Climate Models and Interest Rate Risk:

How interest rate risk affects inter-generational equity in climate models

Version 2.6

http://www.christian-fries.de/finmath/intergenerationalequity

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LMU

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* The views expressed in this work are the personal views of the authors
and do not necessarily reflect the views or policies of current or previous employers.

Note

Some parts of the slides will be skipped.

It is sufficient to get a basic understanding of the main idea and these slide are there for further reading.

Christian Fries

- Professor for Computational Insurance and Financial Mathematics (part time).
- Head of Model Development at Group Risk Control of a German Bank (part time).
- Lectures: Numerical Methods, Computational Finance, Applied Mathematical Finance.
- LMU quantLab: Organization of Workshops and Special Events.
- Maintaining a large Software Library of Numerical Methods, Open Source, [1].
- Research Interest (selection):
  - Numerical Methods: Monte-Carlo Method, Algorithmic Differentiation, ...
  - Modelling: Interest Rate Models, Hybrid Models, Valuation Adjustments, ...
  - Implementation: Object-Functional Implementation, Smart Derivative Contracts. High-Performance Computing, ...
- Book: *Mathematical Finance: Theory, Modeling, Implementation* (Wiley), [4].

- Current Research Project related to Climate Models: [5] with Lennart Quante, *Potsdam Institute for Climate Impact Research (PIK)*), (see last part of this presentation).

Outline

Code Sessions and Numerical Experiments

We will (very shortly) discuss an implementation of an integrated assessment model - the DICE model.

This implementation allows us to conduct some numerical experiments. You can checkout the code (open source) and run the experiments yourself or create new experiments.

You can find the code of the numerical experiment in the
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

You can find the code of the model implementation in the
`package net.finmath.climate.models.dice`
at `https://github.com/finmath/finmath-lib` (see `src/main/java`).

Outline

INTRODUCTION

Interest Rates

- Interest rates are the tool to **compare a value or cost at different times.**
- The link to climate change is evident:
  - We are facing damages, and, hence associated costs **in the future**.
  - We need to finance a transformation in the **present time** to avoid these damages.

Question

- Why is 100 € now not the same as 100 € in the future?

Question

- Why is 100 € now not the same as 100 € in the future?

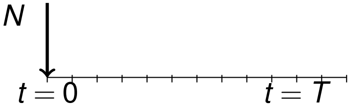
Assume you are a bank, and you lent 100 € each to multiply clients for one year. After one year, you request the 100 € **plus an additional amount.**

This additional amount is...

- **Time Value**: A compensation for the fact that you could have used the amount for something else during that time.
- **Default Premium**: A compensation for the fact that some of your clients failed to pay back (similar to an insurance premium).

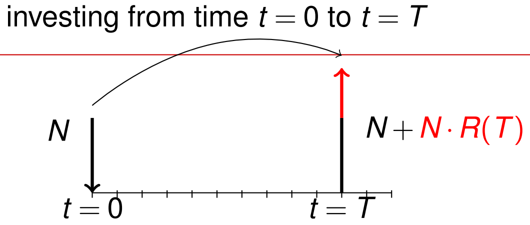
INTEREST RATES

Interest Rates



- If we borrow or lend an amount N today (say at $t = 0$),

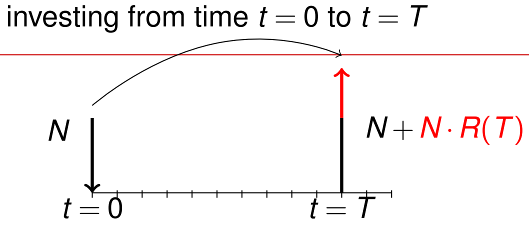
Interest Rates



- If we borrow or lend an amount N today (say at $t = 0$), then you pay or receive back

$$N \cdot (1 + R(T)) \;=\; N + N \cdot R(T) \qquad \text{at in time } T.$$

Interest Rates



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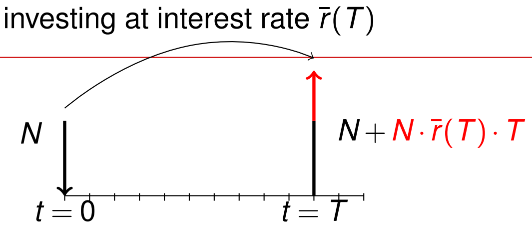
$$N \cdot (1 + R(T)) \; = \; N + N \cdot R(T) \qquad \text{at in time } T.$$

- The additional amount $N \cdot R(T)$ compensates for the time the money has been borrowed or lent. It depends on the time span $t=0$ to $t=T$.

Interest Rates



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$$N \cdot (1 + R(T)) \;=\; N + N \cdot R(T) \qquad \text{at in time } T.$$

- The additional amount $N \cdot R(T)$ compensates for the time the money has been borrowed or lent. It depends on the time span $t = 0$ to $t = T$.
- The term **rate** in *interest rate* refers to the fact that this *additional amount* is expressed in a *per time* basis.
- Expressed on a *per time* basis, we may define the (linear compounded) **interest *rate***

$$\bar{r}(T) \;:=\; \frac{1}{T} R(T)$$

COMPOUNDING

Compounding - Exponential Growth

Assume the interest rate is constant: $\bar{r}(t, T) = \bar{r} = \text{const.}$.

Lets start in $t = 0$. Lending the amount N_0 for 1 year we receive

$$N_0 \cdot (1 + \bar{r}) \qquad\qquad \text{in } t = 1.$$

Compounding - Exponential Growth

Assume the interest rate is constant: $\bar{r}(t, T) = \bar{r} = \text{const.}$.

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$$N_1 \; := N_0 \cdot (1 + \bar{r}) \qquad\qquad \text{in } t = 1.$$

After that we may lend the amount N_1 for another year, such that after two years we receive back

$$N_1 \cdot (1 + \bar{r})$$

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Assume the interest rate is constant: $\bar{r}(t, T) = \bar{r} = \text{const.}$.

Lets start in $t = 0$. Lending the amount N_0 for 1 year we receive

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After that we may lend the amount N_1 for another year, such that after two years we receive back

$$N_2 := N_1 \cdot (1 + \bar{r}) = N_0 \cdot (1 + \bar{r})^2 \qquad \text{in } t = 2$$

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Assume the interest rate is constant: $\bar{r}(t, T) = \bar{r} = \text{const.}$.

Lets start in $t = 0$. Lending the amount N_0 for 1 year we receive

$$N_1 := N_0 \cdot (1 + \bar{r}) \qquad \text{in } t = 1.$$

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After k years we have

$$N_k := N_0 \cdot (1 + \bar{r})^k \qquad \text{in } t = k.$$

The amount grows **exponentially**.

Compounding - Exponential Growth

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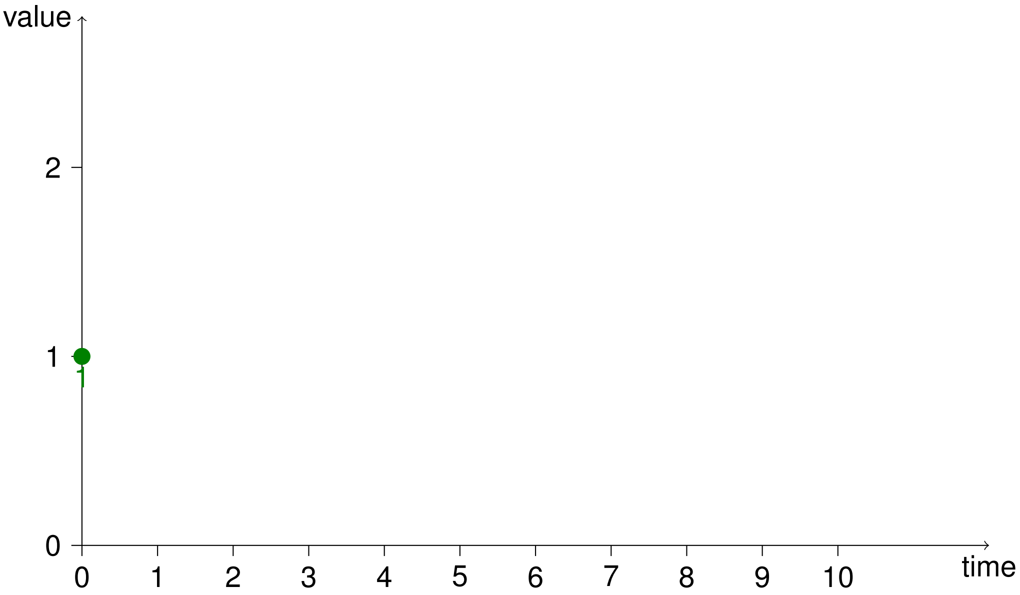
After k years we have

