereas, for a spiral, there will be infinitely many points of intersection. However, the equation $h(x_1, y) = C$ has at most two solutions (in y). Indeed, it implies

$$\log y = \frac{\mathrm{b}}{\mathrm{a}} y + K, \qquad K := \frac{\mathrm{d}x_1 - \mathrm{c}\log x_1 - C}{\mathrm{a}},$$

and the line $\frac{\mathrm{b}}{\mathrm{a}} y + K$ has at most two points of intersection with the curve $\log y$ (**check!**).
As a result, we indeed have a nonlinear centre at $\left(\frac{\mathrm{c}}{\mathrm{d}}, \frac{\mathrm{a}}{\mathrm{b}}\right)$. Note also that both functions $x(t)$ and $y(t)$ are hence periodic. Finally, since $y \to 0$ along the line $x = 0$ (y-axis) and $x \to \infty$ along the line $y = 0$ (x-axis), then the motion goes counter-clockwise.



8.12 Example: Hamiltonian dynamics

Hamiltonian dynamics describes the **one-dimensional** motion of a particle, with position $q = q(t) \in \mathbb{R}$, momentum $p = p(t) \in \mathbb{R}$, and a HAMILTONIAN $H = H(q,p)$, according to the following equations of motion:

$$\begin{cases} q' = \dfrac{\partial H}{\partial p}(q,p), \\ p' = -\dfrac{\partial H}{\partial q}(q,p). \end{cases}$$

Hence, here $(x,y) = (q,p)$, $f = \frac{\partial H}{\partial p}$, $g = -\frac{\partial H}{\partial q}$. Then, choosing $h := H$, we clearly get

$$\frac{\partial h}{\partial q} \cdot \frac{\partial H}{\partial p} + \frac{\partial h}{\partial p} \cdot \left(-\frac{\partial H}{\partial q}\right) = 0;$$

therefofe, all trajectories lie on the levels $H(q,p) = \text{const}$.

8.13 Example

Consider the dynamical system

$$\begin{cases} x' = f(y), \\ y' = g(x), \end{cases} \tag{8.3}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth on $\mathbb{R}$.

We define

$$h(x,y) = \int f(y)\,dy - \int g(x)\,dx.$$

Then

$$\frac{\partial}{\partial x}h(x,y) = -g(x), \quad \frac{\partial}{\partial y}h(x,y) = f(y),$$

and hence $\dfrac{\partial}{\partial x}h \cdot f + \dfrac{\partial}{\partial y}h \cdot g = 0$ on $\mathbb{R}^2$. Therefore, all trajectories of (8.3) lie on the curves $h(x,y) = C$, $C \in \mathbb{R}$.

Nore also that if (x_*, y_*) is a fixed point of (8.3), i.e. if $g(x_*) = f(y_*) = 0$, then the linearisation at (x_*, y_*) is determine by the Jacobian:

$$J(x_*, y_*) = \begin{pmatrix} 0 & f'(y_*) \\ g'(x_*) & 0 \end{pmatrix}.$$

The characteristic equation is then

$$\lambda^2 - g'(x_*)f'(y_*) = 0.$$

Excluding more delicate degenerate case $g'(x_*)f'(y_*) = 0$, we see that if $g'(x_*)f'(y_*) > 0$, (x_*, y_*) is a saddle, and if $g'(x_*)f'(y_*) < 0$, (x_*, y_*) is a linear centre. The next example shows both possibilities.

8.14 Example

Consider the dynamical system

$$\begin{cases} x' = y, \\ y' = x - x^3. \end{cases}$$

The fixed points are $(0,0)$, $(1,0)$, $(-1,0)$.
At $(0,0)$, the eigenvalues are ± 1, the stable manifold is tangent to the main direction $v_1 = (-1,1)^T$, the unstable manifold is tangent to the main direction $v_2 = (1,1)^T$ (**check!**).
Both $(1,0)$ and $(-1,0)$ are linear centres, hence, the local phase portraits require further analysis.
Consider

$$h(x,y) = \int y\,dy - \int (x - x^3)\,dx = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}.$$