$$N_k := N_0 \cdot (1 + \bar{r})^k \qquad \text{in } t = k.$$

The amount grows **exponentially**. Setting $r = \log(1 + \bar{r})$ we have $N(t) \ = \ N_0 \cdot \exp(r \cdot t)$.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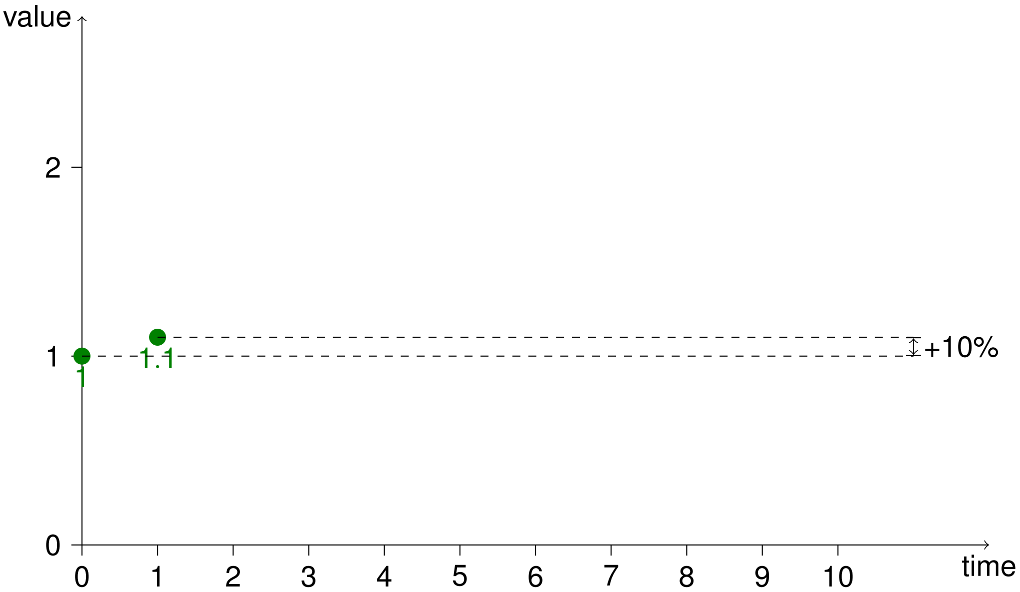
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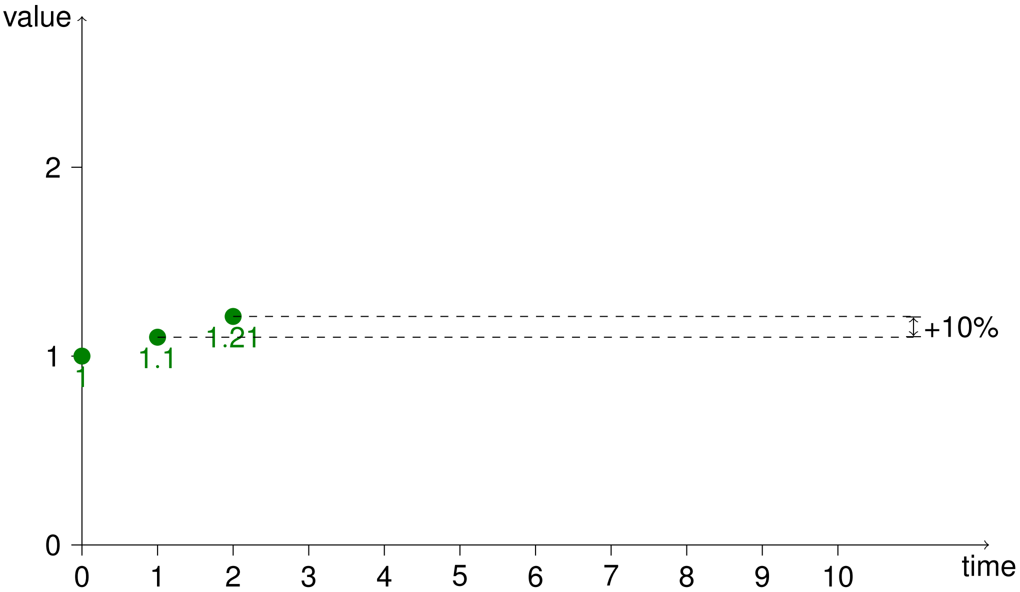
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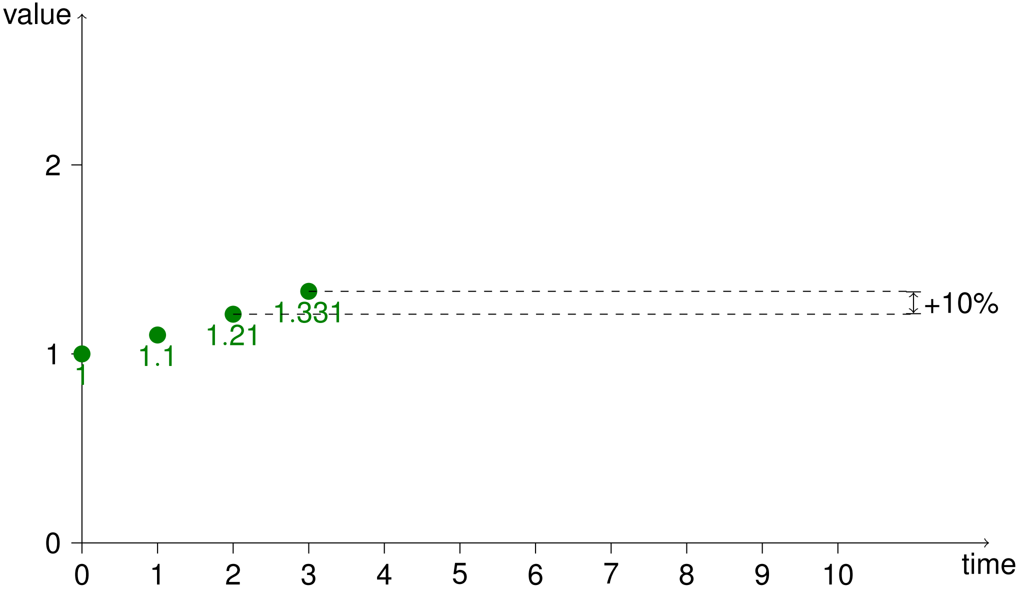
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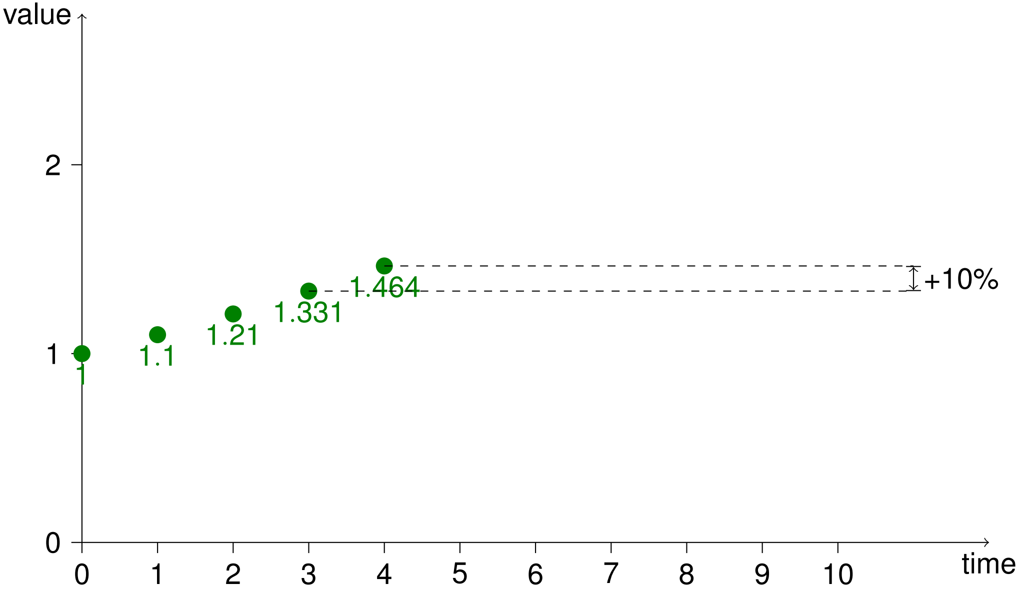
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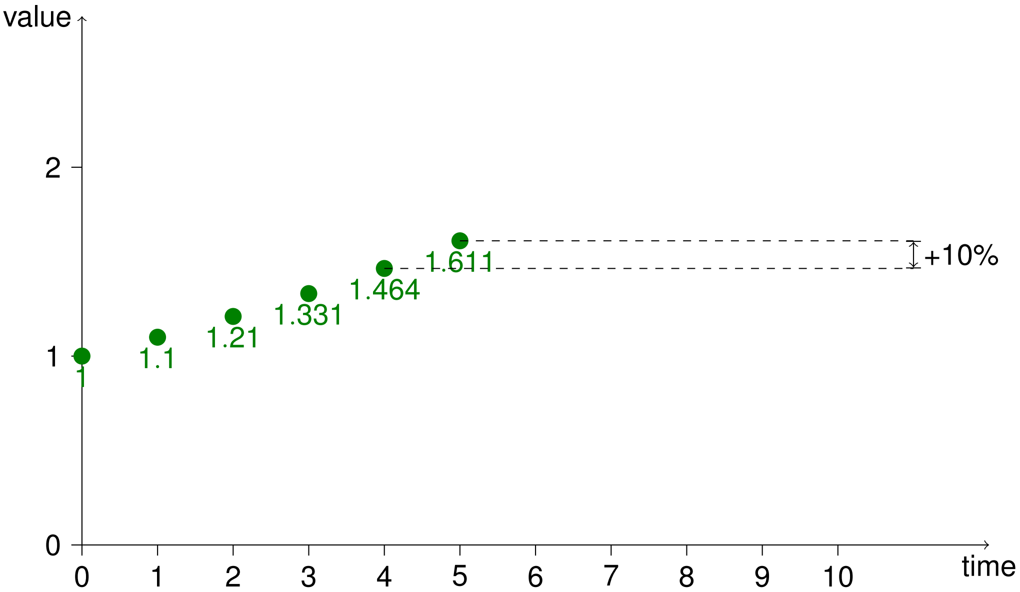
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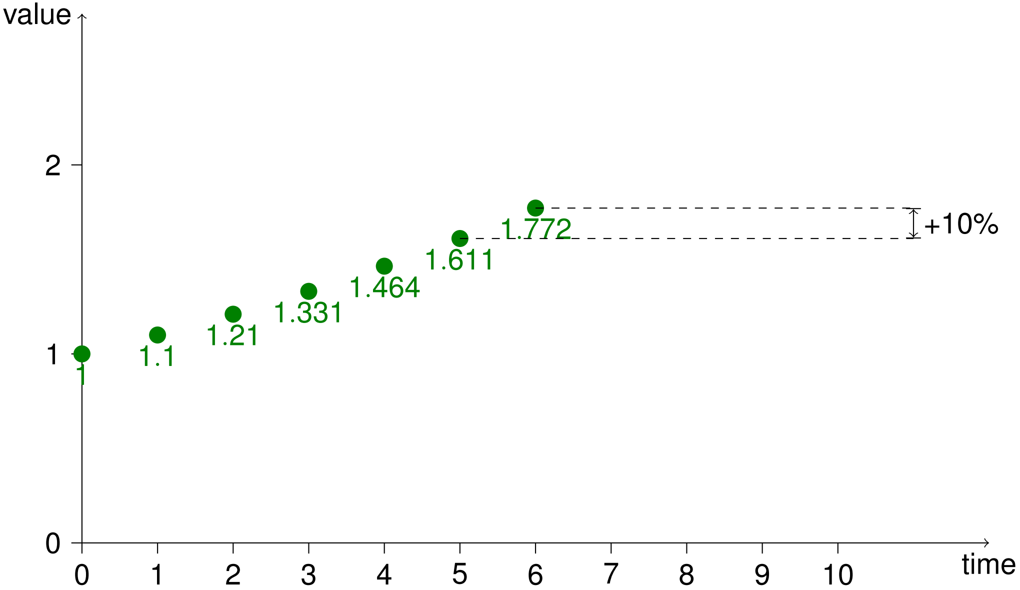
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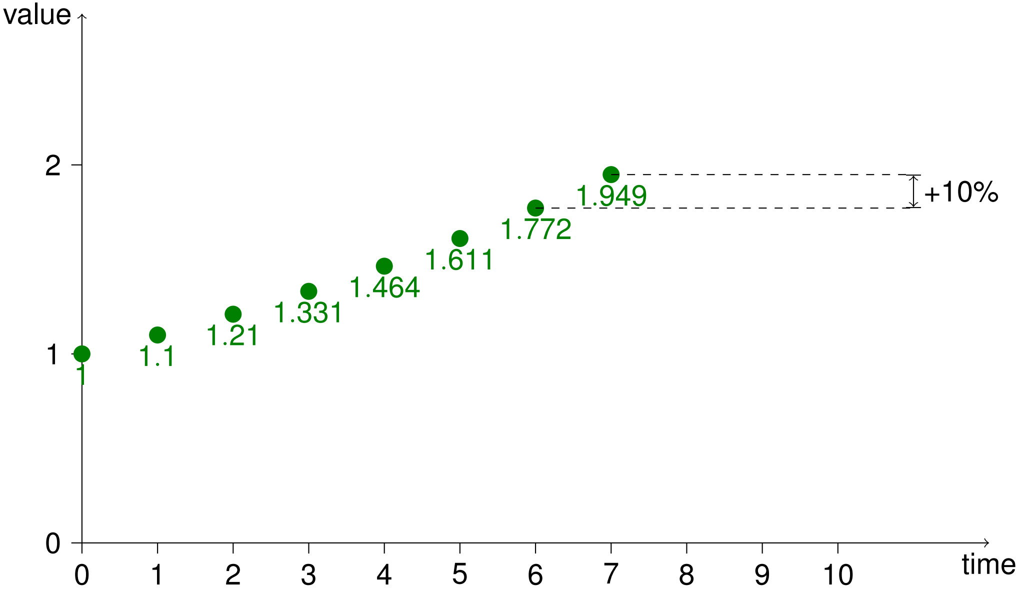
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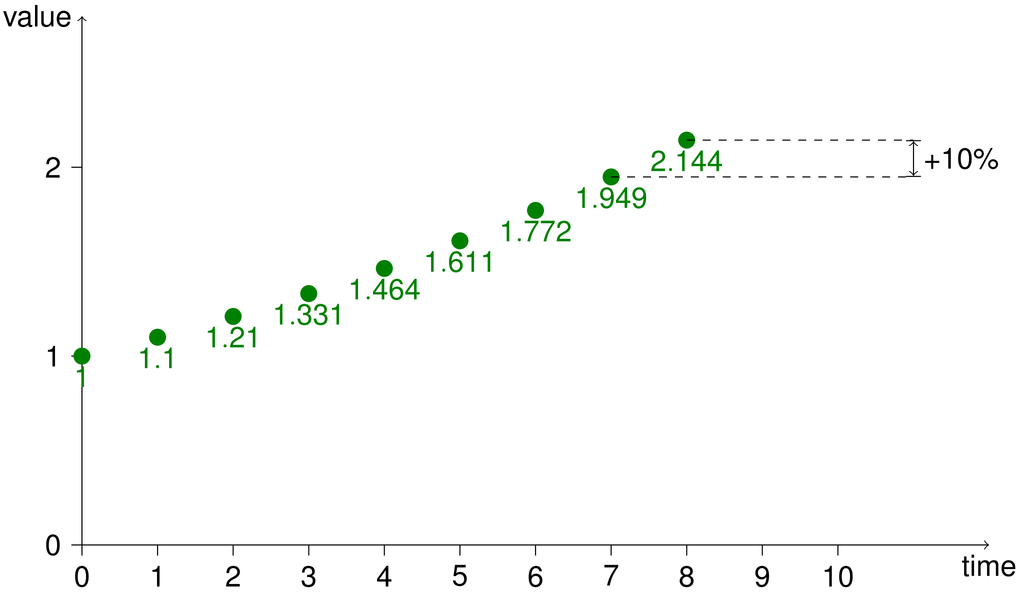
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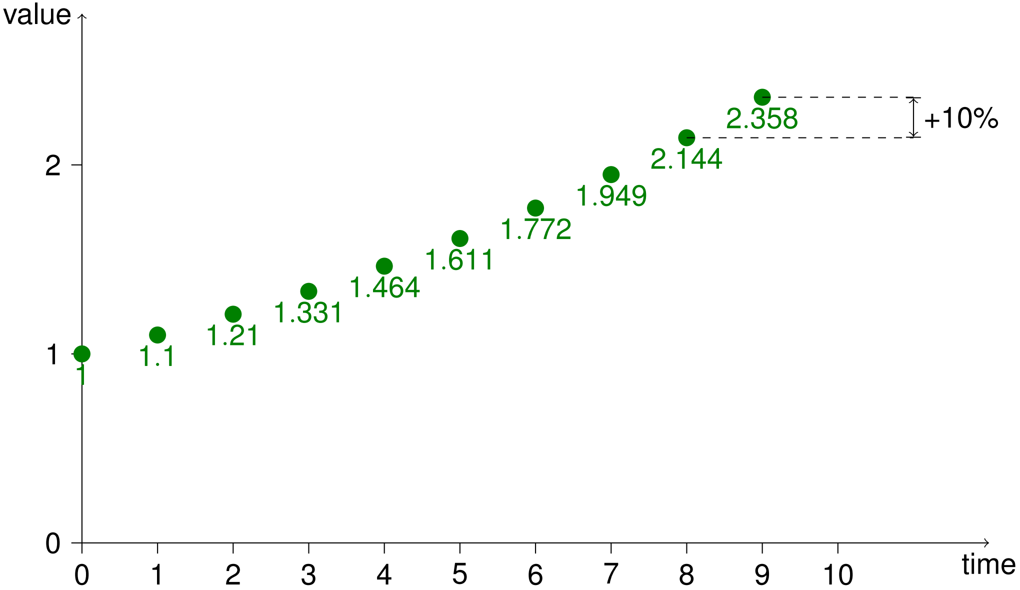
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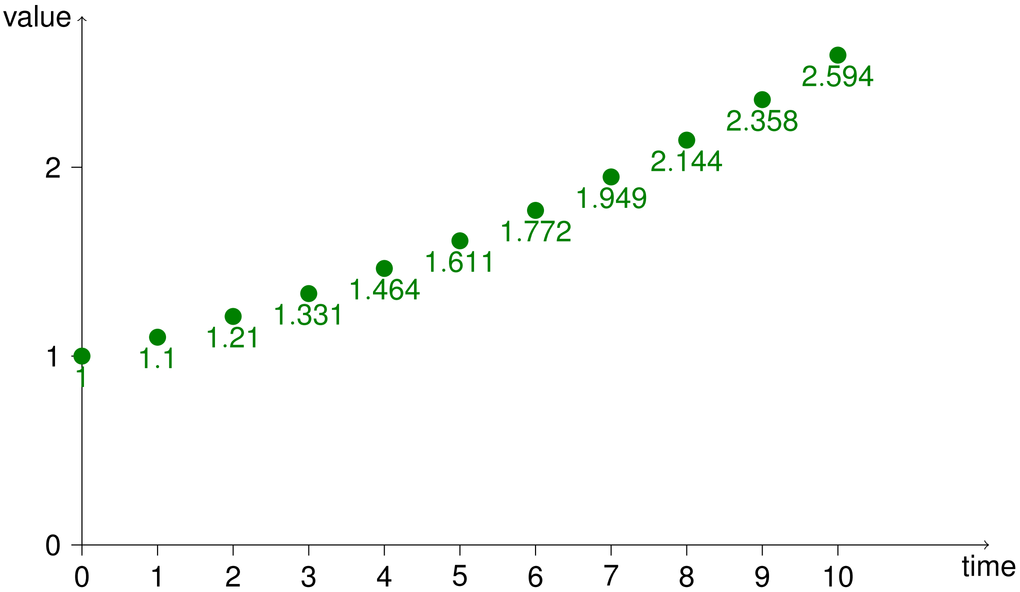
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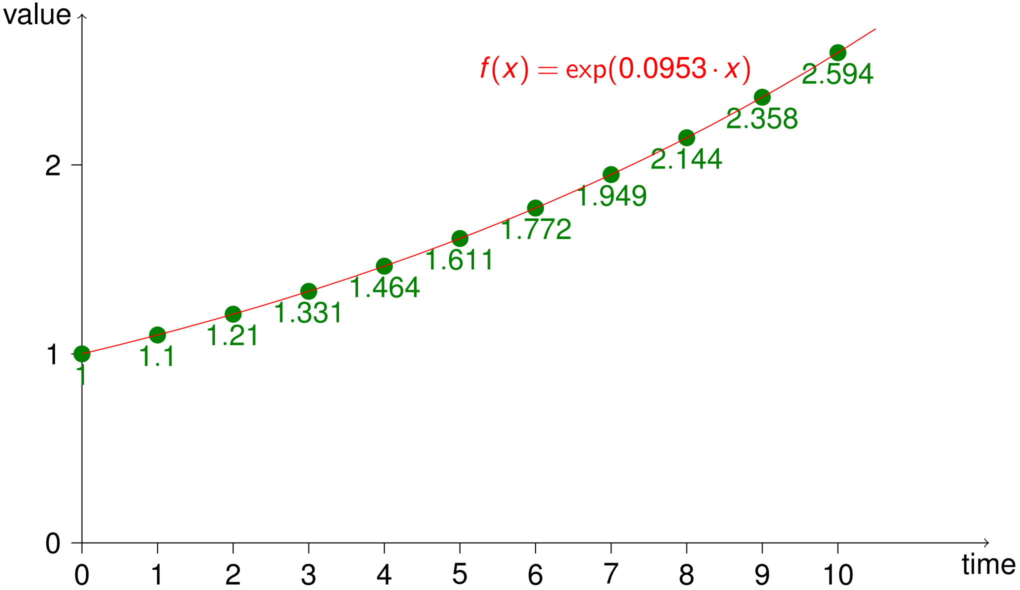
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$f(x) = \exp(0.0953 \cdot x)$
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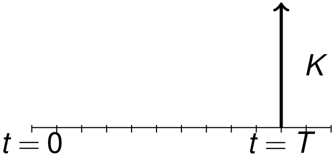
VALUATION (DISCOUNTING)

Valuation

Interest rates tell us how to compare a future value to a today's value.

Why?



- Consider an amount K, being a payment, or liability at the future time T.

Valuation

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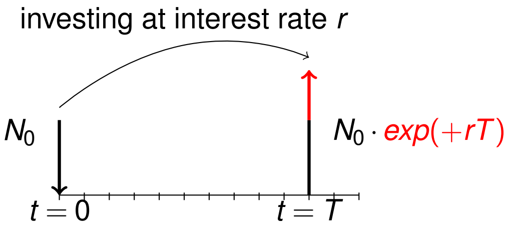


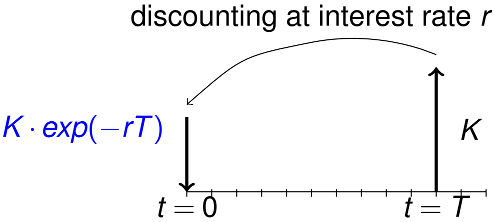
- Consider an amount K, being a payment, or liability at the future time T.
- If we invest N_0 today, assuming interest rate r, we have after T years

$$N(T) \;=\; N_0 \cdot \exp(+r \cdot T)$$

Valuation

Interest rates tell us how to compare a future value to a today's value.

Why?



- Consider an amount K, being a payment, or liability at the future time T.
- If we invest $N_0 = K \cdot \exp(-r \cdot T)$ today, assuming interest rate r, we have after T years

$$N(T) = N_0 \cdot \exp(+r \cdot T) = K \cdot \exp(-r \cdot T) \cdot \exp(+r \cdot T) = K.$$

- Consequently:

$$K \quad \text{in time } t = T \qquad \text{corresponds to} \qquad K \cdot \exp(-r \cdot T) \quad \text{in time } t = 0.$$

RELEVANCE IN CLIMATE MODELS

Relevance in Climate Models

Assume there is a huge damage that is considered to cost K in some future time T. Assume that the market borrows and lends money with a compounding interest rate r.

Then (naively) we would value (i.e., *assess*) this damage today ($t = 0$) as

$$K \cdot \exp(-r \cdot T)$$

The factor $\exp(-rT)$ is called *discount factor*. The valuation is called *discounting*.

We can "repair" the damage K in T (in the future) by investing $K \cdot \exp(-r \cdot T)$ (at the rate r) today.

Relevance in Climate Models

A damage of size K in time T is valued in $t = 0$ as

$$K \cdot \exp(-rT)$$

For a positive interest rate r, the factor $\exp(-rT)$ is smaller than 1.

It makes the amount K ***much smaller***:

- For $T = 100$ and $r = 1\%$ we have $\exp(-rT) = 0.37$.
- For $T = 100$ and $r = 2\%$ we have $\exp(-rT) = 0.14$.
- For $T = 100$ and $r = 4\%$ we have $\exp(-rT) = 0.02$.

That means, to finance a project requiring 100 million in 100 years, we need to put aside only 14 million now into our bank account (at 2%) or only 2 million (at 4%).

It appears as if we can "repair" large damages in the future by very small investments now.

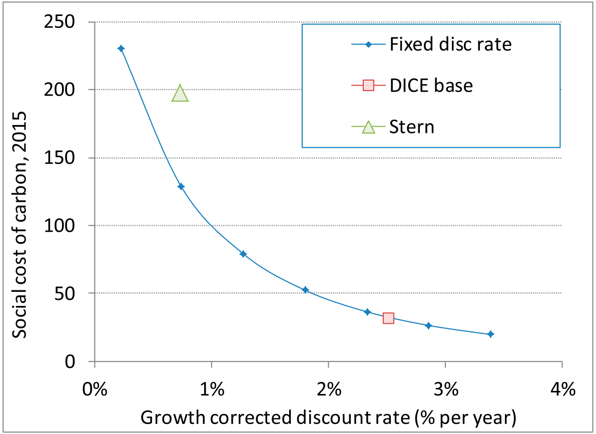


Fig. 3. Social cost of carbon and growth-corrected discount rate in DICE model. The growth-corrected discount rate equals the discount rate on goods minus the growth rate of consumption. The solid line shows the central role of the growth-corrected discount rate on goods in determining the SCC in the DICE model. The square is the SCC from the full DICE model, and the triangle uses the assumptions of *The Stern Review* (10). A further discussion and derivation of the growth-corrected discount rate is given in *Supporting Information*.

Figure taken from [6]:
Dependency of the *social cost of carbon* (kind of a "CO_2" -price) on the discount rate r.

Outline

Integrated Assessment Models

are a class of models that **combine**

- the evolution of climate related (geo-physical) quantities (like emission and temperature levels) with
- economic factors (production, consumption).

Using the model one likes to understand how to find an optimal path, balancing between reducing emission (abatement) and reducing (future) damages.

Integrated Assessment Models: DICE Model

- Nordhaus proposed the DICE model - **Dynamic Integrated Climate-Economy model** around 1992. A revision was published recently, [6].
- The model is simple, but a good example for the class of *integrated assessment models*.
- The model connects several geo-physical and economical quantities: temperature, carbon concentration, emission, population, productivity, GDP, consumption, capital, abatement costs.
- The quantities are connected by simple functional forms: polynomials or power laws, where the parameters are calibrated to match observations or expectations.
- Given the DICE model, one tries to find an optimal saving rate and abatement rate, to maximize a utility function defined on the consumption. This will create an "optimal emission path".

- We give a short introduction to the model (see also [3]).

SKETCH OF THE DICE MODEL

Time

- We measure time in years. Quantities that are *per time* (e.g., emission or consumption) are annualized (i.e., per one year).
- Example: The total amount of CO_2 added over a time step Δt_i from t_i to t_{i+1} is the integral
$$\int_{t_i}^{t_i+\Delta t_i} E(\tau)\mathrm{d}\tau$$
of the emissions $E(t)$ (per time, i.e., annualized emissions).
This will be approximated by $E(t_i)\Delta t_i$ (sometimes called *Euler step*).
- We use a time discretization $0 = t_0 < t_1 < \ldots < t_n$.
- The original model uses a fixed time step of $\Delta t_i = 5\ \mathrm{years}$ and sometimes it may not be clear if a quantity is an annualized value (such that the total amount over the time step has to be multiplied with Δt_i) or is an total amount over the full time-step.

Damage

- The model assumes a **damage function** $\Omega(t) \in [0, 1]$ describing the fraction of the *gross domestic product* being “damaged” by a rise in the atmospheric temperature

$$\Omega(t) = \frac{D(t)}{1 + D(t)} \quad \text{with} \quad D(t) = \phi_1 T_{\mathrm{AT}}(t) + \phi_2 T_{\mathrm{AT}}(t)^2.$$

  The $\Omega(t)$ is between 0 and 1 and models the fraction of the GDP that has to be used to compensate for damages.
- The damage function is a function of the current **atmospheric temperature** T_{AT} above pre-industrial level.
- The coefficients of the original model are $\phi_1 = 0$ and $\phi_2 = 0.00236 \, \frac{1}{K^2}$.

Simplified

Higher **temperature** leads to more **damage** to the GDP.

Temperature

- The temperature is modelled by a vector in $\mathbb{R}^2$

$$T(t) = \begin{pmatrix} T_{\mathrm{AT}}(t) \\ T_{\mathrm{LO}}(t), \end{pmatrix}$$

  where T_{AT} denotes the temperature of the atmosphere and T_{LO} the temperature of land and ocean, both measured in Kelvin (K) above pre-industrial level.
- The unit of the temperature is K (Kelvin). $[T_{\mathrm{AT}}(t)] = \mathrm{K}$, $[T_{\mathrm{LO}}(t)] = \mathrm{K}$.
- The initial value of the temperature is $T_{\mathrm{AT}}(0) = 0.85\ \mathrm{K}$, $T_{\mathrm{LO}}(0) = 0.0068\ \mathrm{K}$.