Then h satisfies $\dfrac{\partial}{\partial x} h \cdot f + \dfrac{\partial}{\partial y} h \cdot g = 0$ on $\mathbb{R}^2$, with $f(x,y) = y$ and $g(x,y) = x - x^3$. Hence h is an integral of motion. Construct then another integral of motions: $4h + 1$ (to get rid of denominators and also to obtain the full square in x): the trajectories lie on the curves $x^4 - 2x^2 + 1 + 2y^2 = \mathrm{const}$, i.e.

$$(x^2 - 1)^2 + 2y^2 = C. \tag{8.4}$$

With necessity, hence, $C \geq 0$.

Recall that, by Theorem 7.17, $(\pm 1, 0)$ may be either nonlinear centres or spirals. When $C = 0$ in (8.4), it reads $x^2 - 1 = y = 0$, and the trajectories are just these fixed points. For small $C > 0$, hence the trajectories are close to the fixed point. Take e.g. $x = 1$ and any small enough $C > 0$, then there are two values of y, such that (8.4) holds. In other words, the vertical line $x = 1$ crosses the trajectory (8.4) with small $C > 0$ at two points only; that would be impossible if $(1, 0)$ were a (nonlinear) spiral. Hence, both $(\pm 1, 0)$ are **nonlinear centres** (**locally!**).

We can analyse here the global phase portrait as well. Since (8.4) implies

$$|x^2 - 1| \le \sqrt{C}, \qquad |y| \le \sqrt{C},$$

the trajectories are bounded subsets of $\mathbb{R}^2$.The trajectories are given by the graphs

$$y = \pm\sqrt{\frac{1}{2}(C - (x^2 - 1)^2)} \quad \text{for } |x^2 - 1| \le \sqrt{C}.$$

The restriction on x yields

$$1 - \sqrt{C} \le x^2 \le 1 + \sqrt{C}. \tag{8.5}$$

Note also that the trajectories are symmetric with respect to both coordinate axes.

Let $x \geq 0$. For $0 < C < 1$, (8.5) reads

$$\sqrt{1-\sqrt{C}} \leq x \leq \sqrt{1+\sqrt{C}},$$

and the trajectories pass through points $(\sqrt{1 \pm \sqrt{C}}, 0)$.Hence the trajectories are indeed cycles around the fixed point $(1, 0)$.

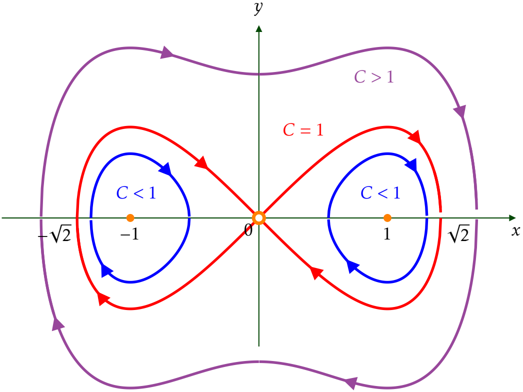
For $C = 1$, (8.5) reads $0 \leq x \leq \sqrt{2}$, and if $x \to 0$ then $y \to 0$ as well. Therefore, if a trajectory starts near the origin, it then pass through $(\sqrt{2}, 0)$ and return to the origin. Since the origin is a fixed point, the trajectory will just tend to the origin, but will not contain it. As a result, the stable manifold at the origin is given by the trajectory $y = -\sqrt{\frac{1}{2}(1-(x^2-1)^2)}$ and the unstable manifold at the origin is given by the trajectory $y = \sqrt{\frac{1}{2}(1-(x^2-1)^2)}$. (**Check** that indeed $y'(0) = \mp 1$ according to the main directions v_1 and v_2.)

For $C > 1$, (8.5) reads

$$0 \leq x \leq \sqrt{1+\sqrt{C}},$$

and the trajectories pass through $(\sqrt{1+\sqrt{C}}, 0)$ and $(0, \pm\sqrt{\frac{1}{2}(C-1)}$.