Temperature Evolution

- The evolution of the temperature vector $T = (T_{\mathrm{AT}}, T_{\mathrm{LO}})^{\top}$ is a function of the so called temperature forcing F. The temperature vector T evolves as

$$\mathrm{d}T(t) \ = \ \left(\Gamma_T \cdot T(t) + (\xi_1 F(t), 0)^{\top}\right) \mathrm{d}t.$$

- The matrix Γ_T models the transport of heat among the different regimes.
- The parameter ξ_1 translates the forcing ($[F] = W/m^2$) to the temperature change per year in K/year.

Simplified

Higher **temperature forcing** leads to higher **temperature**.

Temperature Evolution

- The classical model is recovered when considering an Euler step

$$\begin{aligned} T(t_{i+1}) &= T(t_i) + \left(\Gamma_T \cdot T(t_i) + (\xi_1 F(t_i), 0)^\top\right) \Delta t_i \\ &= (1 + \Gamma_T \Delta t_i) \cdot T(t_i) + (\xi_1 F(t_i), 0)^\top \Delta t_i. \end{aligned}$$

  The matrix $\Phi_T := (1 + \Gamma_T \cdot 5\ \mathrm{year})$ would then correspond to the 5-Y transition matrix for the temperature vector of the original model.
- The original model calibrates the constant ξ_1 to $\xi_1 \Delta t_i = 0.1005 \frac{K/\mathrm{year}}{W/m^2}$ for $\Delta t_i = 5\ \mathrm{year}$.
- Using other time-steps sizes one should re-calculate the time-discrete transition matrix (as a matrix-exponential).

Forcing (temperature change)

- The temperature forcing function $F(t)$ is the (annualized instantaneous) temperature forcing per time ($[F] = \mathrm{K/year}$).
- The forcing is a function of the **atmospheric carbon concentration** M_{AT} and an external forcing F_{EX}. It is

$$F(M;t) \; = \; F(t) \; = \; \eta \; \log_2 \left(\frac{M_{\mathrm{AT}}(t)}{M_{\mathrm{AT},0}} \right) + F_{\mathrm{EX}}(t),$$

  where M is the carbon in atmosphere ($[M] = \mathrm{GtC}$), $M_{\mathrm{AT},0} = 500\ \mathrm{GtC}$, $\eta = 3.6813 \cdot W/m^2$ and $F_{\mathrm{EX}}(t) = 1 \cdot W/m^2$.
- See `https://en.wikipedia.org/wiki/Radiative_forcing`.

Simplified

Higher **carbon concentration** leads to higher **temperature forcing**.

Forcing (temperature change)

- The parameter $M_{\mathrm{AT},0} = 500\ \mathrm{GtC}$ is the **atmospheric carbon concentration** of the year 1750 (i.e., the pre-industrial level).
- Note: 1 $\mathrm{ppm\ CO_2} \mathrel{\hat{=}} 2.12\ \mathrm{GtC}\ =\ 7.76\ \mathrm{GtCO_2}$, i.e. 500 $\mathrm{GtC} \mathrel{\hat{=}} 236\ \mathrm{ppm\ CO_2}$.

Carbon Concentration

- Carbon concentration is given by a vector in $\mathbb{R}^3$

$$M(t) = \begin{pmatrix} M_{\mathrm{AT}}(t) \\ M_{\mathrm{UO}}(t) \\ M_{\mathrm{LO}}(t) \end{pmatrix},$$

  where M_{AT} is the concentration in the atmosphere, M_{UO} is the concentration of the upper ocean, M_{LO} is the concentration of the lower ocean.
- The unit of an element of the vector is $[M_.] = \mathrm{GtC}$ (Gigatons of Carbon).

Carbon Concentration Evolution

- The evolution *carbon concentration* vector $M = (M_{\mathrm{AT}}, M_{\mathrm{UP}}, M_{\mathrm{LO}})^\top$ is a function of the **emissions** $E(t)$.
- The carbon concentration vector M evolves as

$$\mathrm{d}M(t) \ = \ \left(\Gamma_M \cdot M(t) + \frac{3}{11} \frac{\mathrm{C}}{\mathrm{CO_2}} \cdot (E(t), 0, 0)^\top \right) \mathrm{d}t.$$

  Here Γ_M is the infinitesimal generator of the transition matrix and E is the emission per year.
- The unit of the emissions is $[E] = \mathrm{GtCO_2}/\mathrm{year}$. Hence, the conversion factor $\frac{3}{11} \frac{\mathrm{C}}{\mathrm{CO_2}}$.

Simplified

Emissions increase **carbon concentration**.

Carbon Concentration Evolution

- The classical model is recovered when considering an Euler step

$$M(t_{i+1}) \;=\; M(t_i) + \left(\Gamma_M M(t_i) + \frac{3}{11} \frac{\mathrm{C}}{\mathrm{CO_2}} \cdot E(t) \right) \Delta t_i .$$

  Considering a 5Y-Euler-step will recover the classical model with the 5Y transition matrix being $\Phi_M := (1 + \Gamma_M \Delta t_i)$.
- There is a *carbon cycle* that models exchange of these quantities between atmosphere, land and ocean.
- See `https://en.wikipedia.org/wiki/Carbon_cycle`.

Emission

- The emission function $E(t)$ is the (annualized instantaneous) emissions per time ($[E] = \mathrm{GtCO_2/year}$).
- Emissions are split into two components: industrial emissions (proportional to the GDP) E_{GDP} and external emissions E_{EX} (e.g. land use),

$$E(t) = E_{\mathrm{GDP}}(t) + E_{\mathrm{EX}}(t).$$

Emission (Industrial)

- The industrial emission is given as a linear function of the GDP

$$E_{\mathrm{GDP}}(t) \;=\; \sigma(t) \cdot (1 - \mu(t)) \cdot GDP(t).$$

  Here $GDP(t)$ is the (annualized instantaneous) GDP per time (per year) ($[GDP] = 10^{12}\mathrm{USD}/\mathrm{year}$, $[E_{\mathrm{EX}}(t)] = \mathrm{GtCO_2}/\mathrm{year}$.
- In the original model the letter Y is used for the GDP ($Y(t) = GDP(t)$).
- The function μ models our efforts to move to a more sustainable economy. The **abatement**. It describes the fraction of the industrial *GDP* that has been made emission free. Increasing the parameter μ will create costs (see below).
- The function σ models the emission intensity, $[\sigma(t)] = \mathrm{GtCO_2}/10^{12}\mathrm{USD}$, see below.

Simplified

A higher **GDP** comes along with higher **emissions**, reduced by **abatement** factor μ.

Emission (External)

- The external emissions $E_{\mathrm{EX}}(t)$ is an (annualized instantaneous) emissions per time ($[E_{\mathrm{EX}}] = \mathrm{GtCO_2/year}$).

Emission (Total)

- The total emissions is just the sum of $\mathrm{emissionIndustrial}$ and $\mathrm{emissionExternal}$

$$E(t) \;=\; E_{\mathrm{GDP}}(t) + E_{\mathrm{EX}}(t)$$

Emission Intensity

- The emission intensity function $\sigma(t)$, i.e. `emissionPerEconomicOutput`, will follow an exponential decay (reflecting improvement in energy efficiency), with

$$\begin{aligned}\sigma(t) &= \sigma_0 \cdot \exp(-\delta_\sigma(t) \cdot t)\\ \delta_\sigma(t) &= \delta_{\sigma,0} \cdot \exp(-d \cdot t),\end{aligned}$$

where

$$\begin{aligned}\sigma_0 &= \texttt{emissionIntensityInitial} &&= (38.85\ \mathrm{GtCO_2})/(105.5 \cdot 10^{12}\mathrm{USD}),\\ \delta_{\sigma,0} &= \texttt{emissionIntensityRateInitial} &&= 0.0152/\mathrm{year},\\ d &= \texttt{emissionIntensityRateDecay} &&= -\log(1-0.001)/\mathrm{year}.\end{aligned}$$

These are the values used in the original model, re-formulated as exponential rates.[1]

[1] In the original model the `emissionIntensityRate` is expressed as $(1-0.001)^{i \cdot 5}$ for a five year time stepping.

Emission Intensity

- For the units we have $[\sigma(t)] = \mathrm{GtCO}_2/10^{12}\mathrm{USD}$, and all other parameters being rates with unit $1/\mathrm{year}$.
- The discrete time stepping in the original model is

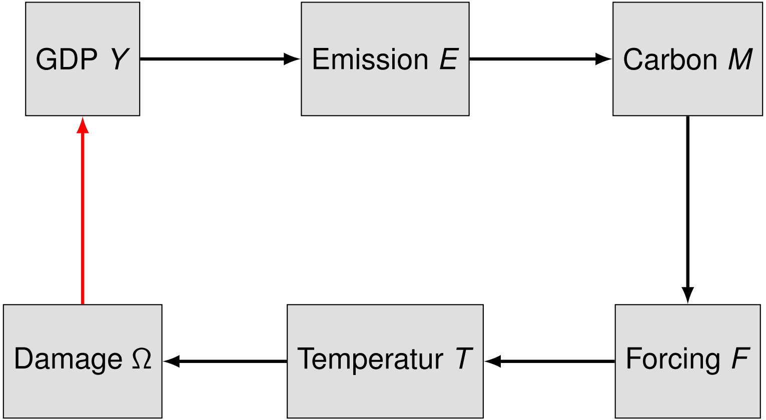
$$\sigma(t_{i+1}) \; = \; \sigma(t_i) \cdot \exp\left(-\delta_\sigma(t_i) \cdot (t_{i+1} - t_i)\right)$$

where the rate of the exponential decay decays

$$\delta_\sigma(t_{i+1}) \; = \; \delta_\sigma(t_i) \cdot (1 - d)^{t_{i+1} - t_i}.$$

We re-calculate the parameter d such that our time-continuous function agrees with the original time-discrete function for the original time step.

At this point, we see that emissions - as a function of the GDP - induce damage to the GDP.



Abatement Cost

Abatement Costs / Backstop Price

- The parameter $\mu(t)$ defines the percentage of emissions we are reducing / abating.
- This does not come for free.
- The *abatement cost* function is a *cost per (abated) emission* and is given by

$$\Lambda(t) \; = \; B(t) \cdot \frac{1}{\theta_2} \mu(t)^{\theta_2},$$

  where $B(t)$ is the *backstop price* and is modelled as

$$B(t) \; = \; B(0) \cdot \exp(-b \cdot t),$$

  where $B(0) = 550.0/1000.0 \cdot 10^{12}\ \mathrm{USD}/\mathrm{GtCO}_2$ and $b = -\log(1-0.025)/5$ and $\theta_2 = 2.6$.
- The unit of Λ is $[\Lambda] = 10^{12}\mathrm{USD}/\mathrm{GtCO}_2 = 1000\mathrm{USD}/\mathrm{tCO}_2$.

Simplified

The larger the **abatement** $\mu(t)$, the larger the **abatement cost** factor $\Lambda(t)$.

- The function $\mu \mapsto \mu^{2.6}/2.6$ is zero for $\mu = 0$ and small for $\mu = 0.03$ (the initial value). For $\mu = 1$ the slope of that function is 1, such that the $B(t)$ can be interpreted as the limit price to achieve full (100%) abatement and that 100% of abatement has an average price of $550/2.6\ \mathrm{USD}/\mathrm{tCO_2}$.
- In other words: $\Lambda|_{\mu=0} = 0$ and $\frac{\mathrm{d}}{\mathrm{d}\mu}\Lambda|_{\mu=0} = 1 = B(t)$
- The abatement cost apply to the industrial emissions only. The abatement costs thus are

$$C_{\mathrm{A}} \;:=\; \Lambda(t) \cdot E_{\mathrm{GDP}}(t).$$

- The unit of C_{A} is 10^{12} *USD* (trillion USD).
- The choice of b will recover the value of the original model, which is

$$\exp(-b \cdot t) \;=\; (1 - 0.025)^{t/5}.$$

- In the original model, Λ includes σ and it is then multiplied with $(1-\mu)GDP$ and not with $E_{\mathrm{GDP}}(t)$.
- Note that these cost apply every year - like the GDP applied for a year.

ECONOMICS

Net GDP (Remaining GDP) and Saving Rate

- A fraction $\Omega(t)$ of the remaining GDP is needed to repair damages (damage function).
- $\Lambda(t)$ is removed from the GDP to pay for abatement (reduction of the emissions).
- The remainder

$$GDP_{\mathrm{net}} := \underbrace{Y(t)}_{\text{GDP}} - \underbrace{\Omega(t) Y(t)}_{\text{Damage Cost } C_{\mathrm{D}}} - \underbrace{\Lambda(t) \cdot E_{\mathrm{GDP}}(t)}_{\text{Abatement Cost } C_{\mathrm{A}}} ,$$

  may be used for investment $I(t)$ or consumption $C(t)$, where the distribution among the two is defined by the savings rate $s(t)$,

$$\begin{aligned} I(t) &:= s(t)\ GDP_{\mathrm{net}}, \\ C(t) &:= (1 - s(t))\ GDP_{\mathrm{net}}. \end{aligned}$$

The savings rate function s is (besides the abatement μ) the second free parameter that is used in the (policy) optimization.