Note also the direction vectors (x', y') at e.g. x-axis, where $y = 0$, are $(x - x^3, 0)$, i.e. they are directed up for $x < -1$ and $0 < x < 1$ and down otherwise. Hence the motion is indeed clockwise (also to be agreed with the motion at the origin).



8.15 Definition

A trajectory which starts and finishes at the same fixed point (but does not contain it) is called a **HOMOCLINIC ORBIT**.

8.16 Example: finalising Example 8.6

Recall that we are dealing with the dynamical system (8.2), i.e.

$$\begin{cases} x' = x(6 - x + 2y) =: f(x,y), \\ y' = y(x - y - 4) := g(x,y), \end{cases}$$

and $C(2,-2)$ is its linear centre which lies inside triangle OAB whose boundaries are $x = 0$, $y = 0$, $y = \frac{2}{3}x - 4$. Rewriting the latter equation as $2x - 3y - 12 = 0$, we consider

$$h(x,y) = x^a y^b (2x - 3y - 12)^c,$$

for some $a,b,c > 0$. We are going to find $a,b,c > 0$, such that, for all $(x,y) \in \mathbb{R}^2$ (it would be enough to have this inside OAB)

$$\frac{\partial}{\partial x} h \cdot f + \frac{\partial}{\partial y} h \cdot g = 0. \tag{8.6}$$

However, since for any integral of motion h, the function $h^{\frac{1}{c}}$ is also an integral of motion, we can assume that $c = 1$, hence,

$$\boxed{h(x,y) = x^a y^b (2x - 3y - 12),}$$

We have

$$\begin{aligned} \frac{\partial}{\partial x} h(x,y) &= x^{a-1} y^b \Big(2(a+1)x - 3ay - 12a \Big), \\ \frac{\partial}{\partial y} h(x,y) &= x^a y^{b-1} \Big(2bx - 3(b+1)y - 12b \Big). \end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial x} h \cdot f + \frac{\partial}{\partial y} h \cdot g \\
&= x^a y^b \Big(\big(2(a+1)x - 3ay - 12a \big)(6 - x + 2y) \\
&\qquad\qquad + \big(2bx - 3(b+1)y - 12b \big)(x - y - 4) \Big).
\end{aligned}$$

To have (8.6) for all x, y, we require

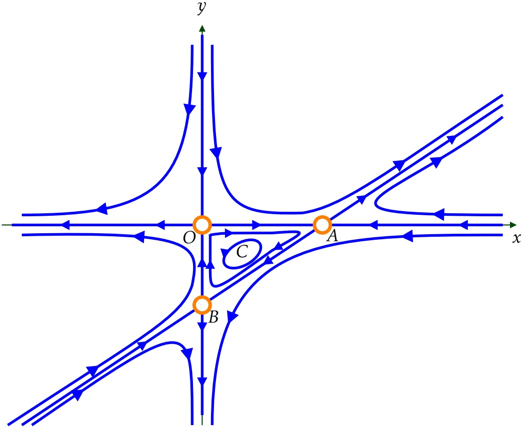
$$\begin{aligned}
&\big(2(a+1)x - 3ay - 12a \big)(6 - x + 2y) \\
&\qquad + \big(2bx - 3(b+1)y - 12b \big)(x - y - 4) = 0.
\end{aligned}$$

The coefficient before x^2 requires: $a + 1 = b$. The coefficient before y^2 requires: $b + 1 = 2a$, Therefore, $b + 1 = 2b - 2$, $b = 3$, then $a = 2$. Then the requirement is satisfied (**check!**). Hence, the curves

$$h(x, y) := x^2 y^3 (2x - 3y - 12) = D, \qquad D \in \mathbb{R},$$

contains (**all**) trajectories of the system (8.2). At the edges of triangle OAB, $h(x, y) = 0$. At the fixed point $C(2, -2)$, $h(2, -2) = 64$. Therefore, by continuity, the levels $h(x, y) = D$ for $D \in [0, 64]$ contain trajectories inside the triangle. (Note that these levels contain other trajectories as well.)