Capital

The investment I increases the capital K as

$$K(t+1) \;=\; (1-\delta_k)K(t)+I(t)$$

and the capital is part of the functions that establishes the GDP

$$Y(t) \;=\; A(t)K(t)^{\gamma}L(t)^{1-\gamma},$$

Population and Productivity

The GDP is modelled as a function of the population L and the productivity A. Both follow simple parametrised functions (see code).

Simplified

Investment is added to the **capital** and capital generates the **GDP** for the next time step.

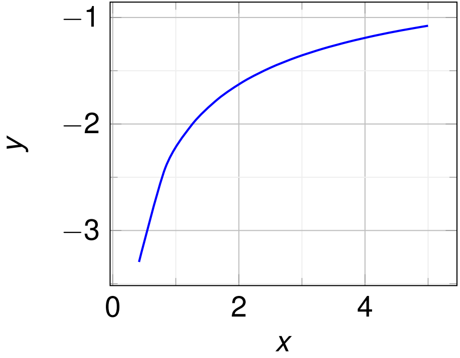
Utility / Welfare

The social welfare
is a function of the utility U depending
on the consumption $c = C/L$ and the population L

$$V(t) \; = \; U(c(t), L(t)) \; = \; L(t) c(t)^{1-\alpha} \frac{1}{1-\alpha}$$

- The original
  model used the utility function with $\alpha = 1.45$.

$y = \frac{x^{1-\alpha}}{1-\alpha}$ with $\alpha = 1.45$



Simplified

Consumption defines the **utility** / the **welfare.**

The social welfare is then valued by summing (integrating) the *discounted V using a discount factor* into a net present value.

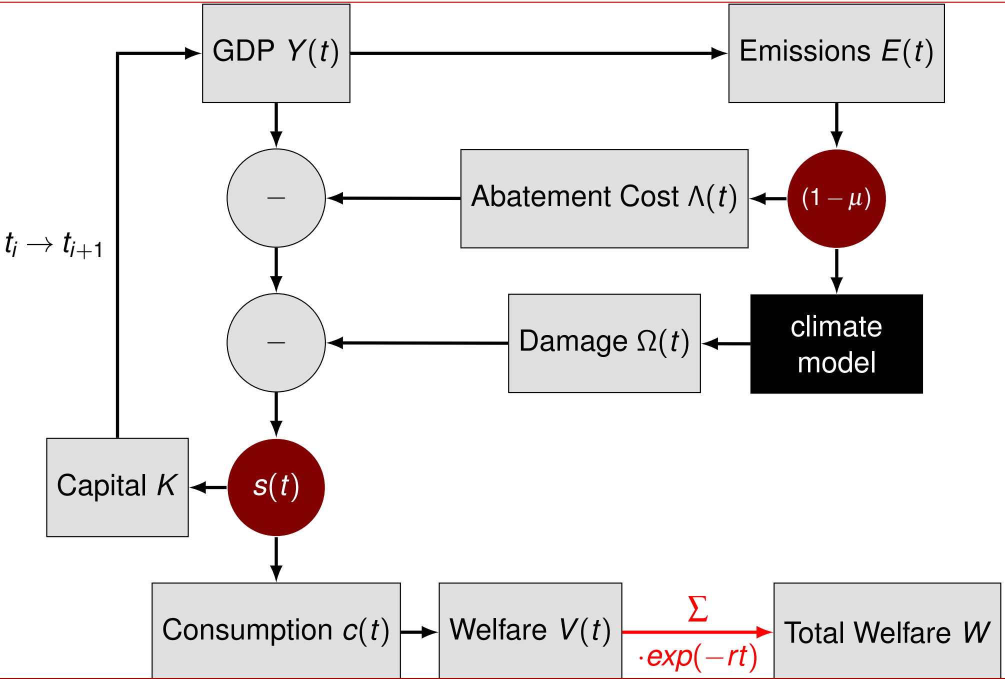
$$W = \sum_{t=1}^{T_{\max}} V(t) \cdot \exp(-rt)$$

The model then tries to find the optimal **emission path** to **maximize the total welfare**. In this maximization we need balance abatement (and abatement cost) and damage.

Controls (Free Parameters):

- $\mu(t)$ will reduce emissions, but increase abatement costs.
- $s(t)$ balances consumption and investment.

We like to find the optimal path $t \mapsto \mu(t)$ (and optimal $s(t)$) that maximizes the *total welfare* .


GDP $Y(t)$
Emissions $E(t)$
$-$
Abatement Cost $\Lambda(t)$
$(1-\mu)$
$t_i \to t_{i+1}$
$-$
Damage $\Omega(t)$
climate model
Capital K
$s(t)$
Consumption $c(t)$
Welfare $V(t)$
Σ
$\cdot exp(-rt)$
Total Welfare W

Calibration

Calibration

- Calibration tries to maximize the total welfare by changing the abatement function $t \mapsto \mu(t)$ and the saving rate function $t \mapsto s(t)$.
- The optimal values of $t \mapsto \mu(t)$ and $t \mapsto s(t)$ may depend on other quantities, e.g., the interest rate r.

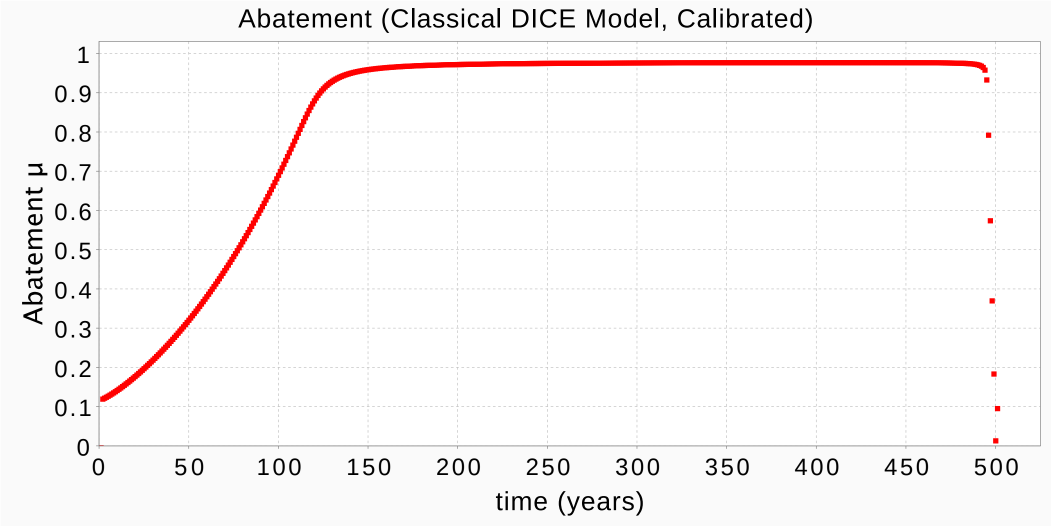


Figure 1: The calibrated (optimal) abatement function $t \mapsto \mu(t)$ for the calibrated classical DICE model.

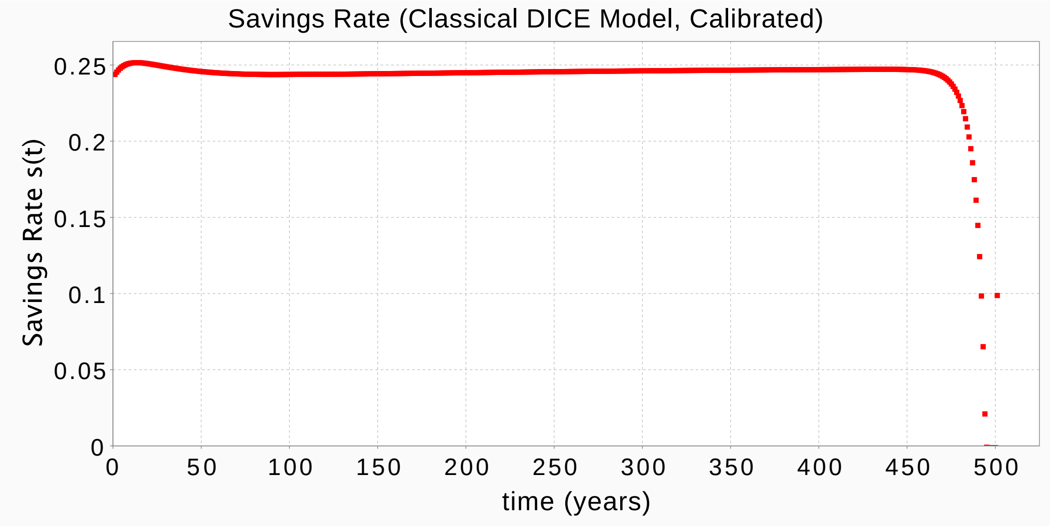


Figure 2: The calibrated (optimal) saving rate function $t \mapsto s(t)$ for the calibrated classical DICE model.

Calibration

Question: Why is the calibration generating this drop to 0 for the optimal abatement function and savings rate function close to $t = 500$?

Calibration

Question: Why is the calibration generating this drop to 0 for the optimal abatement function and savings rate function close to $t = 500$?

Answer: The model has a finite time horizon. This means utility and welfare beyond $t = 500$ are not considered. In this model, it is optimal to stop all abatement and savings, and consume everything before the world ends.

Note: You can check that prolonging the model does not alter the result significantly except for a few years at the end of the model (but one has to check it).

Outline

Interest Rates in the DICE Model

- The DICE model assumes a single discount rate r: does not depend on time, is deterministic.
- The discount factor has a strong impact on the results.
- A smaller discount factor means that damages in a distant future are considered less harmful. It is cheaper to pay abatement in the future than today.

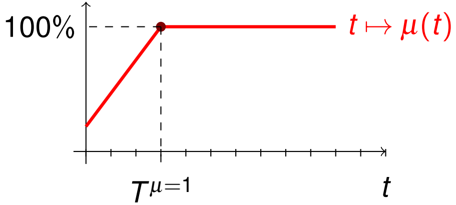
Dependency of Calibrated $T^{\mu=1}$ on the Interest Rate Level r

Consider the simple 1-parametric abatement model:

$$\mu(t) \; = \; \min\left(\mu(0) + \frac{1-\mu(0)}{T^{\mu=1}} \, t, 1.0\right) \, ,$$

where the parameter $T^{\mu=1}$ represents the time for reaching 100% abatement.

Interpretation: The time $T^{\mu=1}$ is the time when the economy becomes 100% carbon neutral.
There is only this parameter.
The abatement policy is pice-wise linear.



The following figure shows how a *calibrated* model, i.e., a calibrated $T^{\mu=1}$ depends on the interest rate level:

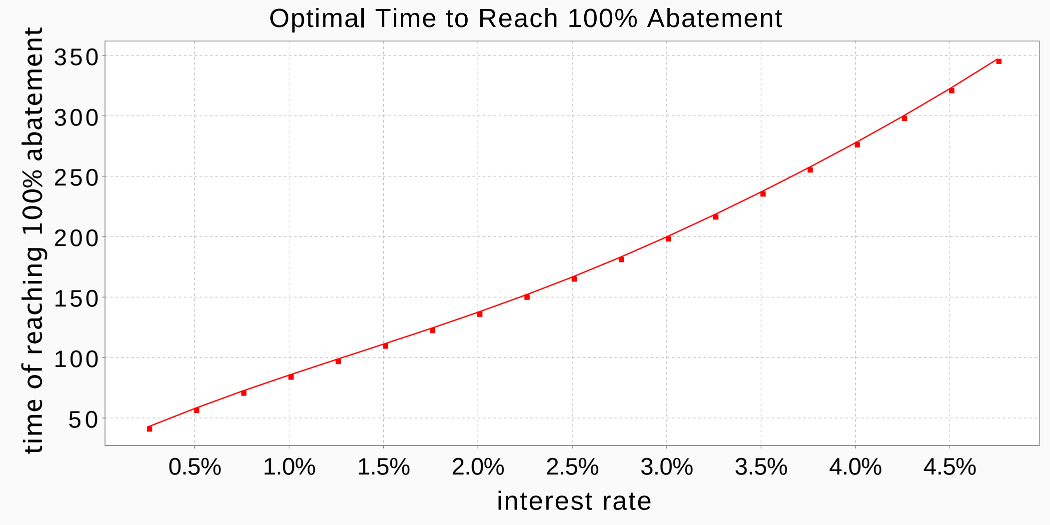


Figure 3: The dependency of $T^{\mu=1}$ on a the interest rate level (for calibrated constant savings rate s_0.

Outline

Implementation / Code Session

- We review a Java implementation of the DICE model and its calibration.
- We will go quickly through all the components that make up our DICE model. These are given in a modular way by *interfaces* and *classes* (e.g., for the damage function of carbon cycle). The corresponding *objects* are then clued together an form the model.
- For those unfamiliar with programming we shortly explain the concepts.
- Based on our code we will conduct several experiments. We try to enable you to run such experiments on your own machine.

- Note: It is not required to have an in-depth understanding of our code to run experiments with the model.

Concepts from Object Oriented Implementation

Concepts from Object Oriented Implementation (Java) (Short)

The main components are

- `interface` - a collection of functions (sometimes called *methods*), that describe **what can be done** with an *object* of a *class*.
  Example: Here: A damage function provides the damage for a given temperature.
- `class` - the implementation of an interface, i.e., the specification **how it is done** and what kind of **data** is associated with the corresponding *object*.
- **object** - an instance of a *class* with concrete **data** (if any), that can be manipulated by the algorithm specified in the `class`. It is created (instanciated) with the keyword `new`.