If we consider e.g. $x = 2$, and if $D < 64$ is close to 64, then there only finite number of y such that that $h(x, y) = D$ holds. Hence $C(2, -2)$ cannot be a spiral fixed point, therefore, it is a **(nonlinear) centre**.



8.17 Remark

More generally, if a linear centre is inside a bounded region which does not contain another fixed points and whose boundary is a union of **straight trajectories** given by linear equations $k_1(x, y) = 0, \ldots, k_n(x, y) = 0$, then one can look for the integral of motion of the form $h = k_1^{p_1} \ldots k_n^{p_n}$ for some $p_1, \ldots, p_n > 0$; one of which can be chosen equal to 1.

8.18 Definition

Recall that (see Remark 6.12) that a fixed point (x_*, y_*) of (8.1) is said to be **Lyapunov stable** (stable in the sense of Lyapunov) if, **for every** neighbourhood $U \subset \mathbb{R}^2$ of (x_*, y_*), there exists a (smaller) neighbourhood $W \subset U$ such that any trajectory which starts in W remains in U for all t.

8.19 Definition

A fixed point (x_*, y_*) of (8.1) is said to be **asymptotically stable** if

- it is Lyapunov stable, and
- **there exists** a neighbourhood $U \subset \mathbb{R}^2$ of (x_*, y_*), such that any trajectory which starts in U converges to (x_*, y_*) as $t \to \infty$.

8.20 Definition

A fixed point (x_*, y_*) of (8.1) is said to be **neutrally stable** if it is stable in the sense of Lyapunov but it is not asymptotically stable.

8.21 Remark

A typical example of a neutrally stable fixed point is hence a **nonlinear centre**.

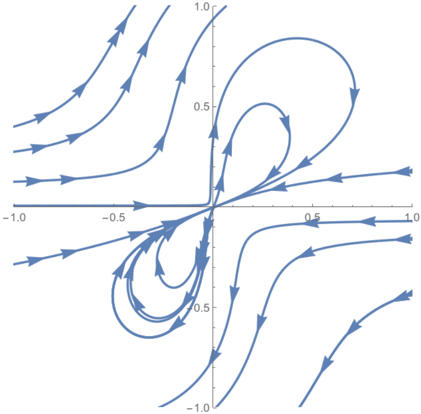
8.22 Remark

Note that even if all trajectories of the system converge to the unique fixed point, it may be not stable in the sense of Lyapunov and hence it will not be asymptotically stable. For example:

$$\begin{cases} x' = \dfrac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)}, \\ y' = \dfrac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)}, \end{cases}$$

has the only fixed point at the origin, that is, however, **non-hyperbolic**: $J(0,0) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$.

The trajectories look as follows:



8.23 Asymptotic stability from linearisation

- If a fixed point (x_*, y_*) of (8.1) is such that the linearisation of (8.1) at (x_*, y_*) has eigenvalues with

$$\operatorname{Re}\lambda_1 < 0 \quad \text{and} \quad \operatorname{Re}\lambda_2 < 0$$

  (and hence, in particular, is hyperbolic) , then (x_*, y_*) is an asymptotically stable fixed point, in some, possibly, small neighbourhood $U \subset \mathbb{R}^2$ of (x_*, y_*).
- This corresponds hence to a stable node or stable spiral of the linearisation which is preserved in the local phase portrait of the nonlinear system (8.1), as well a stable degenerate node or stable star of the linearisation which does not illuminate the phase portrait of (8.1), but still keeps the stability.
- We are going to discuss now how to extend the domain U on which the asymptotic stability takes place to justify/enhance the global phase portrait of (8.1).