In addition there are other helpful constructs , e.g.,

- `package` - a name (prefix) for the `class` to organize them. This is similar to a folder (directory) that organizes files.

A JAVA IMPLEMENTATION OF THE DICE MODEL

A Java implementation of the DICE Model (for Research Purposes)

The following code can be found under the
`package net.finmath.climate.models.dice`
in the repository `https://github.com/finmath/finmath-lib`

- The `package net.finmath.climate.models.dice` contains...
  - the `class DICEModel`.
- The `package net.finmath.climate.models.dice.submodels` contains...
  - the `class`es
    - `AbatementCostFunction`,
    - `DamageFromTemperature`,
    - `EvolutionOfCarbonConcentration`,
    - `EvolutionOfTemperature`,
    - etc.

Time

`interface` `TimeDiscretization` in `package` `net.finmath.time`

`class` `TimeDiscretizationFromArray`

Random Variable

interface RandomVariable in **package** net.finmath.time

- The standard DICE model is not stochastic. All quantities are deterministic. There is no uncertainty.
- However, our implementation allows that quantities may be come stochastic.
- This ability is achieved by using the interface `RandomVariable`.
- For the standard DICE model the `RandomVariable` is *implemented* by floating point number e.g., `Double`.
- We haven an extended version of the DICE model with stochastic interest rate model (and a stochastic abatement model).

Damage

```
interface DoubleUnaryOperator

class DamageFromTemperature
```

Temperature

```
interface Temperature
```

```
class Temperature2DScalar
```

Temperature Evolution

```
class EvolutionOfTemperature
```

Forcing (temperature forcing as a function of carbon concentration)

```
interface BiFunction

class ForcingFunction


interface Function

class ForcingExternalFunction
```

Carbon Concentration

```
interface CarbonConcentration

class CarbonConcentration3DScalar
```

Carbon Evolution

```
interface TriFunction

class EvolutionOfCarbonConcentration
```

Emission

```
interface BiFunction
```

```
class EmissionIndustrialIntensityFunction
```

Abatement Cost

```
interface BiFunction
```

```
class AbatementCostFunction
```

Evolution of Capital

```
interface Function

interface BiFunction

class EvolutionOfCapital
```

Other Elements

- Some other elements are just given by simple floating point numbers, e.g., the current GDP.
- In the simple DICE model the interest rate is a constant r.
- We will later add a stochastic interest rate model

Abatement Model and Savings Rate Model - the free parameters

```
interface AbatementModel

interface SavingsRateModel
```

DICE Model and Calibration

```
interface ClimateModel
```

```
class DICEModel
```

The class DICE Model

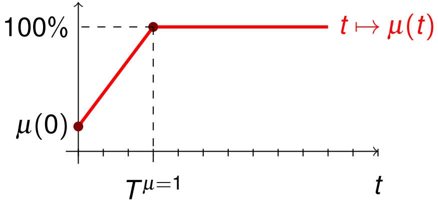
- consumes an abatement model (fixed abatement function) and savings rate model (fixed savings rate function), ...
- ... glues together all the other model components, ...
- ... and calculates the social welfare.

NUMERICAL EXPERIMENT: ONE PARAMETRIC ABATEMENT MODEL

Numerical Experiment: One Parametric Abatement Model

- Instantiate an object of `class DICEModel`.
  - Use a constant savings rate $t \mapsto 0.26$.
  - Use a simple one-parameter abatement model, with the single parameter $T^{\mu=1}$: linear from 0.03 in $t = 0$ to 1.0 in $t = T^{\mu=1}$, then constant 1.0.

$$\mu(t) = \min\left(\mu(0) + \frac{1-\mu(0)}{T^{\mu=1}}\, t, 1.0\right),$$



- Extract from the model the calculated values of
  - the carbon in atmosphere evolution,
  - the temperature evolution,
  - etc.
- Explore the results for different choices of $T^{\mu=1}$ (e.g., 50, 100) and the interest rate r.

Note: We provide some helper code to plot these quantities for a given model.

Code Session

- Explore a DICE model with one-parametric abatement model.

You can find the code of the model implementation in the
package `net.finmath.climate.models.dice`
at `https://github.com/finmath/finmath-lib` (see `src/main/java`).

You can find the code of the numerical experiment in the
class `DICEModelFromGivenParameters`
in the **package** `net.finmath.climateschool.experiments.session1`
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

CALIBRATION

Calibration

In a **calibration**, we like to find the optimal parameters that maximize some given objective function.

- For complex models, this is usually done numerically: change the parameters, check the new value, try to find the maximum value.
- For a one-parameter model without local minima, we may use a simple optimizer, like a bi-section search (e.g., the golden section search): We successively split our search interval, and drop the part with the largest value.
- For multi-parameter models, we can use optimizers like Levenberg-Marquard Optimizer or ADAM.

For the DICE model, calibration means to find the optimal abatement (and savings rate) strategy that maximizes the total social welfare.

NUMERICAL EXPERIMENT: CALIBRATION

Numerical Experiment: Calibration

- Use an optimizer to find the optimal time of reaching maximum abatement in the simple one-parameter DICE model
- Explore the results for different choices for the interest rates r.
- What is the dependency for $r \mapsto T^{\mu=1}$.

Code Session

- Calibrate the DICE Model with one-parametric abatement model.
- Explore the dependency for $r \mapsto T^{\mu=1}$.

You can find the code of the numerical experiment in the
`class DICEModelOneParametricCalibration`
in the `package net.finmath.climateschool.experiments.session2`
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

Code Session

- Calibrate the DICE Model with a full *n*-parametric abatement function

$$\mu(t) \;=\; \mu(t_i) \quad \text{for all } t \text{ with } t_i \leq t < t_{i+1},$$

  that is, we have many free parameters $\mu(t_0), \mu(t_1), \mu(t_2), \ldots$.
- Observe the iterations of the optimizer and interpret the results.

You can find the code of the numerical experiment in the
class `DICEModelCalibration`
in the **package** `net.finmath.climateschool.experiments.session3`
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

Outline

Social Cost of Carbon

The social cost of carbon is a popular concept. It is defined by

$$\mathrm{SCC}(t) = \frac{\partial V(t)}{\partial E(t)} \Big/ \frac{\partial V(t)}{\partial C(t)} \cdot (-1000),$$

where $V(t)$ is the time-t welfare (in Trillion USD = 10^{12} USD), $E(t)$ is the time t emission (in Gt CO_2 = 10^9 t CO_2) and $C(t)$ is the time t consumption (in Trillion USD).

- The unit of $\frac{\partial V(t)}{\partial E(t)}$ is "*utility*" per GtCO_2.
- The unit of $\frac{\partial V(t)}{\partial C(t)}$ is "*utility*" USD per 10^{12} USD.
- Consequently the unit of $\frac{\partial V(t)}{\partial E(t)} \Big/ \frac{\partial V(t)}{\partial C(t)}$ is 1000 USD per tCO_2.
- Consequently the unit of $\mathrm{SCC}(t)$ is 1 USD per 1 tCO_2.
- The $SCC(t)$ is the *marginal* cost in USD per tCO_2.

NUMERICAL EXPERIMENT: SOCIAL COST OF CARBON

Numerical Experiment: Social Cost of Carbon

- Calculate the social cost of carbon for a given DICE model.

Remark

- Since the calculation of the social cost of carbon requires a partial derivative, we may use advanced numerical techniques, like automatic differentiation. Our implementation allow this.
- Alternatively, we may approximate the partial derivative by a finite difference. **Recall:**

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x+h)-f(x)}{h} \qquad \text{with } h \text{ small.}$$

- Our `class DICEModel` allows to apply small shifts to the consumption (for) or the emission (for), then re-valuate the total welfare.
  - The parameter `initialConsumptionShift` applies a given shift to the consumption.
  - The parameter `initialEmissionShift` applies a given shift to the emission.
  - The parameter `isTimeIndexToShift` allows to specify at which time the shift should be applied.

Code Session

- Calculate the social cost of carbon for a given DICE model for different values of the discount rate r.

You can find the code of the numerical experiment in the
class `DICEModelSocialCostOfCarbon`
in the **package** `net.finmath.climateschool.experiments.session4`
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

THE SOCIAL COST OF CARBON DOES NOT AGREE WITH THE SOCIAL COST OF CARBON

- Assume we would pay $SCC(t)$ for every emission $E(t)$. This corresponds to a total value of
$$\int_0^T SCC(t)E(t)\ \exp(-rt)\ \mathrm{d}t$$
- Compare this with the total cost that is associated with climate change. So we compare the cost
$$\int_0^T \mathrm{Cost}(t)\ \exp(-rt)\ \mathrm{d}t$$

Observation: The two do not match.

This is, because the *SCC* is a marginal price in the equilibrium state. The true cost of emissions are far higher.

See [2] for details. A numerical experiment bases on our model is available.

Outline

Intergenerational Equity in Integrated Assessment Models

- The standard calibration of the model induces inter-generational inequality:
- present generations reduce their cost by increasing future generations cost.
- The burden of future generations is larger than that of current generations, the calibration is increasing this effect.

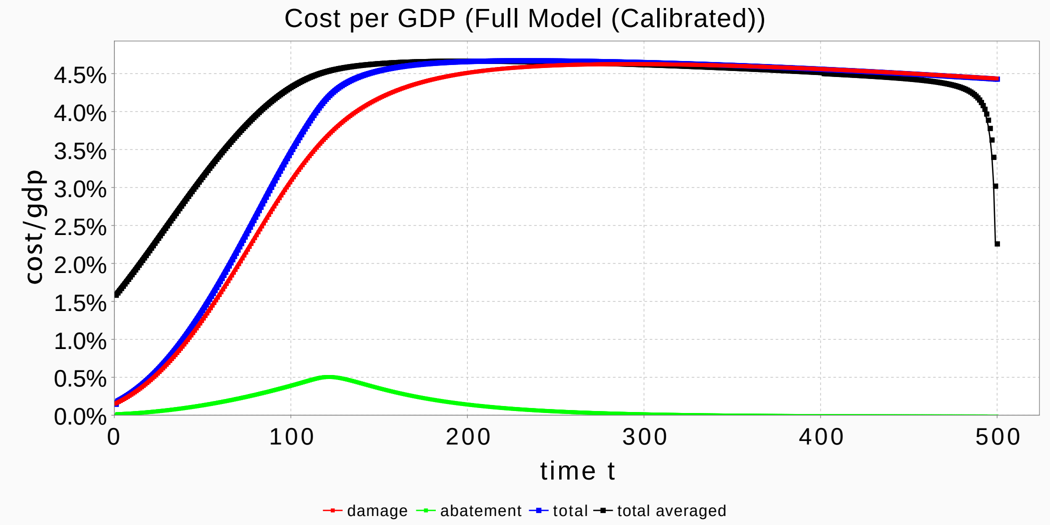


Figure 4: The cost as percentage of the GDP in the calibrated model. Damage cost (red), abatement cost (green) and the sum of the two (blue). Black shows a forward running average over a time window of 100 years.

Why are abatement cost so small, but damage cost so large?

Let $C(t)$ denote the total cost observed in time t. It is the sum of the abatement cost $C_{\mathrm{A}}(t)$ and the damage cost $C_{\mathrm{D}}(t)$. Let's investigate the objective function:

We consider a simple abatement model with a single parameter $T^{\mu=1}$. The calibration will choose the parameter $T^{\mu=1}$ that maximizes $\int_0^T V(t) \frac{N(0)}{N(t)} \mathrm{d}t$. Hence,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}T^{\mu=1}} \int_0^T V(t) \frac{N(0)}{N(t)} \mathrm{d}t &= \int_0^T \frac{\mathrm{d}V(t)}{\mathrm{d}C(t)} \frac{\mathrm{d}C(t)}{\mathrm{d}T^{\mu=1}} \frac{N(0)}{N(t)} \mathrm{d}t \\ &= \int_0^T \frac{\mathrm{d}V(t)}{\mathrm{d}C(t)} \left(\frac{\mathrm{d}C_{\mathrm{D}}(t)}{\mathrm{d}T^{\mu=1}} + \frac{\mathrm{d}C_{\mathrm{A}}(t)}{\mathrm{d}T^{\mu=1}} \right) \frac{N(0)}{N(t)} \mathrm{d}t \overset{!}{=} 0 \qquad (1) \end{aligned}$$

Note that $\frac{\mathrm{d}C_{\mathrm{D}}(t)}{\mathrm{d}T^{\mu=1}} > 0$ and $\frac{\mathrm{d}C_{\mathrm{A}}(t)}{\mathrm{d}T^{\mu=1}} < 0$.

Why are abatement cost so small, but damage cost so large?