8.24 Definition

Let $(x_*, y_*) \in U \subset \mathbb{R}^2$ be a point and its neighbourhood on plane. Let $V : U \to \mathbb{R}$ and

$$V(x_*, y_*) = 0.$$

Then V is said to be

- **POSITIVE DEFINITE** in U, if

$$V(x,y) > 0 \quad \text{for all } (x,y) \in U \setminus \{(x_*, y_*)\};$$

- **NEGATIVE DEFINITE** in U, if

$$V(x,y) < 0 \quad \text{for all } (x,y) \in U \setminus \{(x_*, y_*)\};$$

- **POSITIVE SEMI-DEFINITE** in U, if

$$V(x,y) \geq 0 \quad \text{for all } (x,y) \in U \setminus \{(x_*, y_*)\};$$

- **NEGATIVE SEMI-DEFINITE** in U, if

$$V(x,y) \leq 0 \quad \text{for all } (x,y) \in U \setminus \{(x_*, y_*)\}.$$

8.25 Remark

The notion of positive/negative definite/ semi-definite functions has completely different meaning in some other areas of mathematics (e.g. in probability and analysis).

8.26 Theorem: Lyapunov stability theorem

Let $(x_*, y_*) \in U \subset \mathbb{R}^2$ be a fixed point of (8.1), and U be its neighbourhood. Let $V : U \to \mathbb{R}$ be a **smooth** function on U such that

$$\boxed{V \text{ is positive definite on } U}.$$

Denote $\boxed{\frac{d}{dt}V := \frac{d}{dt}V(x(t), y(t)).}$

1. If

$$\boxed{\frac{d}{dt}V \text{ is negative semi-definite on } U,}$$

   then (x_*, y_*) is Lyapunov stable in U.

2. If

$$\boxed{\frac{d}{dt}V \text{ is negative definite on } U,}$$

   then (x_*, y_*) is asymptotically stable in U;

3. if $\boxed{\frac{d}{dt}V \text{ is negative semi-definite on } U}$,
   U does not contain other fixed points of (8.1), and $\frac{d}{dt}V$ is not equal to 0 **identically on a trajectory** of (8.1), apart from the fixed point (x_*, y_*) itself, then (x_*, y_*) is asymptotically stable in U.

8.27 Lyapunov functions

- A function V which satisfy conditions of the Lyapunov stability theorem is called the LYAPUNOV FUNCTION.
- More precisely, for a positive definite on U function V, if item 1 holds, then V is called a **weak Lyapunov function**, and if item 2 holds, the V is called a **strong Lyapunov function**.
- The following functions are often appear Lyapunov functions in examples:

$$V_1(x,y) = a(x-x_*)^2 + b(y-y_*)^2,$$

$a,b>0$, U is a subset of $\mathbb{R}^2$;

$$V_2(x,y) = a(x-x_*)^{2m} + b(y-y_*)^{2n},$$

$a,b>0$, $n,m\in\mathbb{N}$, U is a subset of $\mathbb{R}^2$;

$$V_3(x,y) = a\Big(x - x_* - x_*\log\frac{x}{x_*}\Big) + b\Big(y - y_* - y_*\log\frac{y}{y_*}\Big),$$

$a,b>0$, U is a subset of $(0,\infty)\times(0,\infty)$. To show that V_3 is positive definite on $(0,\infty)\times(0,\infty)$, note that, for a fixed $s_*>0$, the function $p(s) = s - s_* - s_*\log\frac{s}{s_*}$, $s>0$, has a unique minimum at $s=s_*$ and $p(s_*)=0$.

8.28 Remark

- Typically, the neighbourhood U of (x_*, y_*) is either an infinite set, like $\mathbb{R}^2$ or $(0,\infty)\times(0,\infty)$, or a **level set**

$$\{(x,y)\in\mathbb{R}^2 \mid V(x,y)\le C\}$$

of the Lyapunov function V for certain $C>0$. (Notice the difference between level sets and levels!)

- Often the inequality $\frac{d}{dt}V<0$ (or ≤ 0) holds on some set $D\subset\mathbb{R}^2$ only. In this case, the requirement

$$\{(x,y)\in\mathbb{R}^2 \mid V(x,y)\le C \ (\text{or } <C)\}\subset D$$

justifies the largest level set (i.e. the largest possible $C>0$ to have the latter inclusion) to be used as U in the Lyapunov stability theorem.

8.29 Example

Consider the dynamical system

$$\begin{cases} x' = x(1-y), \\ y' = y(x-1-y), \end{cases} \tag{8.7}$$

for $x\ge 0$, $y\ge 0$.