- From (1) we see that the calibration is *balancing* the marginal abatement cost reduction with the marginal damage cost increase - weighted with the sensitivity of the utility and a discount factor.
- This is an important insight: We are not balancing the cost, we consider the gains.
- The cost sensitivity is weighted by $\frac{\mathrm{d}V}{\mathrm{d}C}$. This weight is small at later times, because the marginal utility gains decay.
- The cost sensitivity is weighted by $\frac{N(0)}{N(t)}$. This weight is the discount factor. It has an exponential decay.

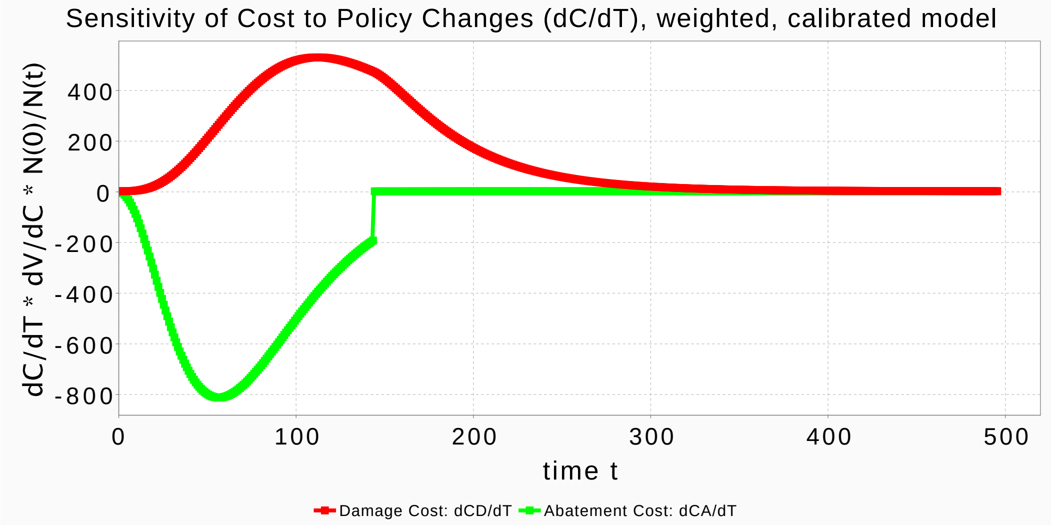


Figure 5: The sensitivity of cost to policy changes, i.e., the two integrands from expression (1).

The Figure 5 is the result of several aspects:

- Abatement cost occur earlier (immediate) than damage cost.
- Once the model has achieved 100% abatement the model is not sensitive to a change of $T^{\mu=1}$.
- The sensitivity is weighted by $\frac{\mathrm{d}V}{\mathrm{d}C}$. This weight is small at later times, because the marginal utility gains decay.
- The sensitivity is weighted by $\frac{N(0)}{N(t)}$. This weight is the discount factor. It has an exponential decay.

Outline

Convexity

- What follows is a repetition of an effect in probability theory which you very likely know from school.
- In mathematical finance, the effect is called *convexity effect*.

Expectation of a Linear Function of a Random Variable

- Consider the mapping $x \mapsto y = f(x) := 6 \cdot x - 4$.
- Assume that X is random with expectation (average) μ, i.e., $\mu = E(X) = 2$
- What is the expectation of the mapped values Y (that is, of $Y = f(X) = 6 \cdot X - 4$?
- The answer is $6 \cdot \mu - 4 = 8$.
- **Let us illustrate this!** $\rightarrow$ *Computer Experiment*.

Observation

Note that the result above depends only on the expectation of X, that is, on μ.
The variance of X does not matter.
But: This is not always the case!

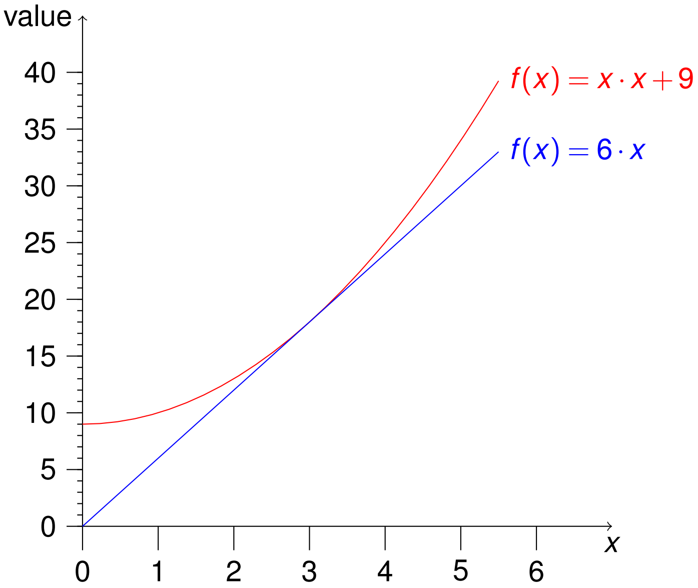
Expectation of a Non-Linear Function of a Random Variable

- Consider the mapping $x \mapsto y = f(x) := x^3$.
- Assume that X is random with expectation (average) μ, i.e., $\mu = E(X) = 2$
- What is the expectation of the mapped values Y (that is, of $Y = f(X) = X^3$?
- The answer depends on the variance!
- **Let us illustrate this!** $\rightarrow$ *Computer Experiment*.

Observation

Note that the expectation $E(Y)$ increases as the variance of X increases.
We may interpret the variance (or standard deviation) of X as some kind of *risk*. The possibility that X deviates from the expectation μ.
If X is the input to a convex (non-linear) function, then more risk increases the expectation of the result.
Hence: using models with expected values may underestimate some results.
We will observe this: uncertainty in interest rates increases the intergenerational in-equality.

Convexity



Code Session: Convextiy

- Illustrate the effect of convexity on the expectation of a function of a random variable.

You can find the code of the numerical experiment in the
class `ConvexityExperiments`
in the **package** `net.finmath.climateschool.experiments.session5`
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

Outline

Stochastic Interest Rates

- Interest rates change over time.
- We do not know how they change in the future - there are different possible values that occur with certain probabilities.
- In mathematical finance there exists a rich family models that model stochastic interest rates.
- In computational finance a popular method is a Monte-Carlo simulation.

- We simulate many (thousands) of szenarios ω for future values (in time$t$) of the interest rate r.
- The $r(t)$ is now a random variable. Here, $t \mapsto r(t)$ is called stochastic process.
- One can then analyse the expectation or some quantile (risk).

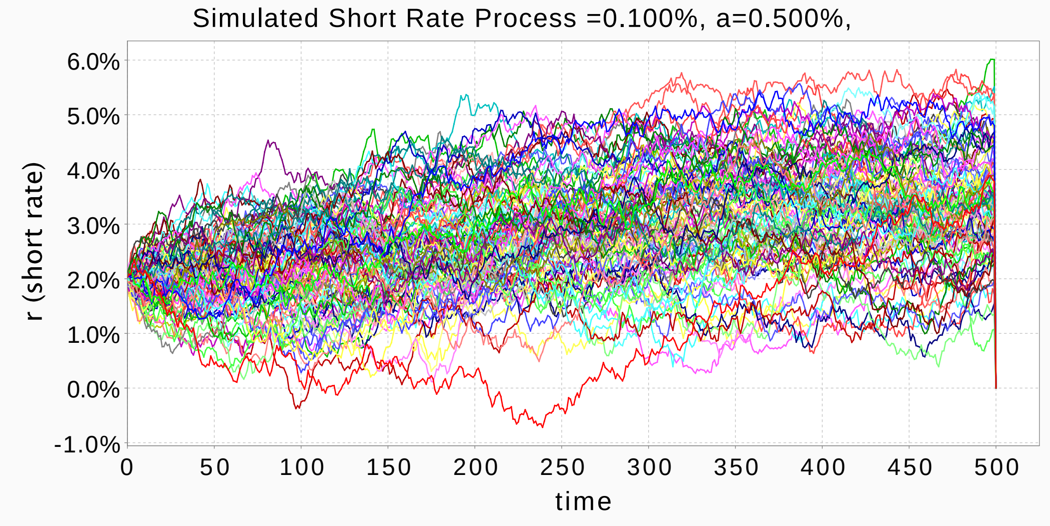


Figure 6: A simulation of the interest rate r (short rate). This is a Hull-White model with parameters $\sigma = 0.1\%$ (volatility) and $a = 0.5\%$ (mean reversion speed).

Code Session: Stochastic Interest Rates

- Generate and plot sample paths of a classical Hull-White model.

You can find the code of the numerical experiment in the
class `StochasticRatesExperiment`
in the **package** `net.finmath.climateschool.experiments.session6`
at `https://github.com/finmath/climate-school-exercises` (see `src/main/java`).

Outline

IAM with Stochastic Interest Rates

- Given that interest rates have a strong impact on the optimal abatement policy, a change in the interest rate curve should lead to a change in the abatement policy, see Figure 3.
- Hence, we introduce a **stochastic abatement model** that reacts on the change of the interest rate level.
- Though this, all model quantities, including abatement cost and damage cost will become stochastic.

Stochastic Abatement Model

We consider a very simple three parametric stochastic model, where a the abatement speed can be reduced or increased as a function of the current interest rate level r (short rate).

The model is

$$\mu(t,\omega) = \min\left(\mu_0 + (a_0 + a_1 \cdot r(t,\omega)) \cdot t, 1.0\right). \tag{2}$$

The parameter a_0 is the speed at which we approach 100 % abatement, but with $a_1 \neq 0$ this speed depends on the interest rate level. Recall Figure 3.

From Figure 3 we expect that a_1 is negative: if interest rate level is larger, we will be slower with abatement.

Stochastic Abatement Model

Having a stochastic abatement model, all quantities will become stochastic: emission, carbon in atmosphere, damage, cost.

The stochastic abatement model introduces another aspect to the model for which we may analyse the temporal distribution (and inter-generational equity): *risk*.

In the following figure we depict the risk (+1 std dev) of the abatement cost and damage cost, following the optimal abatement strategy (within this model).

Surprise: If the abatement policy is adapting to changing interest rate levels, the intergenerational inequality becomes *worse*.

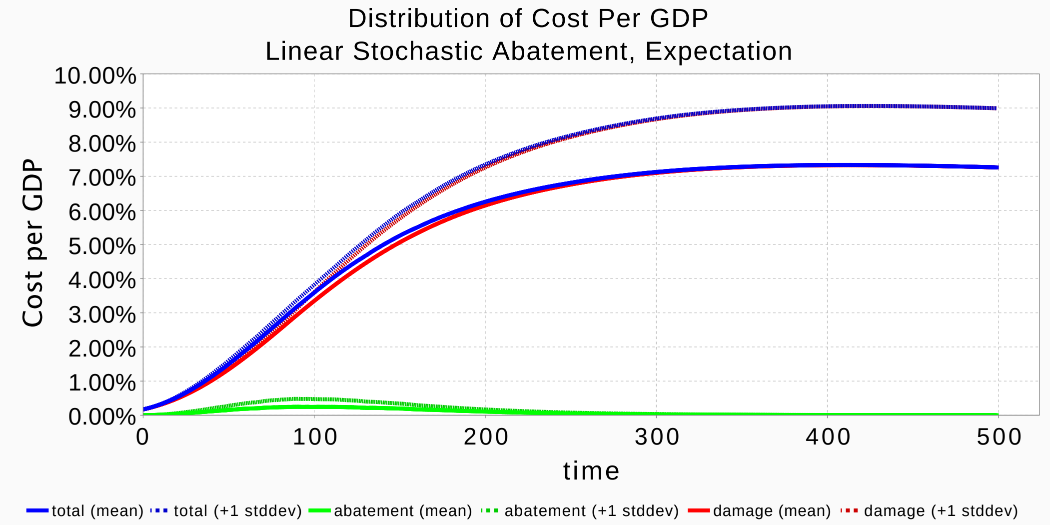


Figure 7: Undiscounted cost per GDP distribution using stochastic interest rates and the abatement model Equation (**??**) with the expectation as objective function.

Stochastic Abatement Model

Insights

Future generations are exposed to higher cost in expectation and to significant **risk** on the damage cost.

So the insight is that repeated adaptation of the abatement strategy to interest rate changes will increase the risk of larger damage cost.

Outline

Extensions that will Improve the Intergenerational Equity

In our recent work [5] we show that two simple (model) extensions improve the intergenerational equity:

- Allowing Funding of Abatement Cost: Abatement cost is funded by a loan.
- Introduce Non-Linear Discounting of Damage Cost: Large damages get a larger weight (a larger discount factor)..

Both extensions have an economic interpretation / motivation.

Extensions that will Improve the Intergenerational Equity

Adding Funding to Integrated Assessment Models

Funding of Abatement Cost

We add the following modification to the model:

- Funding of abatement cost: we allow that time t abatement cost are financed by a load, paid back in time $t + \Delta T$ - with a market forward rate $FR(t, t + \Delta T; t)$.

In a risk neutral valuation model this modification would have not effect - given that the forward rate is derived from the discount factor. For the DICE model, we see an effect. The reason is the change in the sensitivity of the utility.

$\Rightarrow$ Allowing funding,

- improves the intergenerational equity (Figure 8), and
- reduces the risk of cost in a stochastic model (Figure 9).
- The optimal policy is to abate earlier (Figure 10).