This is a modification of the predator-prey model from Example 8.11, here predators compete with rate 1 (the term $-y^2$ in the second equation of (8.7)). Here also the origin O is a saddle fixed point, with the basis vectors as the main directions (**check!**). Since $x = 0$ and $y = 0$ are **both** solutions, the coordinate axes contain trajectories. Again, $x = 0$ is the stable manifold, and $y = 0$ is the unstable manifold (**check!**). The second fixed point is $A(2, 1)$, and this is a stable spiral (**check!**). The third fixed point is $(0, -1)$, and it lies out of the quadrant $\{x \geq 0, y \geq 0\}$.
Hence, A is an asymptotically stable in a small neighbourhood. Our aim is to describe the largest possible area for asymptotic stability of A. Consider the function $V = V_3$ (see above):

$$V(x, y) = a\Big(x - 2 - 2\log\frac{x}{2}\Big) + b\Big(y - 1 - \log y\Big).$$

Then

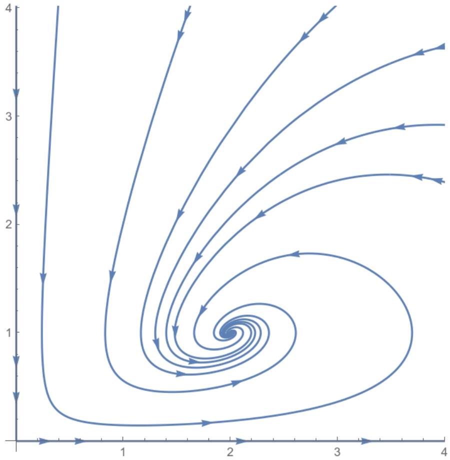
$$\begin{aligned}\frac{d}{dt}V &= a\Big(1 - \frac{2}{x}\Big)x(1 - y) + b\Big(1 - \frac{1}{y}\Big)y(x - 1 - y)\\ &= a(x - 2 - xy + 2y) + b(xy - x + 1 - y^2).\end{aligned}$$

Since we want to have $\dfrac{d}{dt}V < 0$, we take $a = b$ to get rid of $(b - a)xy$.

Then, taking e.g. $a = b = 1$,

$$\frac{d}{dt}V = -(y-1)^2 < 0,$$

unless $y = 1$. However, by the second equation of (8.7), the line $y = 1$ does not contain trajectories of the system unless $x = 2$, that gives the fixed point. Hence, by the third item of the Lyapunov stability theorem, $A(2,1)$ is asymptotically stable on the whole $\{x > 0, y > 0\}$ (the area where V is well-defined and positive-definite). The direction of motion is determined by the motion on the axes:



8.30 Example

Consider the dynamical system

$$\begin{cases} x' = -y - x^3, \\ y' = x - y^3. \end{cases}$$

Here the origin is the only fixed point and this is a linear centre (**check!**). The local phase portrait is hence unclear. Consider the function $V(x,y) = x^2 + y^2$. Then

$$\begin{aligned} \frac{d}{dt}V &= \frac{\partial}{\partial x}V \cdot f + \frac{\partial}{\partial y}V \cdot g \\ &= 2x(-y - x^3) + 2y(x - y^3) \\ &= -2x^4 - 2y^4 < 0, \qquad (x,y) \neq (0,0). \end{aligned}$$

Therefore, V is positive-definite in $\mathbb{R}^2$, and $\dfrac{d}{dt}V$ is negative-definite in $\mathbb{R}^2$, hence, V is a strong Lyapunov function in $\mathbb{R}^2$. Therefore, the origin is asymptotically stable regardless of the initial condition (a.k.a. **globally asymptotically stable**), by the Lyapunov stability theorem. In particular, the trajectories are indeed spirals, at least around the origin.