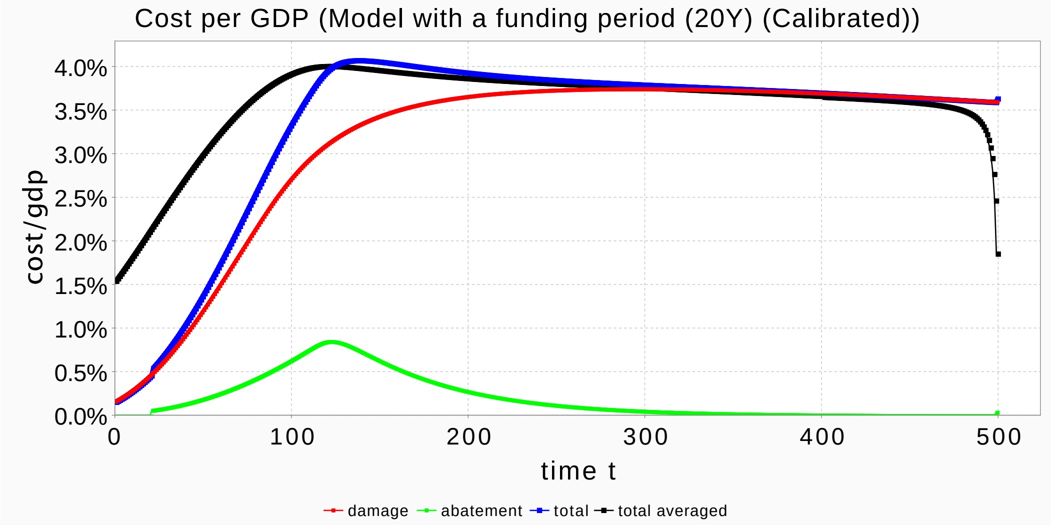


Figure 8: The cost per GDP in the calibrated model using a model with a 20 year funding period for the abatement. Damage cost (red), abatement cost (green) and the sum of the two (blue). Black shows a forward running average over a time window of 100 years.

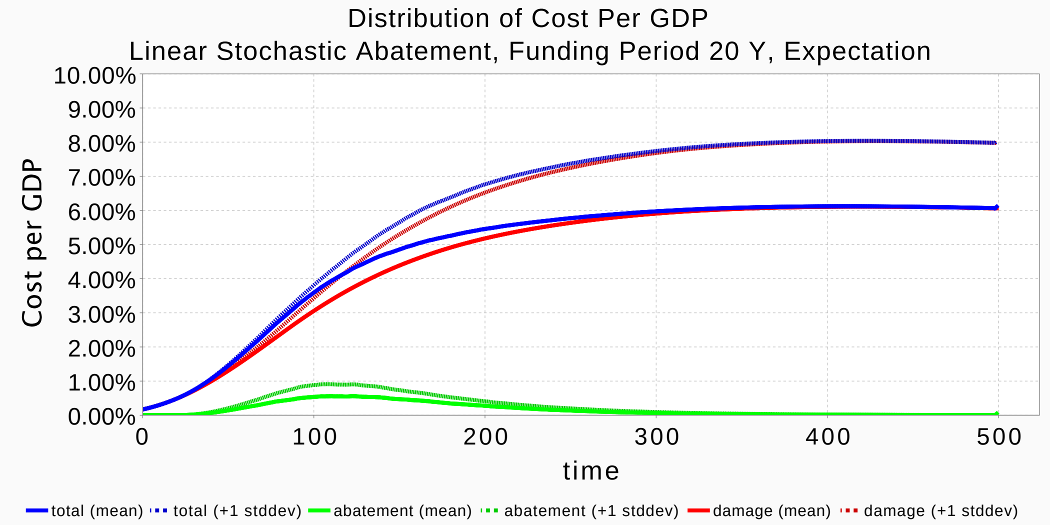


Figure 9: Undiscounted cost per GDP distribution using stochastic interest rates, the abatement model (2) and funding of abatement.

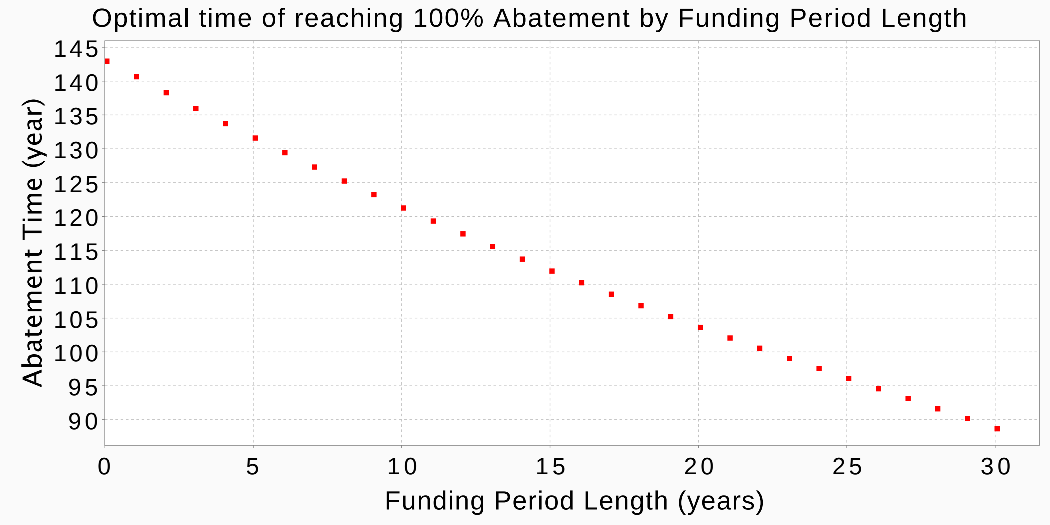


Figure 10: The calibrated (optimal) parameter T^{μ} for models using different funding periods.

A 20 year funding period for the abatement cost would induce a preference of performing abatement roughly 40 years earlier.

ADDING NON-LINEAR DISCOUNTING TO INTEGRATED ASSESSMENT MODELS

We add the following modification to the model:

- Non-Linear Discounting: Larger Cost get a larger discount factor (modelling that large projects / damages are usually over budget).

Adding non-linear discounting of cost is a very simple way to limit cost. Applying a larger weight (discount factor) to large cost than to small cost acts like a penalty. Calibration will favour equidistribution of cost among generations.

We can add the constrain the cost get a larger weight if they grow over 3% of the GDP. The calibration will almost respect this limit:

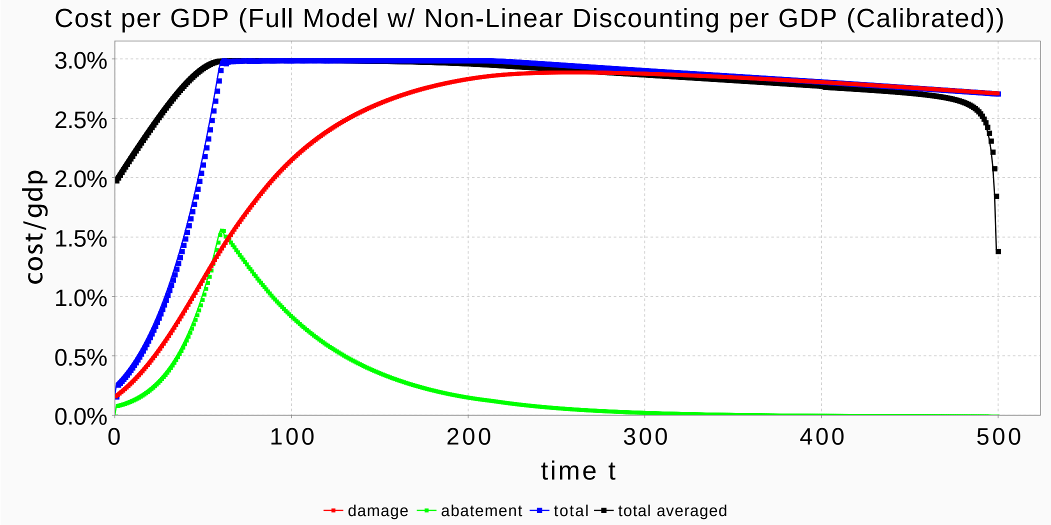


Figure 11: The temporal distribution of cost measured per GDP when using a model with a (strong) non-linear discounting applied to total cost per GDP.

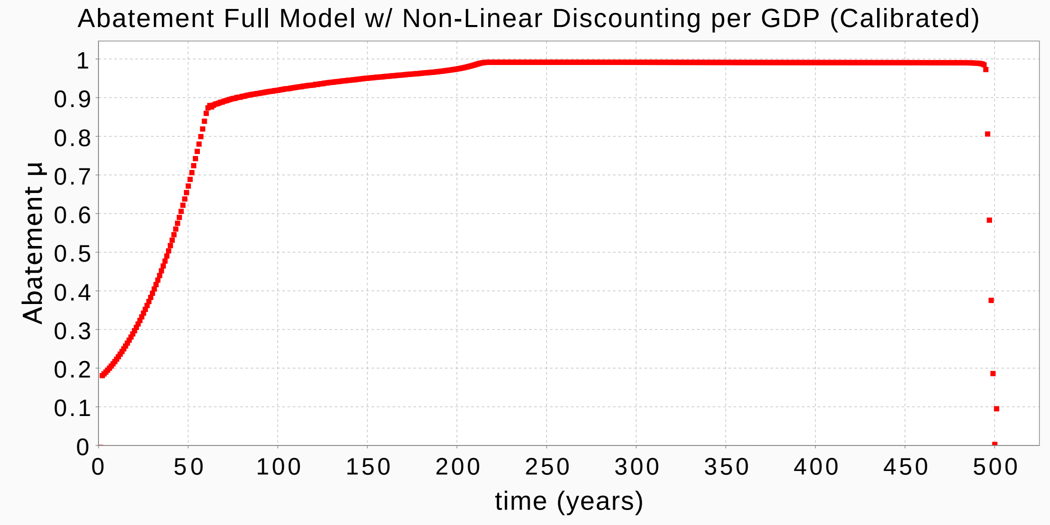


Figure 12: The calibrated (optimal) abatement function $t \mapsto \mu(t)$ for the calibrated model with non-linear discounting applied to the cost (larger than 3% of the GDP)..

Improving Inter-Generational Equity

In other words: to limit the cost for future generations below 3% of the GDP we should ramp up abatement very fast - and contribute our share.

Outline

Summary

- The classical calibration of the DICE model creates inter-generational inequality.
- This is a combined effect of
  - the calibration optimizing the marginal cost changes and
  - the sensitivity of the utility and the discounting.
- Adding stochastic interest rates exhibits that the optimal (stochastic) abatement policy will create risk for future generations: hence, it adds even more inter-generational inequality.
- Allowing funding of abatement, the calibration will choose a strategy with improve inter-generational equity.
- Adding non-linear discounting of cost, that creates a larger weight for large cost, greatly improves inter-generational equity of the optimal climate mitigation pathway.

Outline

Implementation (Open Source)

- You may explore the code of the DICE model, our extensions and all numerical experiments by checking out our open source repositories.

You can find the code of the classical DICE model implementation in the
`package` `net.finmath.climate.models.dice`
at `https://github.com/finmath/finmath-lib` (see `src/main/java`).

You can find our stochastic DICE model and extensions
at `https://gitlab.com/finmath/finmath-climate-nonlinear`.

Source Code

You can find the code of the numerical experiment at `https://github.com/finmath/climate-school-exercises`.

You can also find code with numerical experiments related to the classical DICE model in `class DICEModelExperiment` in the `package net.finmath.experiments.dice` at `https://github.com/finmath/finmath-experiments` (see `src/main/java`).

Outline

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[1] finmath.net. *finmath-lib: Mathematical Finance Library: Algorithms and methodologies related to mathematical finance*. URL: http://finmath.net/finmath-lib.

[2] Christian P. Fries. "Implied CO2-Price and Interest Rate of Carbon". In: *SSRN* (2023). URL: https://ssrn.com/abstract=3284722.

[3] Christian P. Fries. *Introduction to Integrated Assessment Models / DICE Model. Lecture*. URL: https://youtu.be/i_SlqpgVCao.

[4] Christian P. Fries. *Mathematical Finance. Theory, Modeling, Implementation*. John Wiley & Sons, 2007. DOI: 10.1002/9780470179789. URL: http://www.christian-fries.de/finmath/book.

[5] Christian P. Fries and Lennart Quante. "Intergenerational Equity in Models of Climate Change Mitigation: Stochastic Interest Rates introduce Adverse Effects, but (Non-linear) Funding Costs can Improve Intergenerational Equity". In: *SSRN* (2023). URL: https://ssrn.com/abstract=4005846.

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[6] William D. Nordhaus. “Revisiting the social cost of carbon”. In: *Proceedings of the National Academy of Sciences* 114.7 (2017), pp. 1518–1523. ISSN: 0027-8424. DOI: 10.1073/pnas.1609244114. eprint: https://www.pnas.org/content/114/7/1518.full.pdf. URL: https://www.pnas.org/content/114/7/1518.