8.31 Example

Consider the dynamical system

$$\begin{cases} x' = x(y-1), \\ y' = y(x-1). \end{cases}$$

Here there are two fixed points: the origin $O(0,0)$ is a linear stable star and $A(1,1)$ is a saddle (**check!**). Moreover, the coordinate axes and the line $y = x$ contain trajectories (**check!**). We know that the origin will be locally asymptotically stable, and we are going to show how large can be the area of convergence. Consider again the function $V(x,y) = x^2 + y^2$. Then

$$\frac{d}{dt}V = 2x^2(y-1) + 2y^2(x-1) < 0$$
$$\text{for } x < 1, y < 1, (x,y) \neq (0,0).$$

Therefore, V is a strong Lyapunov function on $D := \{(x,y) \in \mathbb{R}^2 \mid x < 1, y < 1\}$. The largest level set of V to be a subset of D is, evidently

$$\{V(x,y) = x^2 + y^2 < 1\} \subset D,$$

i.e. any trajectory which starts inside the unit circle converges to the origin as $t \to \infty$, by the Lyapunov stability theorem.

8.32 Example

Consider the dynamical system

$$\begin{cases} x' = -y - x^5, \\ y' = x^3 - y^3. \end{cases}$$

Here the origin is the only fixed point, however, the local phase portrait is unclear, as the linearisation is $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.

We will look for a Lyapunov function in the form:

$$V(x,y) = ax^{2n} + y^{2m}, \qquad a > 0,\ n,m \in \mathbb{N}.$$

Note that if V is a Lyapunov function, then cV is also a Lyapunov function for any $c > 0$, because of this, we have taken the second coefficient in V equal to 1. Then

$$\begin{aligned} \frac{d}{dt}V &= 2nax^{2n-1}(-y - x^5) + 2my^{2m-1}(x^3 - y^3) \\ &= -2nax^{2n-1}y - 2nax^{2n+4} \\ &\quad + 2mx^3y^{2m-1} - 2my^{2m+2}. \end{aligned}$$

Since we need a negative function, we get rid of odd powers by setting

$$mx^3y^{2m-1} = nax^{2n-1}y.$$

Then with necessity $2n-1=3$, $2m-1=1$, i.e. $n=2$, $m=1$, and $a=\frac{1}{2}$. Therefore,

$$\frac{d}{dt}V = -2x^8 - 2y^4 < 0, \qquad (x,y) \neq (0,0),$$

and V is a strong Lyapunov function in $\mathbb{R}^2$, and hence the origin is globally asymptotically stable by the Lyapunov stability theorem.

8.33 Example

Consider the dynamical system

$$\begin{cases} x' = -y^3 - x + x^5, \\ y' = x. \end{cases} \tag{8.8}$$

Here the origin is the only fixed point, and the linearisation has a line of stable fixed points (**check!**). The local phase portrait is hence unclear. We will look for a Lyapunov function in the form:

$$V(x,y) = ax^{2n} + y^{2m}, \qquad a>0,\ n,m \in \mathbb{N}.$$

Then

$$\begin{aligned} \frac{d}{dt}V &= 2nax^{2n-1}(-y^3 - x + x^5) + 2mxy^{2m-1} \\ &= -2nax^{2n-1}y^3 - 2nax^{2n} \\ &\quad + 2nax^{2n+4} + 2mxy^{2m-1}. \end{aligned}$$

Since we need a negative function, we get rid of odd powers by setting

$$mxy^{2m-1} = nax^{2n-1}y^3,$$

then $n = 1$, $m = 2$, $a = 2$. Hence, $V(x, y) = 2x^2 + y^4$, and

$$\frac{d}{dt}V = -4x^2 + 4x^6 = -4x^2(1 - x^4) < 0,$$

if $|x| < 1$ and $x \neq 0$. The line $x = 0$ does not contain trajectories of (8.8) except the fixed point at the origin, as $x = 0$ in the first equation of (8.8) implies $y = 0$. Hence, we require

$$\{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^4 < C\} \subset \{(x, y) \in \mathbb{R}^2 \mid |x| < 1\},$$

i.e. find the maximal $C > 0$ such that

$$2x^2 + y^4 < C \Longrightarrow |x| < 1.$$

Therefore, $C = 2$. As a result, by the Lyapunov stability theorem (see item 3), any trajectory which starts inside

$$\{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^4 < 2\}$$

converges to the origin as $t \to \infty$.

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